Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 21, Number 2 (2025), pp. 251-264

http://www.ripublication.com/gjpam.htm

Some Common Fixed Point Theorems of Integral Type Contraction on Cone Metric Spaces with Application

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Abstract

In this paper, we establish new common fixed point theorems for mappings satisfying an integral-type contraction condition in cone metric spaces. Detailed examples are provided and applications to nonlinear differential equations with periodic boundary conditions are presented. The results guarantee existence and uniqueness of solutions, thereby extending the fixed point theory framework.

2020 Mathematics Subject Classification (2020) : 47H10, 54H25.

Keywords: Common fixed point, Cone metric spaces, Property E.A, Contraction mappings of integral type, Weakly compatible mappings

1. INTRODUCTION

The development of fixed point theory in various generalized metric frameworks has significantly evolved over the past few decades, driven by both theoretical interest and practical applications. One such generalization is the concept of K-metric and K-normed spaces, initially surveyed by Zabrejko [17] in 1997. This foundational work laid the groundwork for future explorations of metric-like structures and their potential for accommodating fixed point results.

Building upon this foundation, Pant [13] in 1998, introduced new common fixed point theorems for contractive mappings, which expanded the range of applicability of the

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Banach contraction principle. These early explorations catalyzed a wave of research seeking to refine the contractive conditions under which fixed points can be guaranteed.

In 2002, Aamri and El Moutawakil [1] contributed to this effort by presenting novel fixed point theorems under strict contractive conditions, while Branciari [7] proposed an alternative contractive framework involving integral-type conditions. These works not only introduced new contractive forms but also suggested the potential for broader classes of functions to satisfy fixed point conditions.

A major conceptual shift occurred in 2007, when Long-Guang and Zian [11] introduced the framework of cone metric spaces, replacing real-valued distances with those valued in a Banach space ordered by a cone. This new structure allowed for richer geometrical and topological interpretations, particularly suited for applications in nonlinear analysis.

This new direction inspired a surge of interest. In 2008, Ilić and Rakočević [10] studied common fixed points within this cone metric setting, followed by Abbas and Jungck [2], who relaxed continuity assumptions to prove fixed point results for noncommuting mappings. Rezapour and Hamlbarani [14] also provided critical commentary on foundational aspects of cone metric spaces, refining the theoretical underpinnings of this emerging field.

Subsequent years witnessed a deepening of this line of inquiry. Arshad et al. [3] in 2009, and later Azam and Arshad [4], extended common fixed point results for generalized contractive maps within cone metric spaces, thereby enriching the existing theory with broader applicability. By 2010, Haghi and Rezapour [9] expanded these findings to multifunctions in regular cone metric spaces, further enhancing the scope of fixed point theorems.

In 2011, Cho and Bae [8] introduced the use of property E.A in the context of cone metric spaces, offering additional avenues for the existence of common fixed points under more general conditions. This trend of generalization continued into 2019, with Song et al. [16] developing novel fixed point theorems for mappings in cone metric spaces, while Meng [12] contributed to the theoretical base by exploring generalized algebraic cone metric spaces.

Recently, in 2020, Bakr [5] introduced the notion of Theta cone metric spaces, a refinement of the traditional cone metric structure, and provided several fixed point theorems within this framework. In 2023, Sanjay et al. [15] addressed integral type contractions in cone metric spaces, further advancing both the depth and breadth of the theory and exploring applications to differential and integral equations.

2. PRELIMINARIES

Definition 2.1. [11] For a non-empty set X and real Banach space E and $d : X \times X \to E$ be a mapping fulfills all the conditions of metric space. Thus d is called a cone metric space as define by Long-Guang and Xian [11].

Definition 2.2. [11] For a non-empty set X and the mapping $d : X \times X \to E$, a sequence x_n is called
(i) convergent sequence for $c >> 0$ there exist an positive integer n_0 so as to $d(x_n, x) << c$ for all $n > n_0$ and for certain x in X and any c in E.
(ii) A Cauchy sequence if for any $c >> 0$, there is n_0 be a natural number, so as to $d(x_n, x_m) << c$ for all $n,\ m > n_0$, for some x in X and every c in E.
For the completeness of cone metric space (X, d) every Cauchy sequence should be convergent in X.

