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A Function Derived from Euler's Reflection Formula of the Gamma Function

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Abstract

Euler's reflection formula is usually used to derive the Legendre duplication formula. Another function is also usually derived from Euler's reflection formula. A variant of this function is derived and is shown to be relevant to the Poisson probability distribution and the Riemann zeta function.

Keywords : Riemann zeta function, gamma function.

1. INTRODUCTION

Euler's reflection formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$, $z \notin Z$, implies $\Gamma(z-n) = (-1)^{n-1}\frac{\Gamma(-z)\Gamma(1+z)}{\Gamma(n+1-z)}$, $n \in Z$. Equation (3) in section 1.3 of Edward's [1] book is

$$\Pi(s) = \lim_{N\to\infty} \frac{1 \cdot 2 \cdots N}{(s+1)(s+2)\cdots(s+N)}(N+1)^s \tag{1}$$

This equation is valid for all s in the halfplane $\Re s > -1$. Let $\Pi_1(s)$ denote

$$\lim_{N\to\infty} \frac{1 \cdot 2 \cdots N}{(\Re s+1)(\Re s+2)\cdots(\Re s+N)}(N+1)^s \tag{2}$$

$\Pi(z) = \Gamma(z+1) = z\Gamma(z) = \int_0^\infty e^{-t}t^z dt$. Let $R(z,n)$ (or $R(z,n,N)$) denote $\Pi_1(z-n) - (-1)^{n-1}\frac{\Pi_1(-z)\Pi_1(1+z)}{\Pi_1(n+1-z)}$ where $z-1$ is substituted for z (to convert from $\Gamma(z)$ to $\Pi(z)$). A plot of the imaginary part of $R(z,N,n)$ versus the real part for $n=1$, the first non-trivial zeta function zero, and $N = 1, 2, 3, \ldots, 2000$ is

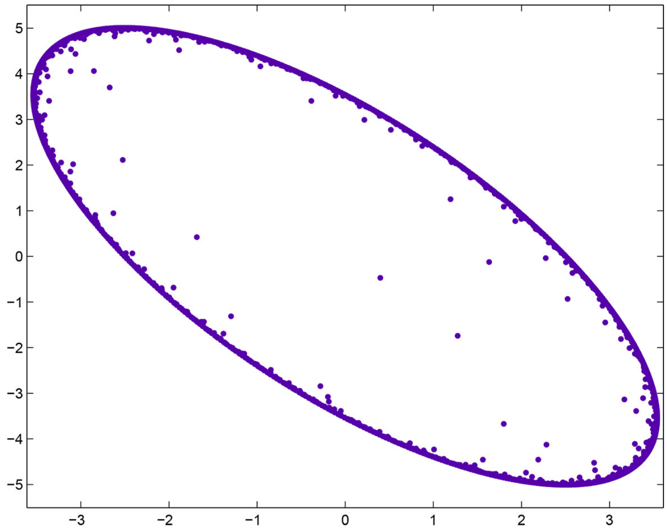

Figure 1

2. THE AMPLITUDE OF $R(z, n)$

A plot of the real and imaginary parts is

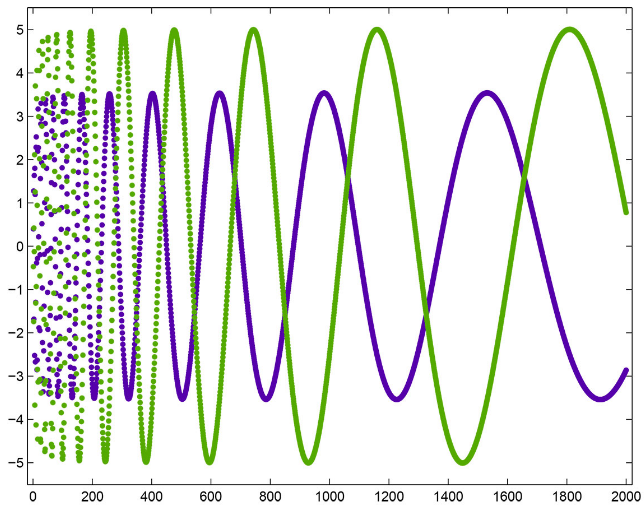

Figure 2

A plot of the logarithms of the N values where the absolute values of the real part crest for $N \leq 10000000$ is

