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Operational Calculus of Extended Laplace-Carson Transform

Fasiyoddin I.Momin[1], Sadikali L.Shaikh[2] and Anil D.Chindhe[*]

[1,2]*Research Center, Department of Mathematics,Maulana Azad College of Arts, Science and Commerce, Chh.Sambhajinagar, Maharashtra, India.*
[*]*Balbhim Arts, Science and Commerce College, Beed, Maharashtra, India.*

Abstract

In this paper,we have constructed the suitable space of pseudoquotient for the Laplace-Carson transform.The extended Laplace-Carson transform has been defined by extending the definition of the Laplace-Carson transform to the space of pseudoquotient.Moreover,we have illustrated some properties of the extended Laplace-Carson transform and developed its operational calculus.

Keywords: The Laplace-Carson Transform,Pseudoqotients,Operational Calculus

AMS Subject Classification: 44A15,44A10

1. INTRODUCTION

One of the most popular mathematical methods for figuring out the solutions to cutting-edge issues in science, engineering,technology and space is integral transformations.It provides the analytical solution of the issue without requiring a significant amount of computation,is the most crucial aspect of these transformations.In this paper we consider one of the integral transform namely Laplace-Carson transform(LC Transform) defined by John Carson[**1**] which is used as normalizer of classical transform like Laplace transform.It has wide range of applications in the fields of physics, engineering and applied mathematics[**2-6**].This paper deals on extension of LC transform on pseudoquotients.The pesudoquotient is the space defined by using basic algebra and elementary theory of calculus.The applications of pesudoquotients is visible in the fields like generalized functions,functional analysis,and abstract harmonic analysis[**7**, **8**].The construction of pesudoquotients was introduced in[**8**] and it is defined as the space of equivalence class of pairs $(x, f) \in A \times G$, where A is the nonempty set and G is a commutative semigroup of injective functions from A to A[**9**].In literature we can see that the Laplce transform is extended to the space of pseudoquotients[10].The authors[11,12] have developed the operational calculus of the extended Sumudu transform and Sadik transform.In this paper we construct the suitable space of pseudoquotient for the extension of LC transform and study some basic properties of pesudoquotients.

The structure of this document is as follows.We define LC transform and its useful identities in section 2.The creation of the space of pseudoquotient is explained and several of the pseudoquotient properties are also illustrated in section 3.In section 4,we define the LC transform and we extend the notion of the LC transform on the space of pseudoquotient.In section 5,we define the generalized derivative,convolution product and generalized multiplication by $'t'$ and create the operational calculus for the extended LC transform.

2. LAPLACE CARSON TRANSFORM

The LC transform is a variant of the Laplace transform (LT), and it is named after John Renshaw Carson (1886-1940),a telecommunication engineer at Bell AT and T Labs.Particularly in the domains of railway engineering and physics, the LC transform has been used.For the analysis of dynamical systems formulated by ordinary/partial differential equations,integral transforms like Fourier, Fourier-Bessel,Mellin,Hilbert and Laplace are effective tools.Since LC transform is equal to the price of an American option with random maturity,it can be used to price a variety of American options[**13-15**].

Definition 1. *Let $f(t)$ be a continuous real-valued function for $t \in \mathbb{R}+ = [0, \infty$),and assume that $|f(t)| \leq Ae^{Bt}(t \geq 0)$ for constants A and B.Then,for $s \in \mathbb{C}(\mathrm{Re}(s) > B)$,define the Laplace transform of $f(t)$ as*

$$\mathcal{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)dt \tag{1}$$

Definition 2. *For $f(t)$ specified above,define its Laplace Carson transform as[**16**]*

$$\mathcal{L}C[f(t)](s) = F(s) = s\int_0^\infty e^{-st} f(t)dt \tag{2}$$

Definition 3. *(Bromwich Integral)*

$$f(t) = \mathcal{L}C^{-1}[F(s)](t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\frac{F(s)}{s}ds = \mathcal{L}^{-1}[\frac{F(s)}{s}](t), t > 0 \quad (3)$$

where a is a real number such that the contour path of integration is in the region of convergence of F(s) .

