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Numercal Treatment of Singularly Perturbed Third Order Differential Equation Using Adaptive Finite Difference Method

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Abstract

The present paper focuses on numerical approximation of Singularly Perturbed third-order differential equation using finite difference technique. Shishkin mesh has been considered for domain discretization. Numerical tests have been carried out which shows the efficiency of proposed adaptive scheme in treating boundary layers arising in the solution as singular perturbation parameter becomes extremely small.

Keywords- Singular perturbation, Shishkin mesh, Boundary layer, Finite Difference method.

Introduction

Singular Perturbation Problems (SPPs) arises frequently in various fields of applied science and engineering. Till date, different numerical strategies have been developed for analyzing singular perturbation problems. Quite a good research work has been done on the qualitative and quantitative analysis of singularly perturbed differential equations. Singularly perturbed differential equations involve study of class of differential equations in which a positive small parameter ϵ is multiplied with the highest order derivative. The solution of singularly perturbed problems exhibits special character: there are thin layers where the solution varies very rapidly known as boundary layer, while away from the layers the solution behaves uniformly and varies slowly known as outer region. The classification of singularly perturbed higher-order problems depends on how the order of the original equation is affected if one sets $\epsilon = 0$, where ϵ is known as perturbation parameter. If the order reduces by two then the problem is said to be reaction-diffusion type and if order is reduced by one then the type of singularly perturbed problem is known as convection-diffusion problem. The simplest example of third order singularly perturbed differential equation is given by

$$\epsilon\, y'''(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0, \qquad x \in [0, 1]$$

subjected to the boundary conditions $y(0) = y(1) = y'(0) = 0$. Here $a(x)$, $b(x)$ and
$c(x)$ represent continuous functions.

Description of Method
Shishkin Mesh
Shishkin mesh is a piecewise equidistant mesh, constructed a prior as a function, that partly resolve the boundary layers. To construct them correctly, it is crucial to have a precise knowledge of the asymptotic behavior of the exact solution. For $r \geq 2^n$, where $n \geq 2$ is an integer, the piecewise uniform Shishkin mesh $\bar{\Omega}_x$ is designed by partitioning the spatial domain Ω_x into two subintervals $\Omega_1 = [0, 1-R]$ and $\Omega_2 = [1-R, 1]$ such that $\Omega_x = \Omega_1 \cup \Omega_2$. Here, transition parameter R is defined by the following function of ϵ, α and r as

$$R = \min\left(0.5, \frac{2\epsilon}{\alpha}\log r\right).$$

Moreover, mesh spacing $\tilde{h}$ in spatial direction is given by

$$\tilde{h} = \begin{cases} h_1' = 2\dfrac{(1-R)}{r}, & \text{if } i = 1, 2, \ldots, \dfrac{r}{2}, \\ h_2' = 2\dfrac{R}{r}, & \text{if } i = \dfrac{r}{2}+1, \ldots, r. \end{cases}$$

Therefore, set of mesh points $\{x_i\}_{i=0}^{r}$ are given by

$$x_i = \begin{cases} h_1' i; & i = 0, 1, 2, \ldots, \dfrac{r}{2} \\ (1-R) + h_2'\left(i - \dfrac{r}{2}\right); & i = \dfrac{r}{2}+1, \ldots r. \end{cases}$$

Thus, when $R = \frac{1}{2}$, the mesh is uniform, otherwise mesh condenses near the right part of the
lateral surface.

Finite Difference Method
Finite Difference Method (FDM) is one of the classical numerical method which is used for solving SPP. The derivatives appearing in the singularly perturbed

problem are replaced by central difference approximations. The whole system is reduced to system of linear equations which can be easily solved by using MATLAB software. Consider the following singularly perturbed problem:

$$-\in y'''(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x), \qquad (1)$$

where $0 \le x \le 1$ with $y(0) = \alpha$, $y(1) = \beta$ and $y' = \gamma$.
Dividing the interval into r parts i.e. $x_0 = 0 < x_1 < \ldots < x_{r-1} < x_r = 1$.

Replacing the derivatives by central difference approximations in (1), we get

$$-\in \left(\frac{y_{i+2}-2y_{i+1}+2y_{i-1}-y_{i-2}}{2h^3}\right) + a(x_i)\left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right) + b(x_i)\left(\frac{y_{i+1}-y_{i-1}}{2h}\right) + c(x_i)y(x_i) = f(x_i), \qquad (2)$$

for $i = 1, 2, 3, \ldots, r-1$ and $x \in (0, 1)$.
On solving Eq. (2), we get matrix of the form,
$[A]\{Y\} = [B]$,
where A is matrix of order $r \times r$ and, B is of order $r \times 1$. The above system is solved numerically to obtain computed solution.

