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The Log Integral Function and the Dirichlet Inverse of Mertens Function

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Abstract

Properties of the sum of Mertens function and its Dirichlet inverse are investigated. The average of the Dirichlet inverse of Mertens function is compared to the log integral function. A Dirichlet product of Mertens function and the log integral function is investigated. A likely upper bound of the absolute value of Mertens function at prime locations is determined.

Keywords: Möbius function, log integral function, Dirichlet inverse, Mertens function

1. INTRODUCTION

The log integral function is

$$li(x) = \lim_{\delta \to +0} \left(\int_0^{1-\delta} + \int_{1+\delta}^{x} \right) \frac{dt}{\log t}, (x > 1). \tag{1}$$

C code for computing $li(n)$ is in the Methods section. The Möbius function is defined as follows. $\mu(1)$ is set to 1. For $n > 1$, write $n = p_1^{a_1} \cdots p_k^{a_k}$. Then $\mu(n) = (-1)^k$ if $a_1 = a_2 = \ldots = a_k = 1$ or 0 otherwise. The Mertens function (denoted by $M(n)$) is the summatory Möbius function. Littlewood [1] proved that the Riemann hypothesis is equivalent to the statement that for every $\epsilon > 0$ the function $M(x)/x^{-(1/2)-\epsilon}$ approaches zero as $n \to \infty$.

2. THE DIFFERENCE BETWEEN $li(n)$ AND THE AVERAGE OF THE DIRICHLET INVERSE OF THE MERTENS FUNCTION

Theorem 2.8 of Apostol's [2] book is

Theorem 1. *If f is an arithmetical function with $f(1) \neq 0$ there is a unique arithmetical function f^{-1}, called the Dirichlet inverse of f, such that $f * f^{-1} = f^{-1} * f = I$. Moreover, f^{-1} is given by the recursion formulas $f^{-1}(1) = 1/f(1)$, $f^{-1}(n) = \frac{-1}{f(1)} \sum_{d|n,d<n} f(\frac{n}{d}) f^{-1}(d)$ for $n > 1$.*

A plot of the Dirichlet inverse of $M(n)$ (denoted by $M'(n)$) for $n = 1, 2, 3, \ldots, 1000$ is

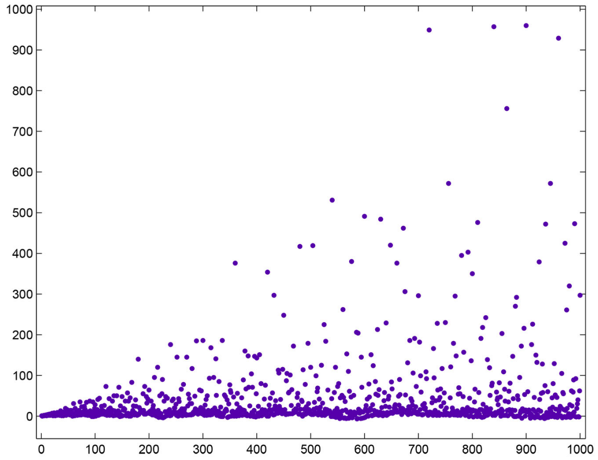

Figure 1

Some empirical results are

(1) If n is a prime, M(n)+M'(n)=0.

(2) If n is a prime squared, $M(n) + M'(n)$ is a square, disregarding sums of 0.

(3) If n is a prime cubed, the absolute value of $M(n) + M'(n)$ has at most two distinct prime factors, disregarding sums of 0 and 1.

(4) If n is the product of two distinct primes, the absolute value of $M(n) + M'(n)$ is even. $M(n) + M'(n) = 0$ if one of the distinct primes is 2.

(5) If n is the product of a prime squared and another distinct prime, the absolute value of $M(n) + M'(n)$ is even or odd.

A normal probability plot of $\sqrt{|M(n) + M'(n)|}$ for $n \leq 200000$ is

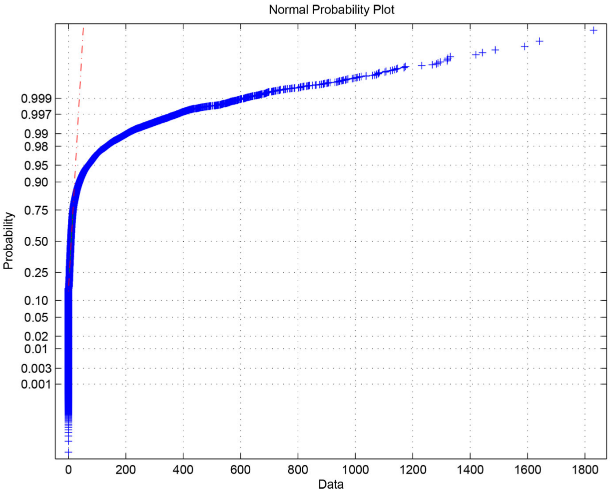

Figure 2

The mean is 18.8965 with a 95% confidence interval of (18.6919, 19.1011) and the standard deviation is 46.6807 with a 95% confidence interval of (46.5365, 46.8258).

Let $A(n)$ denote the average of the $M'(n)$ values. For example, $A(10)$ is the average of the $M'(n)$ values from $n = 1$ to 10. A plot of $li(n)$ and $A(n)$ for $n = 1, 2, 3, \ldots, 10000$ is

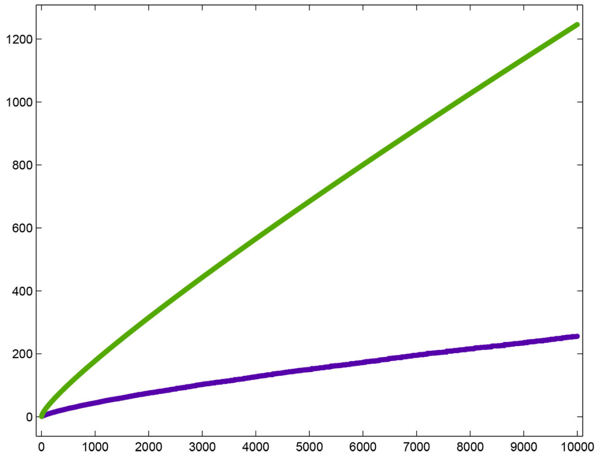

Figure 3

$A(n)$ is the lower curve. A plot of $li(n) - A(n)$ versus $\sqrt{n}$ for $n = 1, 2, 3, \ldots, 2000$ is