Definition 2.3. [11] In a cone metric space (X, d) the point $z \in X$ is called the point of coincidence for the mappings $f, g : X \to X\ if\ w = fz = gz$.

Definition 2.4. [11] Suppose f and g be two self-mappings on cone metric space X, then for weak compatibility of $f,\ g$ the required condition is $fgz = gfz$ when fz is equal to gz for certain z in X.

Definition 2.5. [11] In a cone metric space (X, d), the pair (S, T) fulfills property E.A if there exists x_n in X and a point t in X so as to $lim_{n\to\infty} d(Sx_n, t) = lim_{n\to\infty} d(Tx_n, t) = 0$.

Lemma 2.6. *[11] Suppose that ϕ in Φ and r_n (where n is natural number) with $r_n = a$ (as $\lim\limits_{n\to\infty}$). Then* $\lim_{n\to\infty} \int_0^{r^n} \phi(t)dt = \int_0^a \phi(t)dt$.

Remark 2.7. [11] Let $P,\ Q,\ R\ and\ S$ be four self-mappings in X, such that $P(X)$ is the sub set of $S(X)$ and $Q(X)$ is the subset of $R(X)$, and (Q, S) satisfies "property E.A" and $S(X)$ or $B(X)$ is complete subspace of X. Then $(A,\ B,\ S,\ T)$ is BC's 4-tuple.

The current paper aims to build upon this rich history by exploring new classes of contractive conditions within cone metric spaces, with a particular emphasis on generalization and application. The approach integrates several previous methodologies while introducing novel structures to unify and extend existing results.

3. MAIN RESULT

Now, we establish a fixed point result for a class of generalized contractive conditions involving a new control function $\Delta_2(x, y)$. This result further extends the applicability of BC's 4-tuple framework in cone metric spaces.

Let ψ be the set of all functions $\psi : [0, \infty) \to [0, \infty)$ such that:

1. ψ is continuous and non-decreasing;
2. $\psi(t) < t$ for all $t > 0$;
3. $\psi(0) = 0$.

Theorem 3.1. *Suppose that $K,\ M,\ N,\ L$ is a BC's 4-tuple sustaining*

$$\int_0^{d(Kx,Ly)} \phi(t)dt \ \leq\ \psi\left(\int_0^{\Delta_2(x,y)} \phi(t)dt\right) \tag{3.1}$$

where $(\phi, \psi) \in \phi \times \psi$ and

$$\Delta_2(x, y) = \alpha_1 d(Nx, My) + \alpha_2 d(Nx, Ly) + \alpha_3 d(My, Ly) + \alpha_4 d(x, y) \tag{3.2}$$

for each $x,\ y\ \in X$, where $\alpha_1,\ \alpha_2, \alpha_3,\ \alpha_4\ \geq 0$ and $\alpha_1 +\ \alpha_2 + \alpha_3 +\ \alpha_4\ <\ 1$. Then (K, N) and (L, M) have point of coincidence in X. As well as if they have weak compatibility, then $K,\ M,\ N,\ L$ have a unique common fixed point in X.

Proof. Since (L, M) fulfills property E.A, there exists $\{x_n\}$ in X and $t \in X$, such that $Lx_n = Mx_n = t\ as\ \lim_{n\to\infty}$.
Since $L(X) \subset N(X)$, there exists a sequence $\{y_n\}$ in X, such that $Lx_n = Ny_n$. Hence $lim_{n\to\infty} = t$.
By the completeness of $N(X)$ in X, there exists u in X, such that $t = Nu$.
Thus $d(Ny_n, Nu) = d(Lx_n, Nu) = d(Mx_n, Nu) = 0\ (as\ n \to \infty)$. Putting $x = u$ and $y = x_n$ in contraction conditions of main results, we have

$$\int_0^{d(Ku,Lx_n)} \phi(t)\ \leq\ \psi\left(\int_0^{\Delta_2(u,x_n)} \phi(t)dt\right) \tag{3.3}$$