Numerical Result
In this Section, numerical results are presented to validate the theoretical findings. Computations are carried out by using MATLAB programming language.

3.1 TEST PROBLEM:
Consider the singularly perturbed boundary value problem

$$\in^2 y''' - y' + xy = 0, \quad 0 \le x \le 1 \qquad (3)$$

with $y(0) = y'(0) = y(1) = 1$.

Since $y'(0) = 1$

$$\frac{y_1 - y_0}{h} = 1 \qquad (4)$$

Step 1: Discretization of domain

Region of interest is x-axis from 0 to 1. Divide the interval (0, 1) into r parts i.e. $x_0 = 0 < x_1 < \ldots < x_{r-1} < x_r = 1$.

Find the corresponding values of $y_0 = 1, y_1 = 1 + h$.

Step 2: Approximation of derivatives:

Replacing the derivatives appearing in the boundary value problem (3) by finite difference approximations,
we get

$$\epsilon^2\left(\frac{y_{i+2}-2y_{i+1}+2y_{i-1}-y_{i-2}}{2h^3}\right)\quad\left(\frac{y_{i+1}-y_{i-1}}{2h}\right)+x_iy_i=0,$$

where $i=1,2,3,\dots,r-1$ and $h=x_i-x_{i-1}$.

$$y_{i-2}\left(-\frac{\epsilon^2}{2h^3}\right)+y_{i-1}\left(\frac{\epsilon^2}{h^3}+\frac{1}{2h}\right)+y_i(x_i)+y_{i+1}\left(-\frac{\epsilon^2}{h^3}-\frac{1}{2h}\right)+y_{i+2}\left(\frac{\epsilon^2}{2h^3}\right)=0. \quad (5)$$

Put $i=1$ in (5), we get

$$y_{-1}\left(-\frac{\epsilon^2}{2h^3}\right)+y_0\left(\frac{\epsilon^2}{h^3}+\frac{1}{2h}\right)+y_1(x_i)+y_2\left(-\frac{\epsilon^2}{h^3}-\frac{1}{2h}\right)+y_3\left(\frac{\epsilon^2}{2h^3}\right)=0.$$

Using the conditions $y_{-1}=0, y_0=1$ the above equation reduces to

$$y_1(x_i)+y_2\left(-\frac{\epsilon^2}{h^3}-\frac{1}{2h}\right)+y_3\left(\frac{\epsilon^2}{2h^3}\right)=-\left(\frac{\epsilon^2}{h^3}+\frac{1}{2h}\right).$$

Substituting $i=2$ in (5), we get

$$y_0\left(-\frac{\epsilon^2}{2h^3}\right)+y_1\left(\frac{\epsilon^2}{h^3}+\frac{1}{2h}\right)+y_2(x_i)+y_3\left(-\frac{\epsilon^2}{h^3}-\frac{1}{2h}\right)+y_4\left(\frac{\epsilon^2}{2h^3}\right)=0.$$

Using the condition $y_0=1$ the above equation becomes

$$y_1\left(\frac{\epsilon^2}{h^3}+\frac{1}{2h}\right)+y_2(x_i)+y_3\left(-\frac{\epsilon^2}{h^3}-\frac{1}{2h}\right)+y_4\left(\frac{\epsilon^2}{2h^3}\right)=\left(\frac{\epsilon^2}{2h^3}\right).$$

Put $i=3$ in (5), we have

$y_1\left(-\frac{\epsilon^2}{2h^3}\right)+y_2\left(\frac{\epsilon^2}{h^3}+\frac{1}{2h}\right)+y_3(x_i)+y_4\left(-\frac{\epsilon^2}{h^3}-\frac{1}{2h}\right)+y_5\left(\frac{\epsilon^2}{2h^3}\right) = 0.$

Substituting $i = 4$ in (5), we get

$y_2\left(-\frac{\epsilon^2}{2h^3}\right)+y_3\left(\frac{\epsilon^2}{h^3}+\frac{1}{2h}\right)+y_4(x_i)+y_5\left(-\frac{\epsilon^2}{h^3}-\frac{1}{2h}\right)+y_6\left(\frac{\epsilon^2}{2h^3}\right) = 0$ and so on.