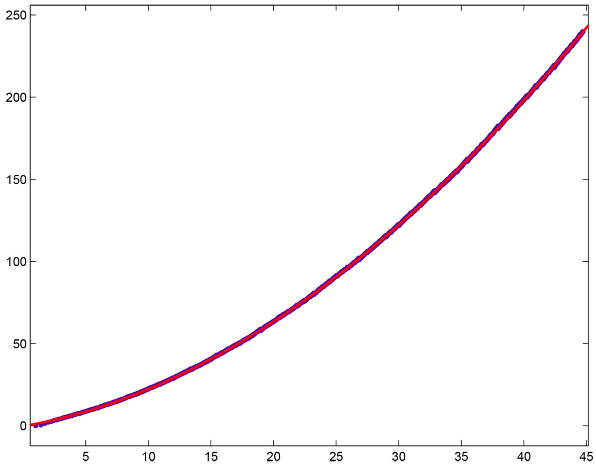

Figure 4

For a cubic least-squares fit of the curve, $p_1 = -0.0001241$ with a 95% confidence interval of $(-0.0001345, -0.0001136)$, $p_2 = 0.09629$ with a 95% confidence interval of (0.09548, 0.0971), $p_3 = 1.306$ with a 95% confidence interval of (1.287, 1.325), $p_4 = -0.2073$ with a 95% confidence interval of $(-0.3415, -0.07306)$, SSE=182.9, R-squared=1, and RMSE=0.3072.

For a cubic least-squares fit of the curve for n up to 200000, $p_1 = -1.093 \cdot 10^{-5}$ with a 95% confidence interval of $(-1.095 \cdot 10^{-5}, -1.092 \cdot 10^{-5})$, $p_2 = 0.07626$ with a 95% confidence interval of (0.07625, 0.07627), $p_3 = 2.889$ with a 95% confidence interval of (2.886, 2.891), $p_4 = -45.05$ with a 95% confidence interval of $(-45.22, -44.88)$, SSE=$3.126 \cdot 10^5$, R-squared=1, and RMSE=3.953. In general, the p_1 parameter approaches 0 from below, the p_2 parameter decreases, the $p3$ parameter increases, and the p_4 parameter decreases as the upper bound of n increases. These parameters do not increase or decrease monotonically. The SSE parameter increases roughly quadratically.

3. A DIRICHLET PRODUCT OF THE LOG INTEGRAL FUNCTION AND MERTENS FUNCTION

Theorem 2.22 (Generalized inversion formula) in Apostol's book is:

If α has a Dirichlet inverse α^{-1} then the equation

$$G(x) = \sum_{n \leq x} \alpha(n) F(\frac{x}{n}) \tag{2}$$

implies

$$F(x) = \sum_{n \leq x} \alpha^{-1}(n) G(\frac{x}{n}) \tag{3}$$

Conversely, (3) implies (2).

A plot of $\sum_{n \leq x} \alpha(n) F(\frac{x}{n})$ where α is the log integral function, F is Mertens function, and $n = 1, 2, 3, \ldots, 1000$ is

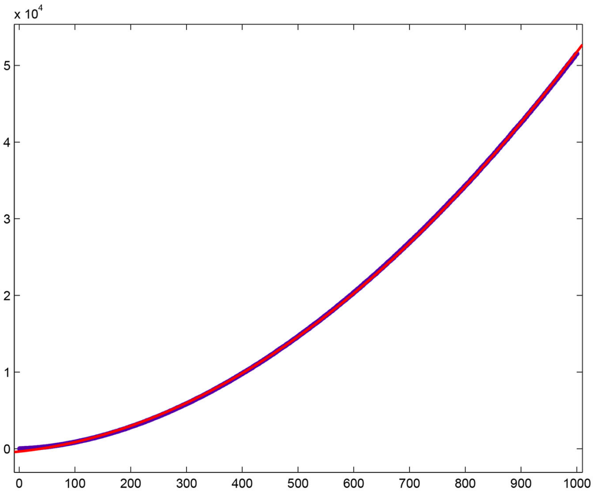

Figure 5

For a quadratic least-squares fit of the curve, $p_1 = 0.04426$ with a 95% confidence interval of (0.04418, 0.04434), $p_2 = 7.843$ with a 95% confidence interval of (7.758, 7.928), $p_3 = -356.4$ with a 95% confidence interval of (-374.8, -338), SSE=$9.667 \cdot 10^6$, R-squared=1, and RMSE=98.47.

A plot of the p_1 parameters for n upper bounds of 1000, 2000, 3000,...,30000 versus $\log(1)$, $\log(2)$, $\log(3)$,...,$\log(30)$ is

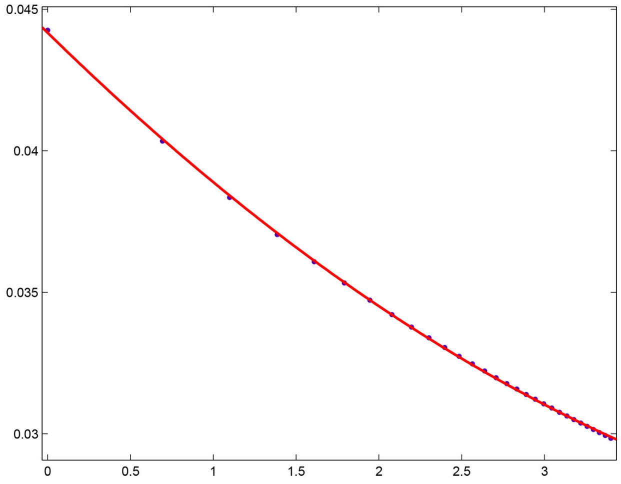

Figure 6

The curve is quadratic.

A plot of the p_2 parameters versus 1, 2, 3,. . .,30 is

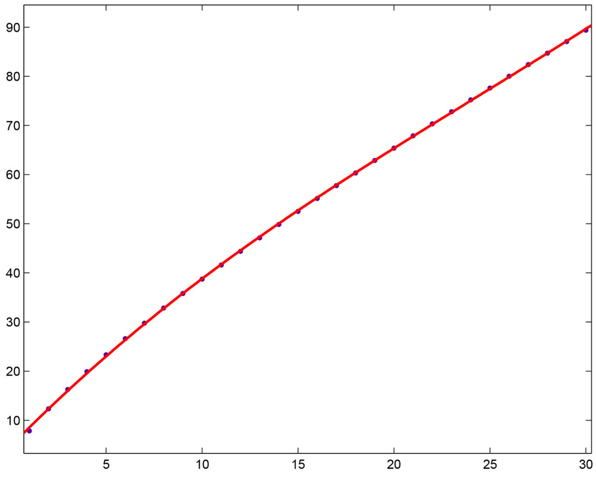

Figure 7

The curve is cubic.