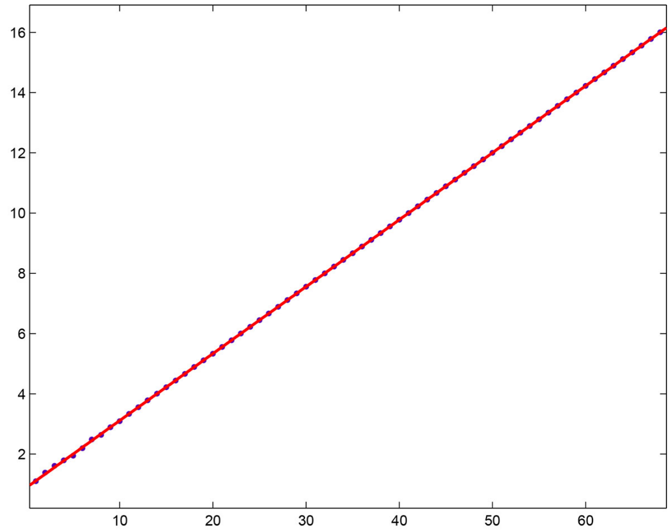

Figure 3

For a linear least-squares fit of the curve, $p_1 = 0.2222$ with a 95% confidence interval of (0.2221, 0.2224), $p_2 = 0.8905$ with a 95% confidence interval of (0.8838, 0.8973), SSE=0.01251, R-squared=1, and RMSE=0.01377.

A plot of the amplitudes at these N values is

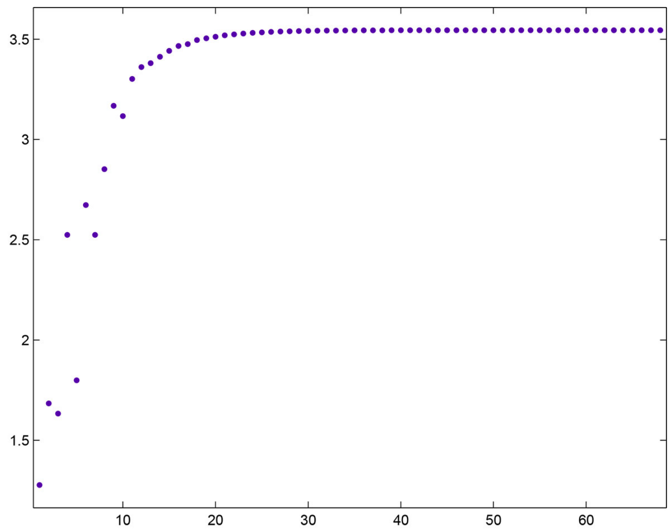

Figure 4

The maximum amplitude is 3.5449.
A plot of the logarithms of the N values where the absolute values of the imaginary part