Definition 4. *A piecewise continuous function on $(-\infty, \infty)$ which vanish on $(-\infty, 0)$ and having exponential order w,where w is any real number is called as Laplace-Carson transformable function.Let $\mathfrak{A}$ be a set of all Laplace-Carson transformable functions.*

2.1. Some Useful Identities

1. $\mathcal{L}C[\frac{d}{dt}f(t)] = s(F(s) - F(0+))$
2. $\mathcal{L}C[f^{(n)}(t)](s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k}F^{(k)}(0+)$, $n \geq 1$
3. $\mathcal{L}C[\int_0^t f(t)dt](s) = \frac{1}{s}F(s)$
4. $\mathcal{L}C[\frac{f(t)}{t}](s) = s\int_s^\infty \frac{F(s)}{s}ds$
5. $\mathcal{L}C[f(t-a)1_{\{t\geq a\}}(t)](s) = e^{-as}F(s)$, $a > 0$
6. $\mathcal{L}C[e^{at}f(t)](s) = \frac{s}{s-a}F(s-a)$
7. $\mathcal{L}C[f(\frac{t}{a})](s) = F(as)$, $a > 0$
8. $\mathcal{L}C[\int_0^t f_1(y)f_2(t-y)dy](s) = \frac{1}{s}F_1(s)F_2(s)$

3. SPACE OF PSEUDOQUOTIENT

In this section,we construct the space of pseudoquotient.We require the set of continuous functions with at most exponential growth,since we intend to extend the LC transform on the space of pseudoquotient.M.Khosravi [**8**] thoroughly examined the creation of pseudoquotient.In order to construct space of pseudoquotient,we require two sets $(X; G)$.Consider the non-trivial function $g(t) = e^{\frac{-1}{t}}$ for $t > 0$ and $g(t) = 0$ for $t = 0$.Since $g(t)$ is continuous and bounded in $(0, \infty)$.So that $g(t) \in$ set of Laplace-Carson transformable function i.e.$g(t) \in \mathfrak{A}$

Let $G(s)$ be the LC transform of $g(t)$ and Let $G_1 = \{g^n/n \in \mathbb{N}\}$ Note that here, $g^n = g(t) * g(t) * g(t) * ... * g(t)$(n times) and $g(t) = e^{\frac{-1}{t}}$

Definition 5. *The action of $g(t)$ on $\mathfrak{A}$ is denoted by $(g * f)(t)$ and is defined by equation*

$$(g^n * f)(t) = \int_0^t g^n(x-t)f(x)dx \tag{4}$$

for all $f \in \mathfrak{A}$

Lemma 1. *An algebraic structure $(G_1, *)$ is a commutative semigroup.*

Proof. Consider $\Phi, \Psi, \chi \in G_1$,then we have $\Phi = g^{n_1}, \Psi = g^{n_2}, \chi = g^{n_3}$ for some $n_1, n_2, n_3 \in \mathbb{N}$. Clearly G_1 is closed under convolution product $*$

$$\begin{aligned} \text{Consider } \Phi * (\Psi * \chi) &= g^{n_1} * (g^{n_2} * g^{n_3}) \\ &= g^{n_1} * (g^{n_2+n_3}) \\ &= g^{n_1+n_2+n_3} \\ &= (g^{n_1+n_2}) * g^{n_3} \\ &= (\Phi * \Psi) * \chi, \text{for all } \Phi, \Psi, \chi \in G \end{aligned}$$

Also, $\Phi * \Psi = g^{n_1} * g^{n_2} = g^{n_1+n_2} = g^{n_2+n_1} = g^{n_2} * g^{n_1} = \Psi * \Phi$ for all $\Phi, \Psi \in G$
Hence an algebraic structure $(G_1, *)$ is a commutative semigroup. □

Lemma 2. *The action of $(G_1, *)$ on $\mathfrak{A}$ is injective.*

Proof. Let $f(t) \in \mathfrak{A}$ and $g(t) \in G_1$
Our aim is to prove that if $f * g = 0$ then $f = 0$
Consider $f * g = 0$
Applying Laplace-Carson transform on both sides,we get
$LC[f * g] = 0$
$\frac{1}{s}LC[f(t)] * LC[g(t)] = 0$
But $LC[g(t)] = s\int_0^\infty e^{\frac{-1}{t}}e^{-st}dt > 0$,for $s > 0$
Hence $LC[f(t)] = 0$ which implies $f(t) = 0$
In general $g^n * f = g * [g^{n-1} * f] = 0$
which implies $g^{n-1} * f = 0$,using inductive argument, we obtain $f = 0$
Hence action of $(G_1, *)$ is injective on $\mathfrak{A}$. □

Definition 6. *A product set* $\mathfrak{A} \times G_1$ *is defined by*

$$\mathfrak{A} \times G_1 = \{(f, \phi)/f \in \mathfrak{A}, \phi \in G_1\} \tag{5}$$

Definition 7. *Let* $m, n \in N$ *and* $f_1, f_2 \in \mathfrak{A}$ *then* (f_1, g^m) *and* (f_2, g^n) *are said to be equivalent if and only if* $f_1 * g^n = f_2 * g^m$
i.e. $(f_1, g^m) \sim (f_2, g^n)$ *if and only if* $f_1 * g^n = f_2 * g^m$
Hence $(\mathfrak{A}, G_1)$ *is a* $\sum -pair$