A plot of the p_3 parameters is

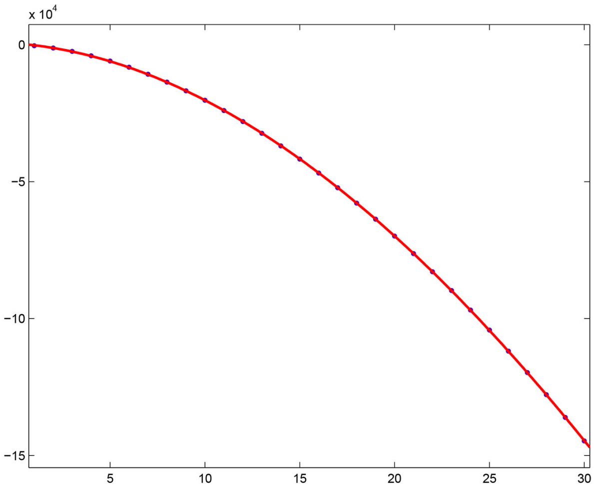

Figure 8

The curve is cubic.

A plot of the SSE values is

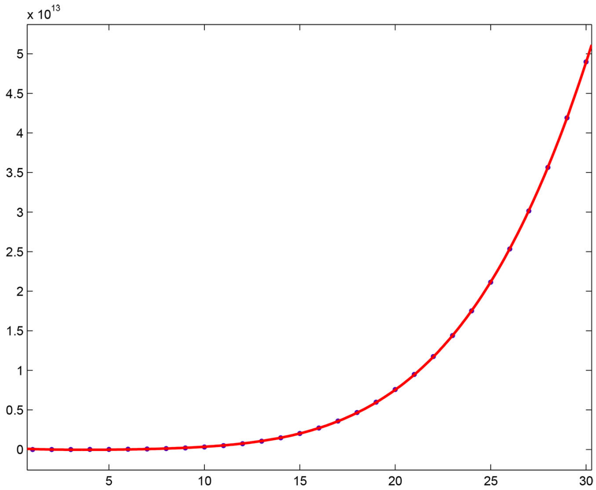

Figure 9

The curve is quartic.

A plot of the RMSE values is

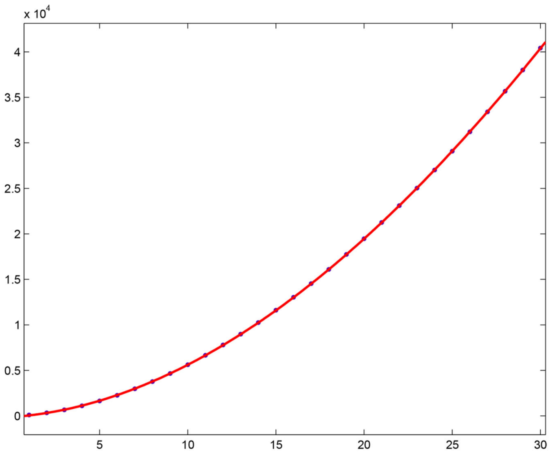

Figure 10

The curve is cubic.

4. THE DIRICHLET INVERSE OF $A(n)$

Let $A'(n)$ denote the Dirichlet inverse of $A(n)$. Some empirical results are
(6) If n is a prime, $A(n) + A'(n) = 0$.
(7) If n is the square of a prime, $A(n) + A'(n)$ is a rational number with a denominator of the square of the prime and a numerator of the square of a naural number. The first few numerators are 1, 2^2, 5^2, 8^2, 16^2, 23^2, 34^2, 41^2, 57^2, 51^2, 85^2, 98^2,.... The numerator may not be relatively prime to the denominator. For a denominator of 5^2, the numerator is 5^2. For a denominator of 17^2, the numerator is $2^2 \cdot 17^2$.
A plot of $A(n) + A'(n)$ versus $\sqrt{n}$ for prime-squared n and $n < 200000$ is

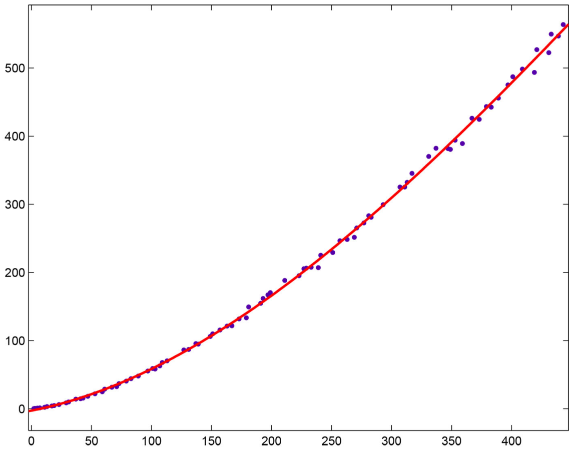

Figure 11

For a cubic least-squares fit of the curve, $p_1 = -1.749 \cdot 10^{-6}$ with a 95% confidence interval of $(-2.421 \cdot 10^{-6}, -1.077 \cdot 10^{-6})$, $p_2 = 0.002844$ with a 95% confidence interval of (0.0024, 0.003288), $p_3 = 0.3448$ with a 95% confidence interval of (0.24, 0.4256), $p_4 = -2.737$ with a 95% confidence interval of $(-6.479, 1.006)$, SSE=2484, R-squared=0.999, and RMSE=5.504.

(8) If n is the cube of a prime, $A(n) + A'(n)$ is a rational number with a denominator of the cube of the prime. The first few numerators are 5, $2^2 \cdot 11$, $3^2 \cdot 13$, $2^{10} \cdot 3$, $2^9 \cdot 61$, $23 \cdot 3343$, $2^4 \cdot 3 \cdot 7 \cdot 53$, $5 \cdot 11 \cdot 41 \cdot 241$,....