where $(\phi, \psi) \in \phi \times \psi$ and

$$\Delta_2(u, x_n) = \alpha_1 d(Nu, Mx_n) + \alpha_2 d(Nu, Lx_n) + \alpha_3 d(Mx_n, Lx_n) + \alpha_4 d(u, x_n) \tag{3.4}$$

taking $n \to \infty$ as upper limit in both the equations, we have

$$\lim_{n\to\infty} \int_0^{d(Ku,Lx_n)} \phi(t)dt\ \leq\ \psi\left(\ \lim_{n\to\infty}\left(\int_0^{\Delta_2(u,x_n)} \phi(t)dt\right)\right)$$

or

$$\int_0^{d(Ku,Nu)} \phi(t)dt \ \leq \ \psi\left(\lim_{n\to\infty}\left(\int_0^{\Delta_2(u,x_n)} \phi(t)dt\right)\right)$$

implies $\int_0^{d(Ku,Nu)} \phi(t)dt \ \leq 0$. Hence

$$Ku = Nu. \tag{3.5}$$

Since $K(X) \subset M(X)$, then there exists v in X, such that

$$Ku = Mv. \tag{3.6}$$

Interchanging x with u and y with v in equation (3.1), we get

$$\int_0^{d(Ku,Lv)} \phi(t)dt \ \leq \ \psi\left(\int_0^{\Delta_2(u,v)} \phi(t)dt\right) \tag{3.7}$$

where

$$\begin{aligned}\Delta_2(u,v) &= \alpha_1 d(Nu,Mv) + \alpha_2 d(Nu,Lv) + \alpha_3 d(Mv,Lv) + \alpha_4 d(u,v)\\ &= \alpha_1 d(Mv,Mv) + \alpha_2 d(Mv,Lv) + \alpha_3 d(Mv,Lv) + \alpha_4 d(u,v)\\ &= 0 + (\alpha_2+\alpha_3) d(Mv,Lv) + \alpha_4 d(u,v).\end{aligned} \tag{3.8}$$

Since $Ku = Mv$, from $(3.6),\ (3.7),\ and\ (3.8)$, we have
$\int_0^{d(Ku,Lv)} \phi(t)dt \ \leq \ \psi(\int_0^{\Delta_2(u,v)} \phi(t)dt)$.
Which is possible only when $d(Mv,Lv) = 0$. Thus $Mv = Lv$, hence

$$Ku \ = \ Mv \ = \ Nu \ = \ Lv. \tag{3.9}$$

Since (K,N) is weakly compatible, $KNu = NKu$. Then, we have

$$KKu = KNu = NKu = NNu, \tag{3.10}$$

and

$$MMv = MLv = LMv = LLv, \tag{3.11}$$

put $x = Ku$ and $y = v$ in (3.1) and (3.3), we have

$$\int_0^{d(KKu,Lv)} \phi(t)dt \ \leq \ \psi\left(\int_0^{\Delta_2(Ku,v)} \phi(t)dt\right) \tag{3.12}$$

where

$$\begin{aligned}\Delta_2(Ku, v) &= \alpha_1 d(NKu, Mv) + \alpha_2 d(NKu, Lv) + \alpha_3 d(Mv, Lv) + \alpha_4 d(Ku, v),\\ &= (\alpha_1 + \alpha_2) d(KKu, Ku) + 0 + \alpha_4 d(Ku, v),\\ &= (\alpha_1 + \alpha_2) d(KKu, Ku) + 0 + \alpha_4 d(Ku, v).\end{aligned} \tag{3.13}$$

from (3.12) and (3.13), we get
$\int_0^{d(KKu,Lv)} \phi(t)dt \ \leq \ \psi \left(\int_0^{(\alpha_1+\alpha_2)d(KKu,Ku)+\alpha_4 d(Ku,v)} \phi(t)dt \right)$.
Thus, from (3.9),
$\int_0^{d(KKu,Ku)} \phi(t)dt \ \leq \ \psi \left(\int_0^{(\alpha_1+\alpha_2)d(KKu,Ku)+\alpha_4 d(Ku,v)} \phi(t)dt \right)$
implies $d(KKu, Ku) = 0$.
Which yields $KKu = Ku \ and \ Ku = KKu = NKu$. Similarly, we have Lv is the common fixed point of L and M. Hence $LLv = MLv = Lv$, since $Ku = Lv$ implies $KKu = NKu = MKu = LKu$.
Thus, Ku is common fixed point for all the four mappings.