Definition 8. *The set of all pseudoqotients is the the set* $\mathfrak{P}(\mathfrak{A}, G_1)$ *of equivalence classes* $[(f, g^n)]$ *i.e* $\mathfrak{P}(\mathfrak{A}, G_1) = \{\frac{f}{g^n} : f \in \mathfrak{A}, g^n \in G_1\}$
Now,we define addition$(+)$*,convolution*$(*)$ *and scalar multiplication* $(.)$ *for each element of* $\mathfrak{P}(\mathfrak{A}, G_1)$ *by following way*

1. $\frac{f_1}{g^m} + \frac{f_2}{g^n} = \frac{g^n * f_1 + g^m * f_2}{g^m * g^n}$
2. $\frac{f_1}{g^m} * \frac{f_2}{g^n} = \frac{f_1 * f_2}{g^m * g^n}$
3. $\alpha \frac{f_1}{g^m} = \frac{\alpha f_1}{g^m}$*,where* $\alpha \in \mathbb{C}$

Theorem 1. *The operations addtion* $(+)$*,convolution* $(*)$ *and scalar multiplication* $(.)$ *defined in* $\mathfrak{P}(\mathfrak{A}, G_1)$ *are well defined.*

Proof. In order to prove that the operation addition$(+)$ is well defined,we have to show that $\frac{f_1}{g^m} \sim \frac{f_2}{g^q}$ and $\frac{f_3}{g^n} \sim \frac{f_4}{g^r}$ implies $\frac{f_1}{g^m} + \frac{f_3}{g^n} \sim \frac{f_2}{g^q} + \frac{f_4}{g^r}$

$$\begin{aligned}
(g^n * f_1 + g^m * f_3) * (g^q * g^r) &= (g^n * f_1 * g^q * g^r) + (g^m * f_3 * g^q * g^r) \\
&= g^n * (f_2 * g^m) * g^r + g^m * (f_4 * g^n) * g^q \\
&= f_2 * (g^n * g^m * g^r + f_4 * (g^m * g^n * g^q) \\
&= (f_2 * g^r) * (g^m * g^n) + (f_4 * g^q) * (g^m * g^n) \\
(g^n * f_1 + g^m * f_3) * (g^q * g^r) &= (g^r * f_2 + g^q * f_4) * (g^m * g^n)
\end{aligned}$$

Hence the operation addition$(+)$ is well defined.
Similarly ,we can easily show that convolution $(*)$ and scalar multiplication $(.)$ is well defined in $\mathfrak{P}(\mathfrak{A}, G_1)$ □

Theorem 2. $\mathfrak{P}(\mathfrak{A}, G_1)$ *is a commutative linear algebra with respect to operations addition* $(+)$*,convolution* $(*)$ *and scalar multiplication* $(.)$ *with unity.*

Proof. Let $x, y, z \in \mathfrak{P}(\mathfrak{A}, G_1)$ then
$x = \frac{f_1}{g^m}, y = \frac{f_2}{g^n}, z = \frac{f_3}{g^r}$
By the definition of addition,we have
$\frac{f_1}{g^m} + \frac{f_2}{g^n} = \frac{g^n * f_1 + g^m * f_2}{g^m * g^n} \implies x + y \in \mathfrak{P}(\mathfrak{A}, G_1)$
Similarly $(x * y) = \frac{f_1}{g^m} * \frac{f_2}{g^n} = \frac{f_1 * f_2}{g^m * g^n}$
Since $f_1 * f_2 = f_2 * f_1$,we have
$(x * y)(t) = (y * x)(t)$
Hence convolution product is commutative in $\mathfrak{P}(\mathfrak{A}, G_1)$.
Similarly,we can easily show that $\mathfrak{P}(\mathfrak{A}, G_1)$ is closed under scalar multiplication $(.)$
Consider

$$\begin{aligned}
x * (y + z) &= \frac{f_1}{g^m} * (\frac{g^r * f_2 + g^n * f_3}{g^n * g^r}) \\
&= \frac{f_1 * (g^r * f_2 + g^n * f_3)}{g^m * (g^n * g^r)} \\
&= \frac{f_1 * g^r * f_2 + f_1 * g^n * f_3}{g^m * g^n * g^r} \\
&= \frac{f_1 * g^r * f_2}{g^m * g^n * g^r} + \frac{f_1 * g^n * f_3}{g^m * g^n * g^r} \\
&= \frac{f_1 * f_2}{g^m * g^n} + \frac{f_1 * f_3}{g^m * g^r}
\end{aligned}$$