A plot of $A(n) + A'(n)$ versus the cube root of n for prime-cubed n and $n < 200000$ is

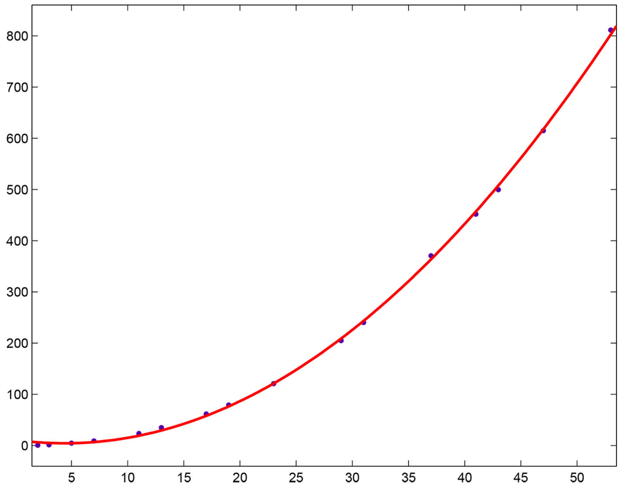

Figure 12

For a quadratic least-squares fit of the curve, $p_1 = 0.338$ with a 95% confidence interval of (0.3247, 0.3514), $p_2 = -2.979$ with a 95% confidence interval of $(-3.694, -2.2644)$, $p_3 = 11.11$ with a 95% confidence interval of (3.639, 18.57), SSE=375, R-squared=0.9996, and RMSE=5.371.

A plot of $A(n)+A'(n)$ for $n=1,2,3,\ldots,1000$ is

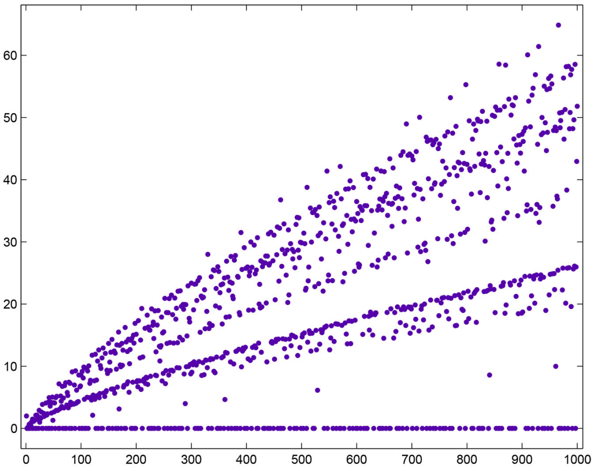

Figure 13

The bottom curve is for prime n values. The next-to-bottom curve is for prime-squared n values. The group of curves above this is for n values that are the product of two distinct primes. For n values that are twice a prime, the denominator is the prime (the numerator may not be relatively prime to the denominator).

A plot of $A'(n)$ versus $n/\log(n)$ at prime n is

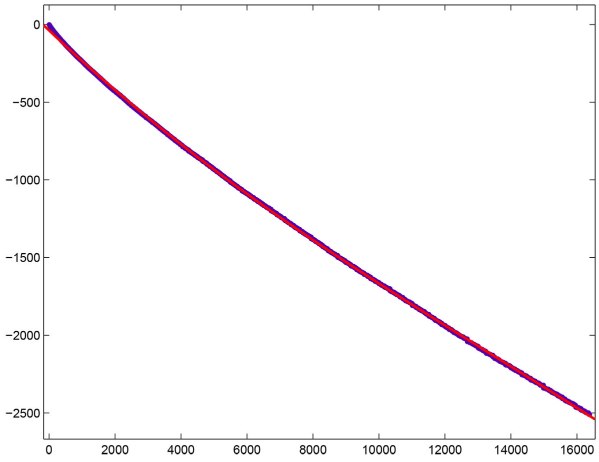

Figure 14

For a cubic least-squares fit of the curve, $p_1=-1.275\cdot 10^{-10}$ with a 95% confidence interval of ($-1.286\cdot 10^{-10}$, $-1.246\cdot 10^{-10}$, $p_2=5.16\cdot 10^{-6}$ with a 95% confidence interval of ($5.132\cdot 10^{-6}$, $5.187\cdot 10^{-6}$), $p_3=-0.2018$ with a 95% confidence interval of $(-0.202, -0.2016)$, $p_4 = -37.89$ with a 95% confidence interval of $(-38.25, -37.53)$, RMSE=$7.046 \cdot 10^5$, R=squared=0.9999, and RMSE=6.226.

A plot of $A'(n)$ versus $\sqrt{n}$ at prime-squared n is

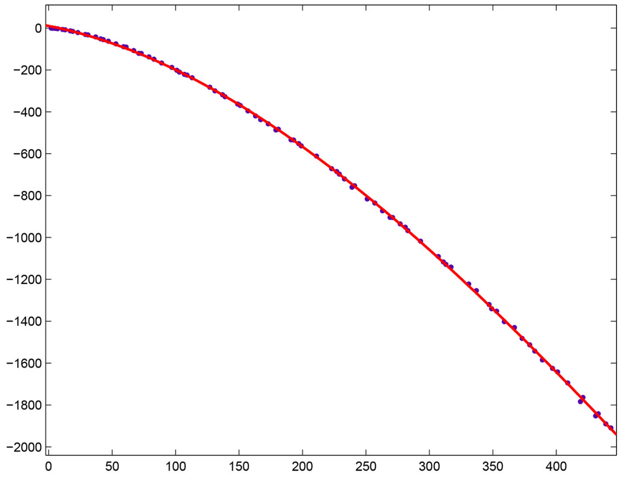

Figure 15

For a cubic least-squares fit of the curve, $p_1 = 5.491 \cdot 10^{-6}$ with a 95% confidence interval of $(4.749 \cdot 10^{-6}, 6.233 \cdot 10^{-6})$, $p_2 = -0.00935$ with a 95% confidence interval of $(-0.01003, -0.009045)$, $p_3 = -1.196$ with a 95% confidence interval of $(-1.285, -1.107)$, $p_4 = 9.934$ with a 95% confidence interval of $(5.83, 14.06)$, SSE=3026, R-squared=0.9999, and RMSE=6.075.