Uniqueness: Suppose that another common fixed point of all four mappings is w, put x is equal to z and y is equal to w in (3.1) and (3.2), we have

$$\int_0^{d(Kz,Lw)} \phi(t)dt \ \leq \ \psi \left(\int_0^{\Delta_2(z,w)} \phi(t)dt \right) \tag{3.14}$$

where

$$\begin{aligned}\Delta_2(z, w) &= \alpha_1 d(Nz, Mw) + \alpha_2 d(Nz, Lw) + \alpha_3 d(Mw, Lw) + \alpha_4 d(z, w),\\ &= \alpha_1 d(z, w) + \alpha_2 d(z, w) + 0 + \alpha_4 d(z, w),\\ &= (\alpha_1 + \alpha_2 + \alpha_4) d(z, w).\end{aligned} \tag{3.15}$$

from (3.14) and (3.15),

$\int_0^{d(Kz,Lw)} \phi(t)dt \ < \ \left(\int_0^{(\alpha_1+\alpha_2+\alpha_4)d(z,w)} \phi(t)dt \right)$
implies $d(z, w) = 0$. Hence $z = w$. □

The following example illustrates the applicability of our result by verifying the required conditions for specific mappings. This demonstrates the effectiveness of the established theorem in concrete settings.

Example 3.2. Let $X = [0, \infty)$ and define a cone metric $d : X \times X \to \mathbb{R}_+$ by

$$d(x, y) = |x - y|, \quad \text{for all } x, y \in X.$$

Let $E = \mathbb{R}$ and define the cone $P = [0, \infty)$. Then (X, d) is a cone metric space over E with the cone P, which is closed, convex and normal with nonempty interior.

Define the self-mappings $K,\ M,\ N,\ L : X \to X$ as follows:

$$K(x) = \frac{x}{6},\ M(x) = \frac{x}{4},\ N(x) = \frac{x}{5},\ L(x) = \frac{x}{3},\ \text{for all } x \in X.$$

Let $\phi : [0, \infty) \to [0, \infty)$ be defined by $\phi(t) = 1$, and $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{1}{2}t$. Clearly, ϕ is Lebesgue integrable on every compact subset of $[0, \infty)$ and ψ is continuous, non-decreasing and satisfies $\psi(t) < t$ for all $t > 0$.

Choose $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{6}$, so that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \frac{2}{3} < 1$.

Now,

$$d(Kx, Ly) = \left|\frac{x}{6} - \frac{y}{3}\right| = \left|\frac{x-2y}{6}\right|,\ d(Nx, My) = \left|\frac{x}{5} - \frac{y}{4}\right| = \left|\frac{4x-5y}{20}\right|,$$

$$d(Nx, Ly) = \left|\frac{x}{5} - \frac{y}{3}\right| = \left|\frac{3x-5y}{15}\right|,\ d(My, Ly) = \left|\frac{y}{4} - \frac{y}{3}\right| = \frac{y}{12}.$$

Then,

$$\begin{aligned}\Delta_2(x,y) &= \alpha_1 d(Nx, My) + \alpha_2 d(Nx, Ly) + \alpha_3 d(My, Ly) + \alpha_4 d(x,y)\\ &= \frac{1}{6}\left|\frac{4x-5y}{20}\right| + \frac{1}{6}\left|\frac{3x-5y}{15}\right| + \frac{1}{6}\cdot\frac{y}{12} + \frac{1}{6}|x-y|\\ &= \left|\frac{4x-5y}{120}\right| + \left|\frac{3x-5y}{90}\right| + \frac{y}{72} + \frac{1}{6}|x-y|.\end{aligned}$$

Also

$$\int_0^{d(Kx,Ly)} \phi(t)\,dt = \left|\frac{x-2y}{6}\right|,$$

and

$$\psi\left(\int_0^{\Delta_2(x,y)} \phi(t)\,dt\right) = \frac{1}{2}\cdot\Delta_2(x,y).$$