$\therefore x * (y + z) = (x * y) + (y * z)$
This proves that convolution is distributive in $\mathfrak{P}(\mathfrak{A}, G_1)$.

$$\begin{aligned}
\text{Consider, } \beta(x * y) &= \beta \frac{(f_1 * f_2)}{g^m * g^n} \\
&= \frac{\beta f_1 * f_2}{g^m * g^n} = \frac{\beta f_1}{g^m} * \frac{f_2}{g^n} \\
\therefore \beta(x * y) &= \beta x * y
\end{aligned}$$

Similarly we can show that $\beta(x * y) = (x * \beta y)$
Hence $\mathfrak{P}(\mathfrak{A}, G_1)$ creates commutative linear algebra,Here $\sigma = \frac{g}{g}$ is unit element in $(\mathfrak{P}(\mathfrak{A}, G_1), +, *, .)$ □

4. EXTENDED LAPLACE-CARSON TRANSFORM

In this section we extend the definition of the Laplace Carson transform on the space of pseudoquotients and illustrate its auxiliary results.

Definition 9. *Let $x = \frac{f}{g^n}$ be any element of $\mathfrak{P}(\mathfrak{A}, G_1)$ then extended Laplace-Carson transform denoted by $LC[x(t)]$ is defined by*

$$LC[x(t)] = LC[\frac{f}{g^n}] = s^n.\frac{LC[f(t)]}{LC[g^n(t)]} = s^n.\frac{F_c(s)}{G_c^n(s)} \tag{6}$$

Preposition 1. *The extended Lapalace -Carson transform of each element of $\mathfrak{P}(\mathfrak{A}, G_1)$ is well defined.*

Proof. Let $\frac{f_1}{g^n}, \frac{f_2}{g^m} \in \mathfrak{P}(\mathfrak{A}, G_1)$ such that $\frac{f_1}{g^n} \sim \frac{f_2}{g^m}$
Hence ,we have $f_1 * g^m = f_2 * g^n$
Applying classical Laplace -Carson transform to both sides of above equation,we get
$LC[f_1 * g^m] = LC[f_2 * g^m]$
Using convolution theorem of classical Laplace-Carson transform,we have

$$\frac{1}{s}LC[f_1].LC[g^m] = \frac{1}{s}LC[f_2].LC[g^n]$$
$$\frac{1}{s}LC[f_1].\frac{1}{s^{m-1}}.LC[g]^m = \frac{1}{s}LC[f_2].\frac{1}{sn-1}LC[g]^n$$
$$\frac{1}{s^m}LC[f_1]LC[g]^m = \frac{1}{s^n}LC[f_2].LC[g]^n$$
$$s^n\frac{LC[f_1]}{LC[g]^n} = s^m\frac{LC[f_2]}{LC[g]^m}$$
$$s^n\frac{F_1(s)}{G_C^n(s)} = s^m\frac{F_2(s)}{G_C^m(s)}$$

where $F_1(s)$ and $F_2(s)$ are classical Laplace-Carson transform of $f_1(t)$ and $f_2(t)$ and $G(s)$ is classical Laplace-Carson transform of $g(t)$.Hence the extended Laplace -Carson transform is well defined. □

Theorem 3. *(Uniqueness Theorem) If $\frac{f_1}{g^n}, \frac{f_2}{g^m} \in \mathfrak{P}(\mathfrak{A}, G_1)$ such that $LC[\frac{f_1}{g^n}] = LC[\frac{f_2}{g^m}]$ for $R_e(s) > 0$ then $\frac{f_1}{g^n} = \frac{f_2}{g^m}$*

Proof. We have $LC[\frac{f_1}{g^n}] = LC[\frac{f_2}{g^m}]$
By definition ,we get

$$\frac{F_1(s)}{G^n(s)} = s^m\frac{F_2(s)}{G^m(s)}$$
$$\frac{1}{s^m}F_1(s)G^m(s) = \frac{1}{s^n}F_2(s)G^n(s)$$
$$LC[f_1 * g^m(t)] = LC[f_2(t) * g^(t)]$$

Since classical Laplace-Carson transform possesses uniqueness property,we have

$$(f_1 * g^m) = (f_2 * g^n)(t)$$
$$\frac{f_1}{g^n} = \frac{f_2}{g^m}$$

Hence the uniqueness theorem is proved. □

Definition 10. *Define* $H_a : \mathfrak{P}(\mathfrak{A}, G_1) \to \mathfrak{P}(\mathfrak{A}, G_1)$ *by*
$H_a(\frac{f}{g^n}) = \frac{H_a f}{a^n (H_a g)^n}$ *for all* $\frac{f}{g^n} \in \mathfrak{P}(\mathfrak{A}, G_1)$ *where* $H_a f(t) = f(at), a \epsilon R_+$