A plot of $A'(n)$ versus the cube root of n at prime-cubed n is

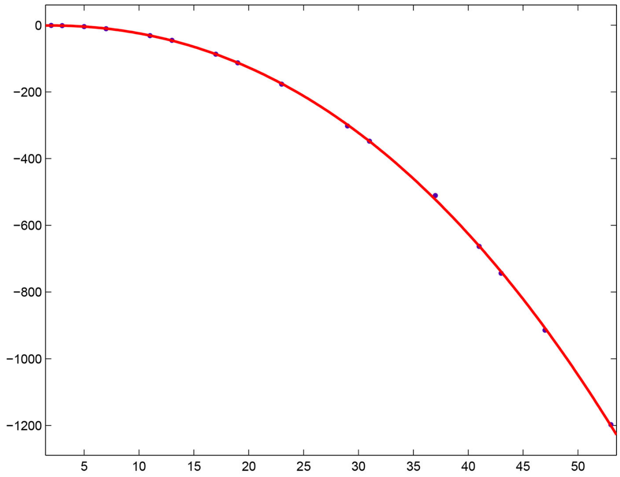

Figure 16

For a cubic least-squares fit of the curve, $p_1 = -0.002227$ with a 95% confidence interval of $(-0.003001, -0.001452)$, $p_2 = -0.3339$ with a 95% confidence interval of $(-0.3969, -0.2708)$, $p_3 = 1.339$ with a 95% confidence interval of $(-0.09687, 2.774)$, $p_4 = -2.4$ with a 95% confidence interval of $(-10.66, 5.859)$, SSE=227.9, R-squared=0.9999, and RMSE=4.358.

5. ROUNDED DIRICHLET INVERSE OF $A(n)$

Let $B(n)$ denote the Dirichlet inverse of $A(n)$ where the values are rounded to the nearest integer. A plot of $B(n)$ for $n = 1, 2, 3, \ldots, 1000$ is

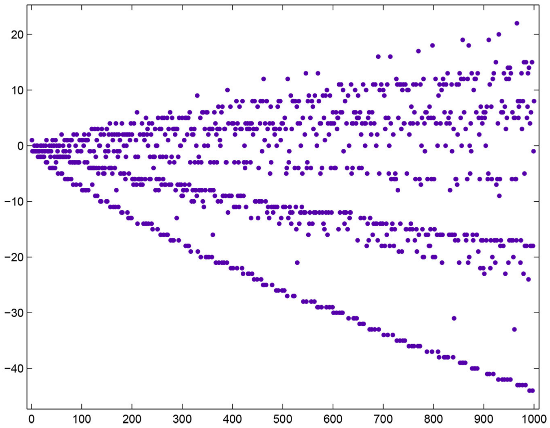

Figure 17

A plot of four times the square roots of the absolute values of the $B(n)$ values and the absolute values of Mertens function values at prime n values less than 20000 is

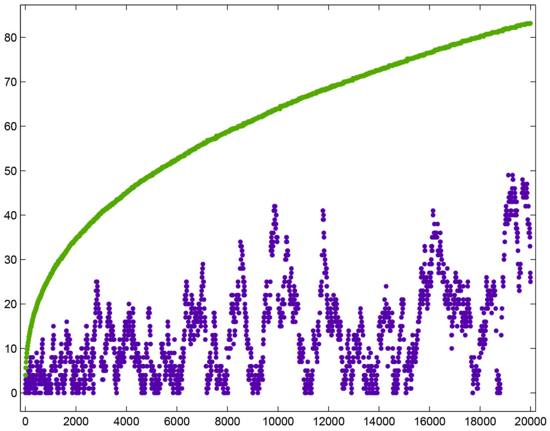

Figure 18

A plot of four times the square roots of the absolute values of the $B(n)$ values and the absolute values of Mertens function values at prime-squared n values less than 200000 is

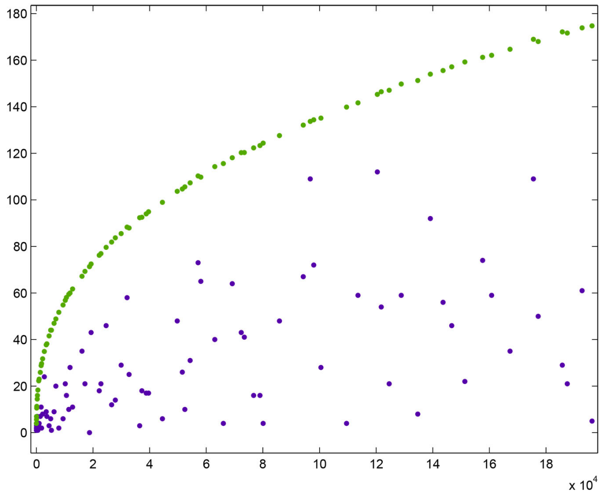

Figure 19

6. THE DIRICHLET INVERSE OF $B(n)$

Let $B'(n)$ denote the Dirichlet inverse of $B(n)$. A plot of $B'(n)$ for $n = 1, 2, 3, \ldots, 1000$ is

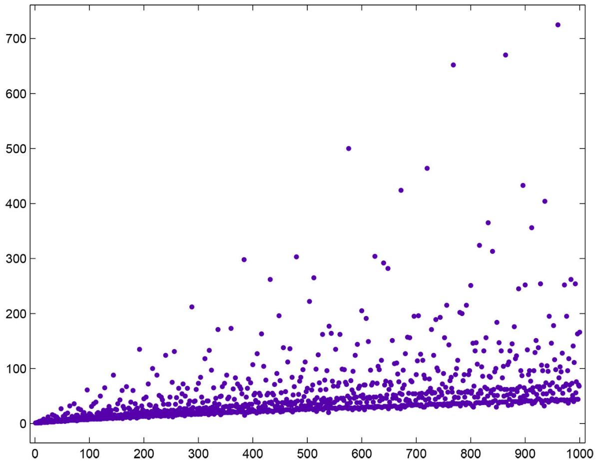

Figure 20

Some empirical results are
(8) If n is prime, $B(n) + B'(n) = 0$.

(9) If n is a prime squared, $B(n) + B'(n)$ is the square of a natural number. A plot of $B(n) + B'(n)$ for the 86 prime squares less than 200000 is

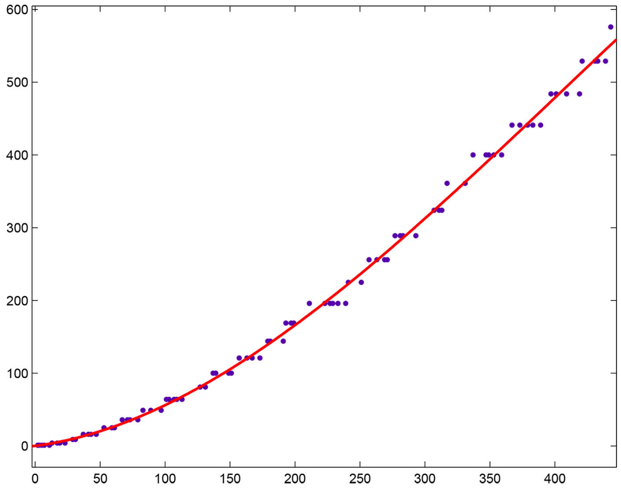

Figure 21

The squares of the natural numbers 1, 2, 3,. . .,... are duplicated to generate a step function.