Thus, the inequality

$$\int_0^{d(Kx,Ly)} \phi(t)\,dt \le \psi\left(\int_0^{\Delta_2(x,y)} \phi(t)\,dt\right)$$

becomes

$$\left|\frac{x-2y}{6}\right| \le \frac{1}{2}\left(\left|\frac{4x-5y}{120}\right| + \left|\frac{3x-5y}{90}\right| + \frac{y}{72} + \frac{1}{6}|x-y|\right),$$

which is satisfied for all $x, y \in X$.

Since the mappings K, M, N, L are continuous and weakly compatible, all conditions of Theorem 3.1 are satisfied. The unique common fixed point of K, M, N, L is 0.

Now, we prove a fixed point theorem involving integral type contractive conditions in cone metric spaces. This result ensures the existence and uniqueness of a common fixed point for a pair of weakly compatible self-mappings under suitable assumptions.

Theorem 3.3. *Let (X, d) be a cone metric space over a Banach space E. Let $T,\ S : X \to X$ are two self-mapping satisfying following conditions*

1. *There exists a functions $\phi : [0, \infty) \to [0, \infty)$, such that for all $x,\ y \in X$, $\int_0^{d(Tx,Sy)} \phi(t)dt \leq \alpha \int_0^{L(x,y)} \phi(t)dt$, where $L(x, y) = max\{d(x, y),\ d(Tx, x),\ d(Sy, y),\ d(Tx, Sy)\}$, $\alpha \in [0, 1)$,*
2. *$S(X)$ is complete subspace of X and $T(X) \subseteq S(X)$,*
3. *T and S are weakly compatible.*
   Then T and S have a unique common fixed point in X.

Proof. Let us choose an arbitrary point $x_0 \in X$ and define a sequence $\{x_n\}$ in X by setting

$$x_1 = Tx_0, \quad x_2 = Sx_1, \quad x_3 = Tx_2, \quad x_4 = Sx_3, \quad \dots$$

so that we generate the sequence recursively as,

$$x_{2n+1} = Tx_{2n}, \quad x_{2n+2} = Sx_{2n+1} \quad \text{for all } n \in \mathbb{N}.$$

We will prove that $\{x_n\}$ is a Cauchy sequence in $S(X)$.
For all $x, y \in X$, we have,

$$\int_0^{d(Tx,Sy)} \phi(t)dt \leq \alpha \int_0^{L(x,y)} \phi(t)dt,$$

where

$$L(x, y) = \max\{d(x, y), d(Tx, x), d(Sy, y), d(Tx, Sy)\}, \quad \alpha \in [0, 1).$$

Applying this to the sequence elements, for each $n \in \mathbb{N}$, let us denote $x = x_{2n}$, $y = x_{2n+1}$. Then:

$$\int_0^{d(x_{2n+1},x_{2n+2})} \phi(t)dt = \int_0^{d(Tx_{2n},Sx_{2n+1})} \phi(t)dt \ \leq \alpha \int_0^{L(x_{2n},x_{2n+1})} \phi(t)dt.$$

Similarly, since ϕ is non-negative and non-decreasing, and $\alpha < 1$, we get a sequence of integrals, implying that

$$\int_0^{d(x_{n+1},x_n)} \phi(t)dt \leq \alpha^n \int_0^{L(x_0,x_1)} \phi(t)dt.$$

Hence,

$$\lim_{n\to\infty} \int_0^{d(x_{n+1},x_n)} \phi(t)dt = 0 \; implies \; \lim_{n\to\infty} d(x_{n+1},x_n) = 0,$$

since $\phi(t) > 0$ for all $t > 0$.

Now we show that $\{x_n\}$ is Cauchy. For $m > n$, we have:

$$d(x_n,x_m) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{m-1},x_m).$$

Since $d(x_{n+1},x_n) \to 0$, the series converges and $\{x_n\}$ is a Cauchy sequence in $S(X)$, which is complete. So, there exists $z \in S(X)$ such that $x_n \to z$.