Preposition 2. $H_a : \mathfrak{P}(\mathfrak{A}, G_1) \to \mathfrak{P}(\mathfrak{A}, G_1)$ *defined by*
$H_a(\frac{f}{g^n}) = \frac{H_a f}{a^n (H_a g)^n}$ *is well defined.*

Proof. Consider $\frac{f_1}{g^m} = \frac{f_2}{g^n}$
$\therefore f_1 * g^n = f_2 * g^m$
Applying classical Laplace-Carson transform,we obtain

$$LC[f_1 * g^n] = LC[f_2 * g^m]$$
$$\frac{1}{s^n} F_1(s) G^n(s) = \frac{1}{s^m} F_2(s) G^m(s)$$

Now replacing s by as,we obtain

$$\frac{1}{(as)^n} F_1(as) G^n(as) = \frac{1}{(as)^m} F_2(as) G^m(as)$$
$$(as)^m \frac{F_1(as)}{G^m(as)} = (as)^n \frac{F_2(as)}{G^n(as)}$$
$$a^m \frac{s^m F_1(as)}{G^m(as)} = a^n \frac{s^n F_2(as)}{G^n(as)}$$
$$a^m LC[\frac{H_a f_1(t)}{H_a g^m(t)}] = a^n LC[\frac{H_a f_2(t)}{H_a g^n(t)}]$$

By uniqueness theorem of extended Laplace-Carson theorem.We obtain

$$a^m \frac{H_a f_1(t)}{H_a g^m(t)} = a^n \frac{H_a f_2(t)}{H_a g^n(t)}$$
$$H_a(\frac{f_1}{g^m}) = H_a(\frac{f_2}{g^n})$$

Hence the operator H_a is well defined. □

Definition 11. *Define* $K_a : \mathfrak{P}(\mathfrak{A}, G_1) \to \mathfrak{P}(\mathfrak{A}, G_1)$ *by*

$K_a(\frac{f}{g^n}) = \frac{K_a f(t)}{[K_a g(t)]^n}$ *for all* $\frac{f}{g^n} \in \mathfrak{P}(\mathfrak{A}, G_1)$, *where* $K_a f(t) = e^{atf(t)}$ *and* $a \in \mathbb{R}$

Preposition 3. $K_a : \mathfrak{P}(\mathfrak{A}, G_1) \to \mathfrak{P}(\mathfrak{A}, G_1)$ *defined by*
$K_a(\frac{f}{g^n}) = \frac{K_a f(t)}{[K_a g(t)]^n}$ *for all* $\frac{f}{g^n} \in \mathfrak{P}(\mathfrak{A}, G_1)$ *is well defined.*

Proof. Consider $\frac{f_1}{g^m} = \frac{f_2}{g^n}$
$\therefore f_1 * g^n = f_2 * g^m$
Applying classical Laplace-Carson transform,we get

$$LC[f_1 * g^n] = LC[f_2 * g^m]$$
$$\frac{1}{s^n} F_1(s) G^n(s) = \frac{1}{s^m} F_2(s) G^m(s)$$

Now replacing s by $\dfrac{s}{s-a}$,we get

$$\frac{1}{(\frac{s}{s-a})^n} F_1(\frac{s}{s-a}) G^n(\frac{s}{s-a}) = \frac{1}{(\frac{s}{s-a})^m} F_2(\frac{s}{s-a}) G^m(\frac{s}{s-a})$$
$$(\frac{s-a}{s})^n F_1(\frac{s}{s-a}) G^n(\frac{s}{s-a}) = (\frac{s-a}{a})^m F_2(\frac{s}{s-a}) G^m(\frac{s}{s-a})$$
$$\frac{F_1(\frac{s}{s-a})}{(\frac{s-a}{a})^m G^m(\frac{s}{s-a})} = \frac{F_2(\frac{s}{s-a})}{(\frac{s-a}{a})^n} . G^n(\frac{s}{s-a})$$
$$LC[\frac{K_a f_1(t)}{(K_a g(t))^m}] = LC[\frac{K_a f_2(t)}{[K_a g(t)]^m}]$$

By uniqueness theorem of extended Laplace -Carson transform,we obtain

$$\frac{K_a f_1(t)}{[K_a g(t)]^m} = \frac{K_a g(t)}{[K_a g(t)]^n}$$
$$K_a[\frac{f_1}{g^n}] = K_a[\frac{f_2}{g^m}]$$