(10) If n is a prime cubed, $B(n) + B'(n) = 2 \cdot n^3$.

A plot of $B'(n)$ and $li(n)$ at prime n less than 200000 is

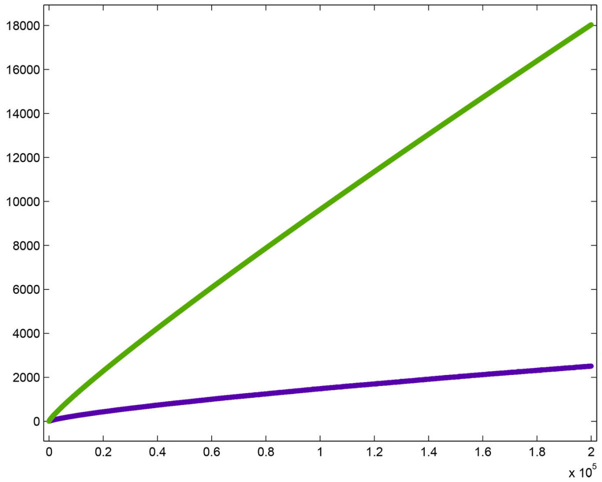

Figure 22

The upper curve is the log integral function. A plot of $li(n) - B'(n)$ versus $\sqrt{n}$ is

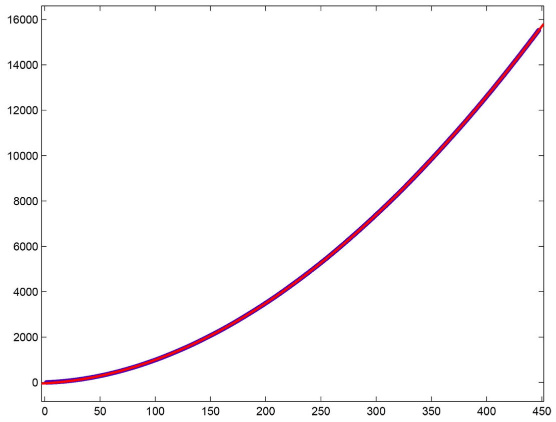

Figure 23

For a cubic least-squares fit of the curve, $p_1 = -1.14 \cdot 10^{-5}$ with a 95% confidence interval of $(-1.145 \cdot 10^{-5}, -1.136 \cdot 10^{-6})$, $p_2 = 0.07664$ with a 95% confidence interval of (0.07661, 0.07667), $p_3 = 2.796$ with a 95% confidence interval of (2.788, 2.804), $p_4 = -38.65$ with a 95% confidence interval of $(-39.16, -38.13)$, SSE=$3.179 \cdot 10^5$, R-squared=1, and RMSE=4.205.

A plot of $li(n) - B'(n)$ versus $\sqrt{n}$ at prime-squared n less than 200000 is

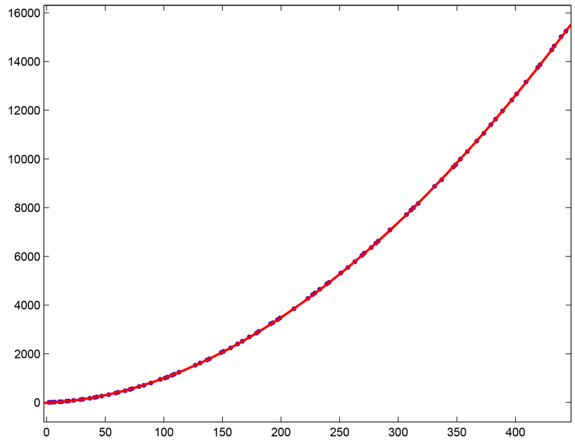

Figure 24

For a cubic least-squares fit of the curve, $p_1 = -1.29 \cdot 10^{-5}$ with a 95% confidence interval of $(-1.421 \cdot 10^{-5}, -1.16 \cdot 10^{-5})$, $p_2 = 0.07789$ with a 95% confidence interval of (0.07703, 0.07876), $p_3 = 2.486$ with a 95% confidence interval of (2.329, 2.643), $p_4 = -18.61$ with a 95% confidence interval of $(-25.89, -11.34)$, SSE=9836, R-squared=1, and RMSE=10.7.

A plot of $li(n) - B'(n)$ versus $\sqrt{n}$ at n values that are 3 times the primes other than 3 and less than 200000

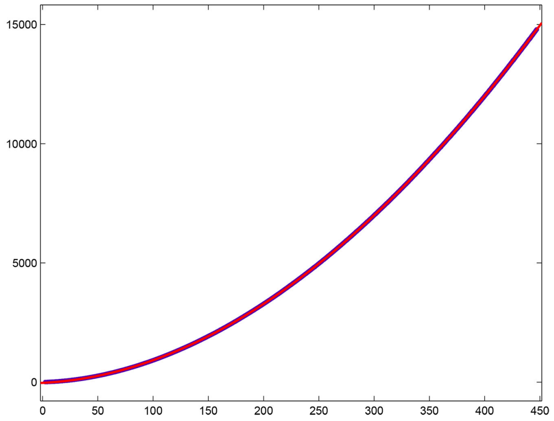

Figure 25

For a cubic least-squares fit of the curve, $p_1 = -1.005 \cdot 10^{-5}$ with a 95% confidence interval of $(-1.012 \cdot 10^{-5}, -9.983 \cdot 10^{-6})$, $p_2 = 0.0736$ with a 95% confidence interval of (0.07355, 0.07365), $p_3 = 2.24$ with a 95% confidence interval of (2.229, 2.252), $p_4 = -28.9$ with a 95% confidence interval of $(-29.66, -28.14)$, SSE=$9.896 \cdot 10^4$, R-squared=1, and RMSE=3.861.

There are then analogues of $A(n)$ at subcurves.