Now, since $T(X) \subseteq S(X), Tz \in S(X)$. Also, from continuity of the integral inequality and limits, we get

$$\begin{aligned} \int_0^{d(Tz,Sz)} \phi(t)dt \leq& \alpha \int_0^{L(z,z)} \phi(t)dt \\ =& \alpha \int_0^{d(Tz,Sz)} \phi(t)dt. \end{aligned}$$

This implies,

$$\begin{aligned} (1-\alpha)\int_0^{d(Tz,Sz)} \phi(t)dt \leq 0 \; implies \; & \int_0^{d(Tz,Sz)} \phi(t)dt = 0 \\ & implies \; d(Tz,Sz) = 0. \end{aligned}$$

Hence, $Tz = Sz$. Since T and S are weakly compatible, it follows that $T(Sz) = S(Tz)$. Therefore, z is a common fixed point of both T and S.

Finally, the uniqueness follows from the contraction inequality. Suppose there exist two common fixed points $z_1, z_2 \in X$ such that $Tz_1 = Sz_1 = z_1$ and $Tz_2 = Sz_2 = z_2$. Then

$$\begin{aligned} \int_0^{d(z_1,z_2)} \phi(t)dt &= \int_0^{d(Tz_1,Sz_2)} \phi(t)dt \\ &\leq \alpha \int_0^{L(z_1,z_2)} \phi(t)dt \\ &\leq \alpha \int_0^{d(z_1,z_2)} \phi(t)dt. \end{aligned}$$

Which yields,

$$(1-\alpha)\int_0^{d(z_1,z_2)}\phi(t)dt \le 0 \, implies \int_0^{d(z_1,z_2)}\phi(t)dt = 0 \, implies \, d(z_1,z_2) = 0 \, implies \, z_1 = z_2.$$

Hence, T and S have a unique common fixed point in X. □

The next example illustrate that all the conditions of the Theorem 3.3 are satisfied.

Example 3.4. Let $X = [0,\infty)$ and define a cone metric $d : X \times X \to \mathbb{R}_+$ by

$$d(x,y) = |x-y|, \quad \text{for all } x, y \in X.$$

Let $E = \mathbb{R}$ and define the cone $P = [0,\infty)$, which is a closed, normal cone with nonempty interior. Then (X,d) is a cone metric space over the Banach space E.

Define two self-mappings $T, S : X \to X$ as follows:

$$T(x) = \frac{x}{4}, \quad S(x) = \frac{x}{2}, \quad \text{for all } x \in X.$$

Let the function $\phi : [0,\infty) \to [0,\infty)$ be defined by $\phi(t) = 1$, which is Lebesgue integrable on any compact subset of $[0,\infty)$. Choose a constant $\alpha = \frac{1}{2} \in [0,1)$.

We define:

$$L(x,y) = \max\{d(x,y), d(Tx,x), d(Sy,y), d(Tx,Sy)\}, \text{ for all } x, y \in X.$$

Take arbitrary $x, y \in X$. Then

$$d(Tx,Sy) = \left|\frac{x}{4} - \frac{y}{2}\right| = \left|\frac{x-2y}{4}\right|, \, d(Tx,x) = \left|\frac{x}{4} - x\right| = \frac{3x}{4} \, and \, d(Sy,y) = \left|\frac{y}{2} - y\right| = \frac{y}{2},$$

Hence,

$$L(x,y) = \max\left\{|x-y|, \frac{3x}{4}, \frac{y}{2}, \left|\frac{x-2y}{4}\right|\right\}.$$

Now,

$$\int_0^{d(Tx,Sy)}\phi(t)dt = \left|\frac{x-2y}{4}\right|.$$

and

$$\alpha\int_0^{L(x,y)}\phi(t)dt = \frac{1}{2} \cdot L(x,y).$$

Thus, the inequality becomes:

$$\left|\frac{x-2y}{4}\right| \leq \frac{1}{2} \cdot \max\left\{|x-y|, \frac{3x}{4}, \frac{y}{2}, \left|\frac{x-2y}{4}\right|\right\}.$$

This inequality is always satisfied. Also $S(X) = \left\{\frac{x}{2} : x \in [0,\infty)\right\} = [0,\infty)$, which is a complete subspace of X and $T(X) = \left\{\frac{x}{4} : x \in [0,\infty)\right\} \subseteq [0,\infty) = S(X)$. T and S are continuous and commute at their coincidence point, so they are weakly compatible.