Put $i = r-1$ in (5), we get

$y_{r-3}\left(-\frac{\epsilon^2}{2h^3}\right)+y_{r-2}\left(\frac{\epsilon^2}{h^3}+\frac{1}{2h}\right)+y_{r-1}(x_{r-1})+y_r\left(-\frac{\epsilon^2}{h^3}-\frac{1}{2h}\right)+y_{r+1}\left(\frac{\epsilon^2}{2h^3}\right) = 0.$

Using the conditions $y_r = 1, y_{r+1} = 0$, the above equation reduces to

$$y_{r-3}\left(-\frac{\epsilon^2}{2h^3}\right)+y_{r-2}\left(\frac{\epsilon^2}{h^3}+\frac{1}{2h}\right)+y_{r-1}(x_{r-1}) = \left(\frac{\epsilon^2}{h^3}+\frac{1}{2h}\right).$$

The above system of linear equations can be written in matrix form as

$[A]\{Y\} = [B],$

where

$$[A] = \begin{bmatrix} x_1 & -\frac{\epsilon^2}{h^3}-\frac{1}{2h} & \frac{\epsilon^2}{2h^3} & 0 & 0 & 0 & \dots & 0 \\ \frac{\epsilon^2}{h^3}+\frac{1}{2h} & x_2 & -\frac{\epsilon^2}{h^3}-\frac{1}{2h} & \frac{\epsilon^2}{2h^3} & 0 & 0 & \dots & 0 \\ -\frac{\epsilon^2}{2h^3} & -\frac{\epsilon^2}{h^3}-\frac{1}{2h} & x_3 & -\frac{\epsilon^2}{h^3}-\frac{1}{2h} & \frac{\epsilon^2}{2h^3} & 0 & \dots & 0 \\ 0 & -\frac{\epsilon^2}{2h^3} & \frac{\epsilon^2}{h^3}+\frac{1}{2h} & x_4 & -\frac{\epsilon^2}{h^3}-\frac{1}{2h} & \frac{\epsilon^2}{2h^3} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & x_{r-1} \end{bmatrix}$$

$$\{Y\} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ . \\ . \\ . \\ y_{r-1} \end{bmatrix}$$

$$[B] = \begin{bmatrix} -\frac{\epsilon^2}{h^3} + \frac{1}{2h} \\ \frac{\epsilon^2}{2h^3} \\ 0 \\ 0 \\ . \\ . \\ . \\ \frac{\epsilon^2}{h^3} + \frac{1}{2h} \end{bmatrix}$$

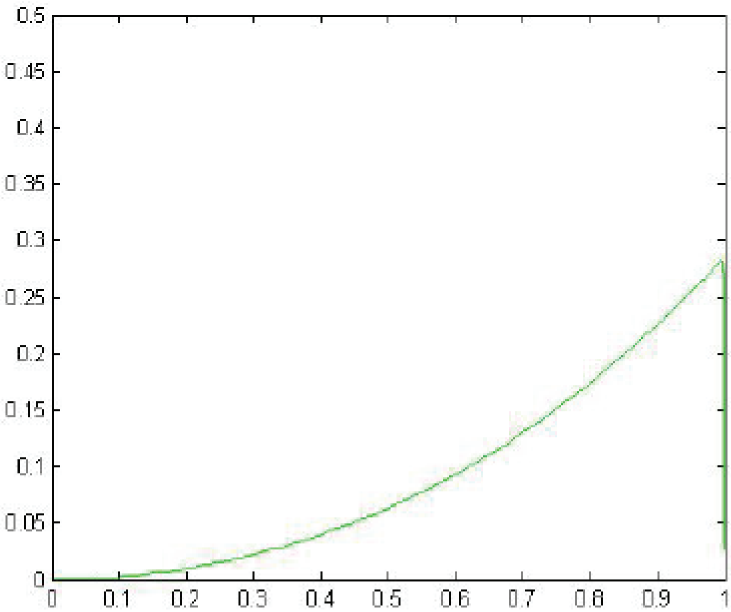

Fig. 1. Numerical Solution Profile for $\epsilon = 2^{-10}$

Conclusion

In the current work, a numerical technique has been proposed for approximating singularly perturbed third order differential equation based on adaptive finite difference strategy. The problem has been discretized using finite difference approximations on uniform Shishkin mesh. Numerical computations have been carried out. It has been shown that the proposed scheme is quite efficient in capturing sharp boundary layers arising in the solution as perturbation parameter ϵ approaches to zero.

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