7. THE DIRICHET INVERSE OF THE LOG INTEGRAL FUNCTION

Let $li'(n)$ denote the Dirichlet inverse of $li(n)$. A plot of $li'(n)$ for $n = 1, 2, 3, \ldots, 1000$ is

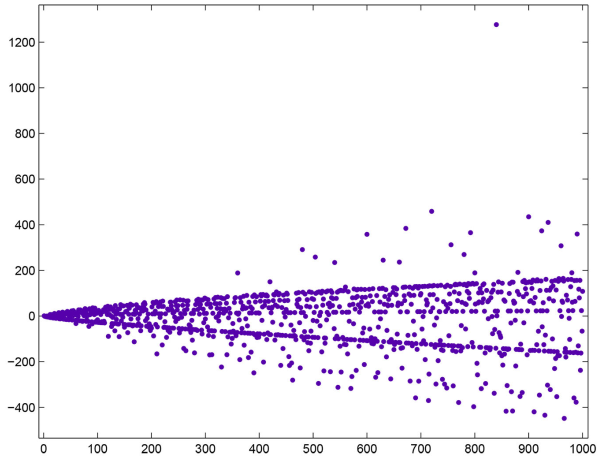

Figure 26

A plot of the 17984 $li'(n)$ values for prime n less than 200000 versus $n/\log(n)$ is

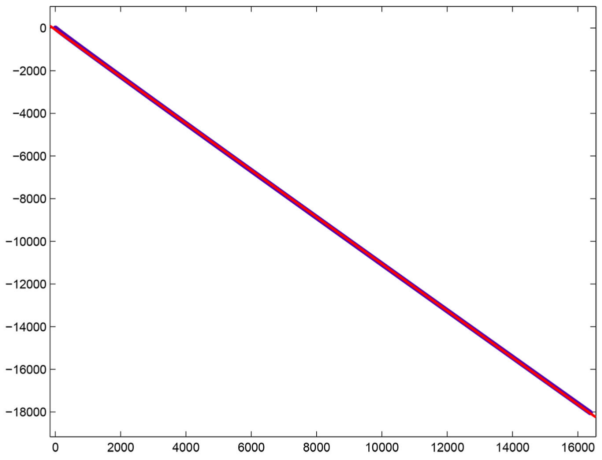

Figure 27

For a linear least-squares fit of the curve, $p_1 = -1.005$ with a 95% confidence interval of $(-1.005, -1.005)$, $p_2 = -72.94$ with a 95% confidence interval of $(-73.43, -72.45)$, SSE=$5.019 \cdot 10^6$, R-squared=1, and RMSE=16.71.

A plot of the 86 $li'(n)$ values for prime-squared n less than 200000 versus $\sqrt{n}$ is

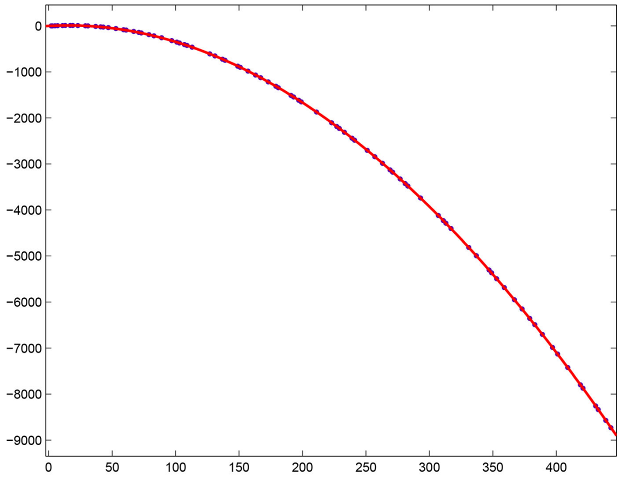

Figure 28

For a cubic least-squares fit of the curve, $p_1 = 5.713 \cdot 10^{-6}$ with a 95% confidence interval of $(5.525 \cdot 10^{-6}, 5.9 \cdot 10^{-6})$, $p_2 = -0.05047$ with a 95% confidence interval of $(-0.05059, -0.05035)$, $p_3 = 1.544$ with a 95% confidence interval of (1.522, 1.567), $p_4 = -2.507$ with a 95% confidence interval of $(-3.55, -1.463)$, SSE=193, R-squared=1, and RMSE=1.534

A plot of the 16 $li'(n)$ values for prime-cubed n less than 200000 versus the cube root of n is

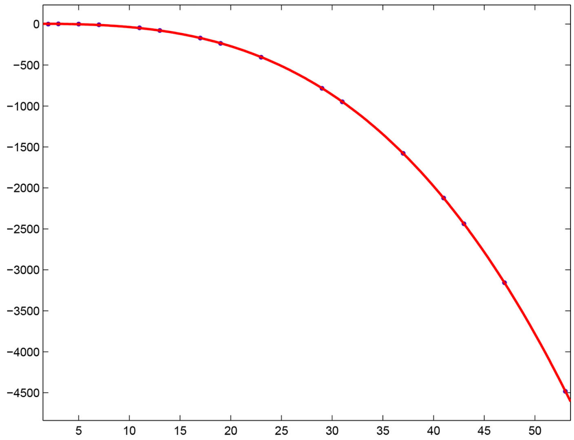

Figure 29

For a cubic least-squares fit of the curve, $p_1 = -0.02747$ with a 95% confidence interval of $(-0.02787, -0.02707)$, $p_2 = -0.1433$ with a 95% confidence interval of $-0.176, -0.1106)$, $p_3 = 0.1215$ with a 95% confidence interval of $(-0.6229, 0.8659)$, $p_4 = 4.446$ with a 95% confidence interval of $(0.1635, 8.729)$, SSE=61.3, R-squared=1, and RMSE=2.26.

A plot of $li(n) + li'(n)$ versus $n/\log(n)$ for the first thousand primes is

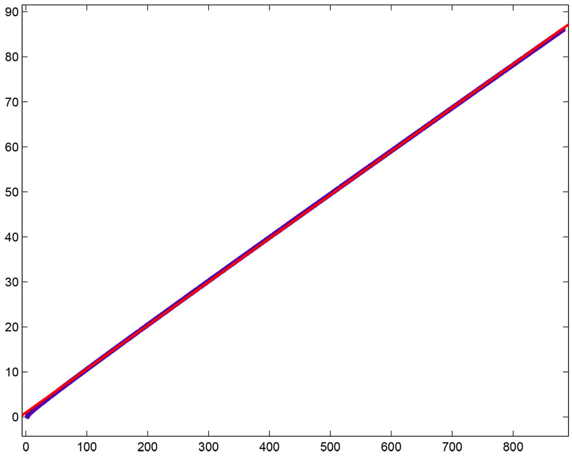

Figure 30

For a linear least-squares fit of the curve, $p_1 = 0.09675$ with a 95% confidence interval of (0.0967, 0.0968), $p_2 = 0.9844$ with a 95% confidence interval of (0.9588, 1.01), SSE=42.88, R-squared=0.9999, and RMSE=0.2073.