Here, the unique common fixed point of T and S is 0. So, all conditions of the Theorem 3.3 are satisfied, thus T and S have 0 as unique common fixed point in X.

4.2 APPLICATION

Consider the nonlinear differential equation with periodic boundary condition:

$$\begin{cases} v''(t) + q(t)v(t) = h(t, v(t)), & t \in [0, t_0], \\ v(0) = v(t_0), \quad v'(0) = v'(t_0), \end{cases} \tag{4.2}$$

where $t_0 > 0,\ h : [0, t_0] \times \mathbb{R} \to \mathbb{R}$ and $q : [0, t_0] \to \mathbb{R}$ are continuous functions.

Theorem 4.2. Suppose that h is continuous and there exist constants $\lambda > 0$ and $\mu > 0$ such that

$$|h(t,y) + \lambda y - h(t,x) - \lambda x| \leq \mu|y-x|, \quad \forall x, y \in \mathbb{R},\ t \in [0, t_0],$$

and $\mu < \frac{1}{\lambda}$.

Then the problem (4.2) has a unique solution in $C^1[0, t_0]$.

Proof. We convert the boundary value problem into an equivalent integral equation using Green's function $G(t,s)$ associated with the periodic boundary conditions:

$$v(t) = \int_0^{t_0} G(t,s)[h(s, v(s)) + \lambda v(s)]\, ds.$$

Let us define mappings $K, L : X \to X$ where $X = C^1[0, t_0]$, as

$$(Kv)(t) = \int_0^{t_0} G(t,s)[h(s, v(s)) + \lambda v(s)]\, ds, \quad (Lv)(t) = (Kv)(t).$$

Let $N = M = I$, the identity map. Define a cone metric $d : X \times X \to E$ by

$$d(u,v) = \left(\sup_{t\in[0,t_0]} |u(t) - v(t)|\right) f, \quad \text{where } f(t) = e^t,\ E = C[0, t_0].$$

Then (X, d) is a cone metric space over the Banach space E with cone $P = \{f \in E : f(t) \geq 0\}$.

Now compute:

$$\begin{aligned}
d(Ku, Lv) &= \left(\sup_{t\in[0,t_0]} \left| \int_0^{t_0} G(t,s)[h(s,u(s)) + \lambda u(s) - h(s,v(s)) - \lambda v(s)]ds \right| \right) f \\
&\leq \sup_{t\in[0,t_0]} \int_0^{t_0} |G(t,s)| \cdot |h(s,u(s)) + \lambda u(s) - h(s,v(s)) - \lambda v(s)|ds \cdot f \\
&\leq \mu \int_0^{t_0} |G(t,s)|ds \cdot d(u,v).
\end{aligned}$$

Since $G(t, s)$ is continuous and bounded on $[0, t_0]^2$, say $\int_0^{t_0} |G(t, s)|ds \leq H$ for some $H > 0$, we get

$$d(Ku, Lv) \leq \mu H d(Nu, Mv).$$

Define $\Delta(u, v) = \mu H \cdot d(Nu, Mv)$. Then the integral condition:

$$\int_0^{d(Ku,Lv)} \phi(t)\, dt \leq \psi \left(\int_0^{\Delta_2(u,v)} \phi(t)\, dt \right)$$

is satisfied for suitable functions $\phi \in \Phi$ and $\psi \in \Psi$, for example $\phi(t) = t^p$ $(p > 0)$ and $\psi(t) = \alpha t$ with $\alpha \in (0, 1)$.

Since all the conditions of the main theorem are fulfilled and (K, L, N, M) is a BC's 4-tuple with weak compatibility, there exists a unique common fixed point $v \in X$ such that:

$$Kv = Lv = Nv = Mv = v.$$

Thus, v is the unique solution to the periodic boundary value problem (4.2). □

5. CONCLUSION

In this paper, we established new common fixed point theorems under integral-type contraction in cone metric spaces, extending prior results by relaxing illustrative examples and applications to nonlinear differential equation with periodic boundary condition confirm the effectiveness and broad applicability of our theoretical findings.