Hence an operator K_a is well defined. □

Definition 12. *Define* $H_a : \mathfrak{P}(\mathfrak{A}, G_1) \to \mathfrak{P}(\mathfrak{A}, G_1)$ *by*
$H_a(\frac{f}{g^n}) = \frac{H_a f(t)}{g^n(t)}$ *for all* $\frac{f}{g^n} \in \mathfrak{P}(\mathfrak{A}, G_1)$
where $H_a f(t) = f(t-a)H(t-a)$ *and* $a \in R_+$ *,*$H(t)$ *is Heaviside unit step function*

Preposition 4. $H_a : \mathfrak{P}(\mathfrak{A}, G_1) \to \mathfrak{P}(\mathfrak{A}, G_1)$ *defined by*
$H_a(\frac{f}{g^n}) = \frac{H_a f(t)}{g^n(t)}$ *for all* $\frac{f}{g^n} \in \mathfrak{P}(\mathfrak{A}, G_1)$ *is well defined.*

Proof. Consider $\frac{f_1}{g^m} = \frac{f_2}{g^n}$
$\therefore f_1 * g^n = f_2 * g^m$

Applying classical Laplace Carson transform,we get

$$LC[f_1 * g^m] = LC[f_2 * g^n]$$
$$\frac{1}{s^m}F_1(P)G^m(s) = \frac{1}{s^n}F_2(P)G^n(s)$$

Now multiplying by e^{-as},we obtain

$$s^n\frac{e^{-as}F_1(s)}{g^n(s)} = s^m\frac{e^{-as}F_2(s)}{g^m(s)}$$
$$s^n\frac{LC[f_1(t-a)(H(t-a)]}{G^n(s)} = s^m\frac{LC[f_2(t-a)(H(t-a))]}{G^m(s)}$$
$$LC[\frac{H_a f_1(t)}{g^n}] = LC[\frac{H_a f_2(t)}{g^m}]$$

By uniqueness theorem of extended Laplace -Carson theorem,we get

$$[\frac{H_a f_1(t)}{g^n}] = [\frac{H_a f_2(t)}{g^m}]$$
$$H_a[\frac{f_1(t)}{g^n}] = H_a[\frac{f_2(t)}{g^m}]$$

Hence an operator H_a is well defined. □

5. OPERATIONAL CALCULUS OF EXTENDED LAPLACE CARSON TRANSFORM

In this section,we will illustrate the operational calculus of extended Laplace Carson transform including the convolution and generalized derivative theorem.

Property 1. *If* $\frac{f_1}{g^m}, \frac{f_2}{g^n} \in \mathfrak{P}(\mathfrak{A}, G_1)$ *and* α, β *are any scalars then*
LC$[\alpha\frac{f_1}{g^m} + \beta\frac{f_2}{g^n}] = \alpha$ *LC*$[\frac{f_1}{g^m}] + \beta$ *LC*$[\frac{f_2}{g^n}]$

Proof. From the definition, we have
$\alpha\frac{f_1}{g^m} + \beta\frac{f_2}{g^n} = \frac{\alpha f_1}{g^m} + \frac{\beta f_2}{g^n} = \frac{g^n*(\alpha f_1)+g^m*(\beta f_2)}{g^m*g^n}$

Applying extended Laplace-Carson transform,we obtain

$$\begin{aligned}LC[\alpha\frac{f_1}{g^m}+\beta\frac{f_2}{g^n}] &= LC[\frac{g^n*(\alpha f_1)+g^m*(\beta f_2)}{g^m*g^n}]\\ &= \frac{\alpha.\frac{1}{s^n}F_1(s).G^n(s)+\beta.\frac{1}{s^m}F_2(s).G^m(s)}{\frac{1}{s^{n+m}(s).}G^{n+m}(s)}\\ &= \frac{s^{m+n}[\alpha\frac{1}{s^n}F_1(s)G^n(s)+\beta\frac{1}{s^m}F_2(s)G^m(s)]}{G^{m+n}(s)}\\ &= \alpha s^m F_1(s)G^m(s)+\beta.s^n F_2(s)G^m(s)\end{aligned}$$

Hence, $LC[\alpha\frac{f_1}{g^m}+\beta\frac{f_2}{g^n}]=\alpha LC[\frac{f_1}{g^m}]+\beta LC[\frac{f_2}{g^n}]$ □

Property 2. *(First Scale Preserving Property)*
Let $x(t)\in\mathfrak{P}(\mathfrak{A},G_1)$ *and LC*$[x(t)]=X(s)$*, then LC* $[M_a x(t)]=X(as)$*,where* $a\in R_+$