For a linear least-squares fit of the curve for the 41538 primes less than 500000, $p_1 = 0.09213$ with a 95% confidence interval of (0.09213, 0.09213), $p_2 = 12.52$ with a 95% confidence interval of (12.46, 12.58), SSE=$3.636 \cdot 10^5$, R-squared=1, and RMSE=2.959.

A plot of twice the square root of the absolute values of the $li'(n)$ values and the absolute values of Mertens function values at prime n is

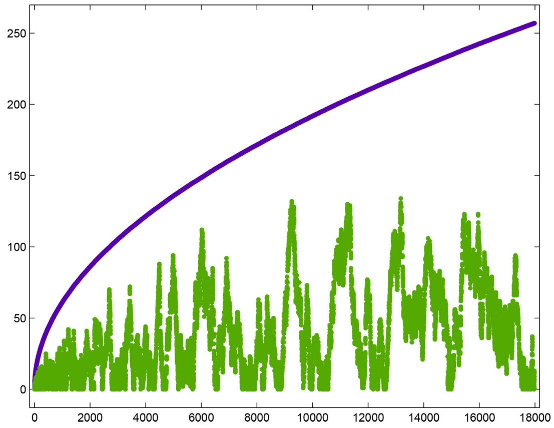

Figure 31

A likely upper bound of the absolute value of the Mertens function at prime n is then about $2\sqrt{\frac{n}{\log n}} + 73$.

A plot of the 17984 $li(n)$ values for prime n less than 200000 versus $n/\log(n)$ is

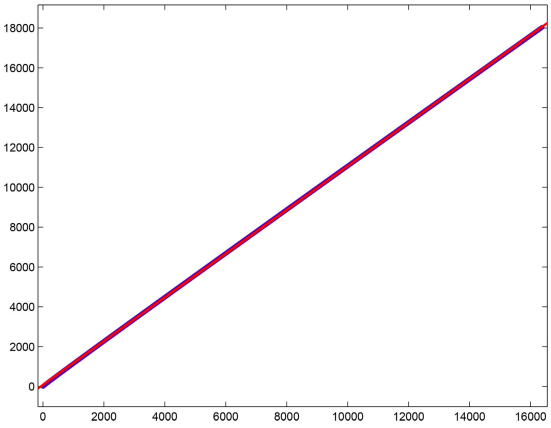

Figure 32

For a linear least-squares fit of the curve, $p_1 = 1.098$ with a 95% confidence interval of (1.098, 1.098), $p_2 = 79.68$ with a 95% confidence interval of (79.15, 80.20), SSE=$5.989 \cdot 10^6$, R-squared=1, and RMSE=18.25.

For a linear least-squares fit of the 41538 $li(n)$ values for prime n less than 500000, $p_1 = 1.09$ with a 95% confidence interval of (1.09, 1.09), $p_2 = 148.1$ with a 95% confidence interval of (147.4, 148.7), SSE=$5.087 \cdot 10^7$, R-squared=1, and RMSE=34.99.

A plot of twice the square root of $li(n)$, four times the square root of $A(n)$, and the absolute values of Mertens function values at prime n for $n < 500000$ is

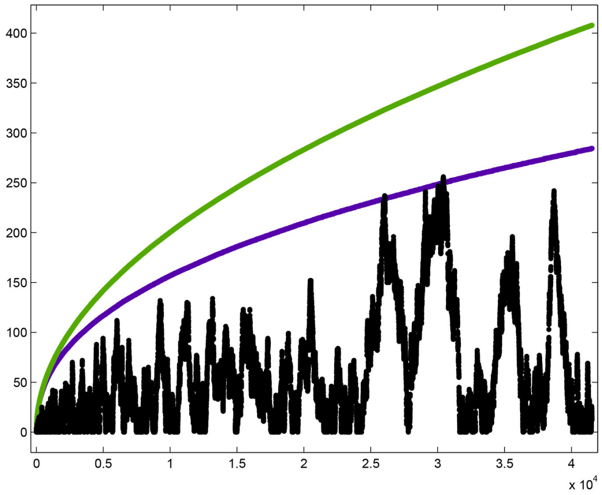

Figure 33

Note that $A(n)$ was derived from Mertens function and $li(n)$ is much greater than $A(n)$.

8. METHODS

compute li(x)

```
#include <math.h>
#include <stdio.h>
void main () {
unsigned int h,MAXN;
int j;
double temp,x,f;
```

```
FILE *Outfp;
Outfp = fopen(”sortz.dat”,”w”);
if (Outfp==NULL)
    return;
fprintf(Outfp,” %.16llf, \n”,1.045164);
for (h=2; h<=75000; h++) {
    MAXN=h;
    f=-1e+99;
    x=log(MAXN);
    temp=x-10;
    if (temp<0.0)
        temp=-temp;
    if (temp>=12.0)
        goto L2;
    if (x==0.0)
        goto L4;
    temp=x;
    if (temp<0.0)
        temp=-temp;
    j=(int)(10.0+2.0*temp);
    f=1.0/(double)((j+1)*(j+1));
    L1: f=(f*(double)j*x+1.0)/(double)(j*j);
    j=j-1;
    if (j!=0.0)
        goto L1;
    temp=x;
    if (temp<0.0)
        temp=-temp;
    f=f*x+log(1.781072418*temp);
    goto L4;
    L2: temp=x;
    if (temp<0.0)
        temp=-temp;
    j=(int)(5.0+20.0/temp);
    f=x;
    L3: f=1.0/(1.0/f-1.0/(double)j)+x;
    j=j-1;
    if (j!=0)
```

```
        goto L3;
    f=exp(x)/f;
    L4: printf(” %d %.16llf \n”,MAXN,f);
    fprintf(Outfp,” %.16llf, \n”,f);
    }
fclose(Outfp);
return;
}

compute Dirichlet inverse
void dirinv(double *input, double *output, unsigned int MAXN) {
unsigned int i,N;
double sum;
output[0]=1.0/input[0];
for (N=2; N<=MAXN; N++) {
    sum=0.0;
    for (i=1; i<N; i++) {
        if (N==(N/i)*i)
            sum=sum+input[N/i-1]*output[i-1];
        }
    output[N-1]=-1.0/input[0]*sum;
    }
return;
}
```

REFERENCES

[1] Littlewood, J. E., Quelques conséquences de l’hypothèse que la fonction $\zeta(s)$ n’a pas de zéros dans le demi-plan $\Re(s) > \frac{1}{2}$. *C. R. Acad. Sci. Paris* **154,** 263-266 (1912).

[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, (1976)