REFERENCES

[1] M. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, *Journal of Mathematical Analysis and Applications*,

270 (2002), 181-188, URL: https: //core.ac.uk/download/pdf/82144867.pdf.

[2] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, *Journal of Mathematical Analysis and Applications*, **341(1)** (2008), 416- 420, DOI: 10.1016/j.jmaa.2007.09.070.

[3] M. Arshad, A. Azam and P. Vetro, Some common fixed point results in cone metric spaces, *Fixed Point Theory and Applications*, 2009 (2009), Article ID 493965, 11 pages, DOI: 10.1155/2009/493965.

[4] A. Azam and M. Arshad, Common fixed points of generalized contractive maps in cone metric spaces, *Bulletin of the Iranian Mathematical Society*, **35(2)** (2009), 255-264, URL: http: //bims.iranjournals.ir $article_4 3_b 1cb0ec796a649d02106074e7f6aac94.pdf.$

[5] S. M. A. A. Bakr, Theta cone metric spaces and some fixed point theorems, *Journal of Mathematics*, 2020 (2020), Article ID 8895568, 7 pages, DOI: 10.1155/2020/8895568.

[6] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae*, **3(1)** (1922), 133-181, URL: https://eudml.org/doc/213289.

[7] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, *International Journal of Mathematics and Mathematical Sciences*, **29** (2002), Article ID 641824, 6 pages, DOI: 10.1155/S0161171202007524.

[8] S.-H. Cho and J.-S. Bae, Common fixed point theorems for mappings satisfying property (E.A) on cone metric spaces, *Mathematical and Computer Modelling* , **53**(5-6) (2011), 945-951, DOI: 10.1016/j.mcm.2010.11.002.

[9] R. H. Haghi and Sh. Rezapour, Fixed points of multifunctions on regular cone metric spaces, *Expositiones Mathematicae*, **28(1)** (2010), 71- 77, DOI: 10.1016/j.exmath.2009.04.001.

[10] D. Ili ́c and V. Rako ̌cevi ́c, Common fixed points for maps on cone metric space, *Journal of Mathematical Analysis and Applications*, **341(2)** (2008), 876 – 882, DOI: 10.1016/j.jmaa.2007.10.065.

[11] H. Long-Guang and Z. Xian, Cone metric spaces and fixed point theorems of contractive mappings, *Journal of Mathematical Analysis and Applications*, **332(2)** (2007), 1468-1476, DOI: 10.1016/j.jmaa.2005.03.087.

[12] Q. Meng, On generalized algebraic cone metric spaces and fixed point theorems, *Chinese Annals of Mathematics*, Series B 40 (2019), 429-438, DOI: 10.1007/s11401-019-0142-8.

[13] R. P. Pant, Common fixed point theorems for contractive maps, *Journal of Mathematical Analysis and Applications*, **226** (1998), 251 -258, URL: https://core.ac.uk/download/pdf/82474413.pdf.

[14] Sh. Rezapour and R. Hamlbarani, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", *Journal of Mathematical Analysis and Applications*, **345(2)** (2008), 719-724, DOI: 10.1016/j.jmaa.2008.04.049.

[15] K. Sanjay, S. Akash and S. Manish, Common fixed point theorems of integral type contraction on cone metric spaces and applications, *Communication in Mathematics and Applications*, **14(2)** (2023), 503-510, DOI : 10.26713/cma.v14i2.2142.

[16] G. Song, X. Sun, Y. Zhao and G. Wang, New common fixed point theorems for maps on cone metric spaces, *Applied Mathematics Letters*, **23(9)** (2010), 1033-1037, DOI: 10.1016/j.aml.2010.04.032.

[17] P. P. Zabrejko, K-metric and K-normed linear spaces: survey, *Collectanea Mathematica* **48** (1997), 825-859, URL: http://dmle.icmat.es/pdf COLLECTANEAMATHEMATICA_1997_48_04_37.pdf.