Proof. For some $n\in\mathbb{N}$ and $f_1\in\mathfrak{A}$,we can take $x(t)=\frac{f_1}{g^n}$

$$\begin{aligned}LC[H_a x(t)] &= LC[\frac{a^n H_a f_1(t)}{(H_a g(t))^n}]\\ &= LC[\frac{a^n f_1(at)}{g^n(at)}]\\ &= \frac{LC[f_1(at)]}{[LC[g(at)]]^n}\\ &= \frac{F_1(as)}{G^n(as)}\\ &= LC[\frac{f_1(t)}{g^n(t)}](as)\end{aligned}$$

$\therefore LC[H_a x(t)]=X(as)$ □

Property 3. *(Convolution theorem of Extended Laplace-Carson transform)*
If $x(t),y(t)\in\mathfrak{P}(\mathfrak{A},G_1)$ *such that LC*$[x(t)]=X(s)$ *and LC*$[y(t)]=Y(s)$*,*
then LC$[(x*y)(t)]=\frac{1}{s}X(s)Y(s)$

Proof. For some $n,m\in\mathbb{N}$ and $f_1,f_2\in\mathfrak{A}$,we can take $x(t)=\frac{f_1}{g^n},y(t)=\frac{f_2}{g^m}$
$\therefore (x*y)(t)=\frac{f_1*f_2}{g^n*g^m}$

Applying extended Laplace-Carson transform on both sides,we get

$$\begin{aligned} LC[(x*y)(t)] &= LC[\frac{f_1 * f_2}{g^n * g^m}] \\ &= \frac{1}{s} s^{n+m} \frac{F_1(s)F_2(s)}{G^{n+m}(s)} \\ &= \frac{1}{s} s^n \frac{F_1(s)}{G^n(s)} . s^m \frac{F_2(s)}{G^m(s)} \\ &= \frac{1}{s} LC[\frac{f_1}{g^n}] LC[\frac{f_2}{g^m}] \\ LC[(x*y)(t)] &= \frac{1}{s} X(s).Y(s) \end{aligned}$$

□

Property 4. *(Second Shifting Property) Let* $x(t) \in \mathfrak{P}(\mathfrak{A}, G_1)$ *and LC*$[x(t)] = X(s)$*, then LC* $[H_a x(t)] = e^{-as}X(s)$*, where* $a \in \mathbb{R}$*.*

Proof. Consider $x(t) = \frac{f_1}{g^n}$ for some $f \in \mathfrak{A}, n \in \mathbb{N}$ then
LC$[H_a x(t)]$ =LC$[H_a(\frac{f_1}{g^n})]$
= LC$[\frac{H_a f_1}{g^n}]$ =LC$[\frac{f_1(t-a)}{g^n(t)}]$
$= s^n \frac{LC[f_1(t-a)]}{LC[g^n(t)]} = s^n . \frac{e^{-as}F_1(s)}{G^n(s)}$
$= e^{-as}$ LC$[\frac{f_1(t)}{g^n(t)}](s)$
LC$[H_a x(t)] = e^{-as}X(s)$ □

Definition 13. *(Generalized Derivative)*
Let D *be the generalized differential operator for the elements of* $\mathfrak{P}(\mathfrak{A}, G_1)$.
If $\frac{f_1(t)}{g^n(t)} \in \mathfrak{P}(\mathfrak{A}, G_1)$*, then its generalized derivative is defined by*
$D(\frac{f_1(t)}{g^n(t)}) = \frac{f_1}{g^n(t)} + \frac{f(0)\delta(t)}{g^n(t)}$

Preposition 5. *The generalized derivative* D *defined as* $D(\frac{f_1(t)}{g^n(t)}) = \frac{f_1}{g^n(t)} + \frac{f(0)\delta(t)}{g^n(t)}$ *is well defined.*

Proof. Let $\frac{f_1}{g^n} = \frac{f_2}{g^m}$
$\implies f_1 * g^m = f_2 * g^n$

Applying classical Laplace-Carson transform,we get

$$\frac{1}{s^m}F_1(s).G^m(s) = \frac{1}{s^n}F_2(s).G^n(s)$$
$$\frac{s^nF_1(s)}{G^n(s)} = \frac{s^mF_2(s)}{G^m(s)}$$
$$\frac{s^nF_1(s)}{\frac{1}{s^{n-1}}G^n(s)} = \frac{s^mF_2(s)}{\frac{1}{s^{m-1}}G^m(s)}$$
$$\frac{s[F_1(s) - F_1(0) + F_1(0)]}{\frac{1}{s^{n-1}}G^n(s)} = \frac{s[F_2(s) - F_2(0) + F_2(0)]}{\frac{1}{s^{m-1}}G^m(s)}$$
$$\frac{LC[f_1'(t)] + f_1(0) + LC[\delta(t)]}{LC[g^n(t)]} = \frac{LC[f_2'(t) + f_2(0) + LC[\delta(t)]}{LC[g^m(t)]}$$
$$s.\frac{LC[f_1'(t)] + f_1(0) + LC[\delta(t)]}{LC[g^n(t)]} = s.\frac{LC[f_2'(t) + f_2(0) + LC[\delta(t)]}{LC[g^m(t)]}$$
$$LC[\frac{[f_1'(t)] + f_1(0)[\delta(t)]}{LC[g^n(t)]}] = LC[\frac{[f_2'(t) + f_2(0)[\delta(t)]}{LC[g^m(t)]}]$$

Using uniqueness theorem,we have

$$\frac{f_1'(t)] + f_1(0)\delta(t)}{g^n(t)} = \frac{f_2'(t) + f_2(0)\delta(t)}{g^m(t)}$$
$$D[\frac{f_1(t)}{g^n(t)}] = D[\frac{f_2(t)}{g^m(t)}]$$

Hence,generalized derivative is well defined. □

Preposition 6. *If $x(t) \in \mathfrak{P}(\mathfrak{A}, G_1)$ and $T(t) = tH(t)$, where $H(t)$ is Heaviside unit step function then , LC$[T.x(t)] = \frac{1}{s^2}\frac{d}{ds}[Z(s)] + \frac{1}{s}Z(s)$*

Proof. For some $n \in \mathbb{N}$ and for some $f_1(t) \in \mathfrak{A}$,
we have $x(t) = \frac{f_1(t)}{g^n(t)} \in \mathfrak{P}(\mathfrak{A}, G_1)$
$\therefore T.x(t) = \frac{Tf_1(t)}{g^n} - \frac{n(f_1 * Tg)(t)}{g^{n+1}}$
Applying extended Laplace Carson transform,we get

$$\begin{aligned} LC[T.x(t)] &= LC[\frac{Tf_1(t)}{g^n}] - nLC[\frac{(f_1 * Tg)(t)}{g^{n+1}}] \\ &= \frac{\frac{1}{s^2}F_1(s) + \frac{1}{s}F_1(s)}{\frac{1}{s^n}G^n(s)} - n[\frac{\frac{1}{s}f_1(s)(\frac{1}{s^2}.G'(s) + \frac{1}{s}G(s))}{\frac{1}{s^{n+1}}G^{n+1}(s)}] \\ &= \frac{\frac{1}{s^2}F_1(s)}{\frac{1}{s^n}G^n(s)} - \frac{n\frac{1}{s}F_1(s)}{\frac{1}{s^n}G^n(s)} - \frac{n\frac{1}{n^2}F_1(s)G'(s)}{\frac{1}{s^n}G^{n+1}(s)} + \frac{\frac{1}{s}F_1(s)}{\frac{1}{s^n}G^n(s)} \\ &= \frac{1}{\frac{1}{s^n}G^n(s)}[\frac{1}{s^2}F_1(s) - n\frac{1}{s}F_1(s) - \frac{n.\frac{1}{s^2}F_1(s)G'(s)}{G(s)}] + \frac{\frac{1}{s}F_1(s)}{\frac{1}{s^n}G^n(s)} \end{aligned}$$

Multiplying numerator and denominator of first term of above equation by $\frac{1}{s^n}G^n(s)$,we get

$$\begin{aligned} LC[T.x(t)] &= \frac{1}{s^2}\frac{d}{ds}[\frac{F_1(s)}{\frac{1}{s^n}G^n(s)}] + \frac{1}{s}\frac{F_1(s)}{\frac{1}{s^n}G^n(s)} \\ &= \frac{1}{s^2}\frac{d}{ds}LC[x(t)] + \frac{1}{s}LC[x(t)] \\ LC[T.x(t)] &= \frac{1}{s^2}\frac{d}{ds}Z(s) + \frac{1}{s}Z(s) \end{aligned}$$

Hence the proof □

6. CONCLUSION

In this article the Laplace Carson transform has been successfully extended on the space of pseudoquotients.The LC transform theorems for generalized multiplication and generalized derivative are helpful in resolving integral and differential problems that require solutions expressed in terms of generalized functions.Additional investigation may reveal that LC transformable distributions and LC transformable pseudoquotients are isomorphic.The suggested distributional approach to the LC-transform on represents a significant theoretical advancement that generalizes the classical LC-transform to a much broader class of objects called distributions.This approach provides rigor to previously heuristic operational calculus and it maintains computational discipline while expanding its applicability in pure as well as in applied mathematics.This generalized approach of LC-transform establish that generalized LC-transform is a robust tool in the modern mathematical treatment of linear dynamical systems and differential equations.This study suggest that similar approaches could be fruitful for other existing integral transforms on suitable distribution spaces.

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