crest for $N \leq 10000000$ is

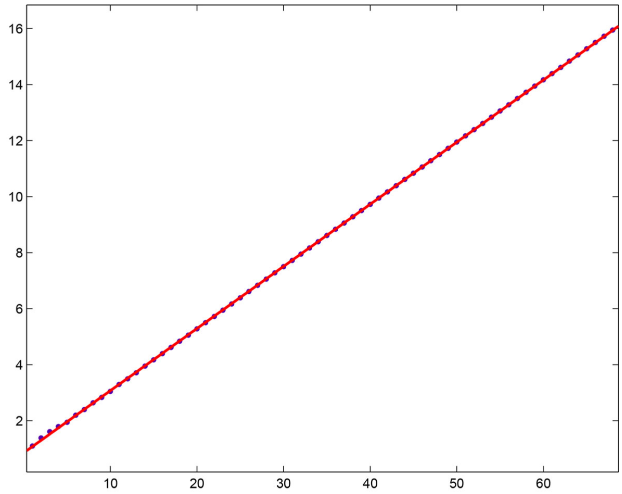

Figure 5

For a linear least-squares fit of the curve, $p_1 = 0.2218$ with a 95% confidence interval of (0.2215, 0.222), $p_2 = 0.8562$ with a 95% confidence interval of (0.8468, 0.8655), SSE=0.02411, R-squared=1, and RMSE=0.01991. The slope is the same as that for the Riemann zeta function (defined in the critical strip) where the imaginary part of the curve crosses the x-axis from above. For a linear least-squares fit of the curve, $p_1 = 0.2218$ with a 95% confidence interval of (0.2215, 0.2222), $p_2 = 2.004$ with a 95% confidence interval of (1.991, 2.018), SSE=0.04178, R-squared=1, and RMSE=0.02617.

A plot of the amplitudes at these N values is

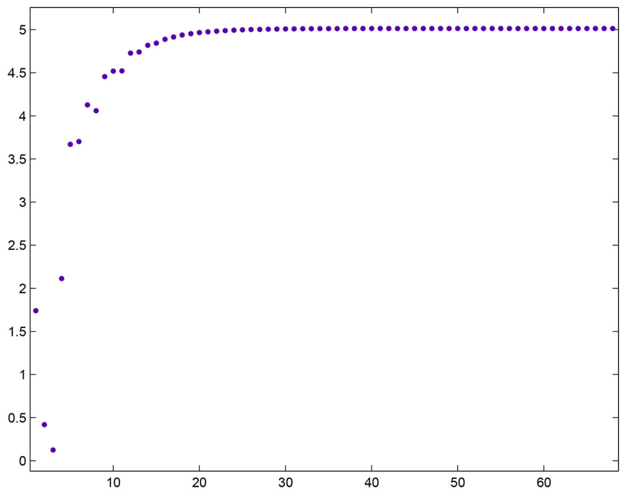

Figure 6

The maximum amplitude is 5.0133. This is $\sqrt{2}$ times the maximum amplitude for the real part.

For the real parts of $R(z,1,N)$, $R(z,2,N)$, $R(z,3,N)$,...,$R(z,15,N)$, $z = (0.9999, 600)$ for even n or $z = (0.9999, 500)$ for odd n, and $N \leq 10000000$, the respective maximum amplitudes are 10000.5763, 9999.5743, 4999.5357, 1666.4557, 416.6033, 83.3184, 13.8862, 1.9837, 0.2480, 0.02755, 0.002755, 0.0002505, 0.00002087, 0.000001605, and 0.0000001146. These values divided by their sum give a Poisson probability distribution ($\frac{\lambda^k e^{-\lambda}}{k!}$) where $\lambda = 1$. A plot of the Poisson probability distribution versus these values is

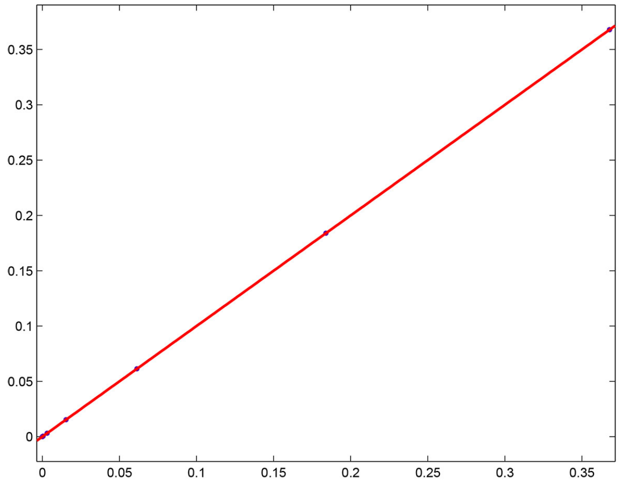

Figure 7

The slope is 1. Relatively large imaginary components of z were chosen so that the amplitudes would converge to their maximum values. Other than this, the amplitudes do not depend on the imaginary component of z. $R(z,n)$ is centered on (0, 0) for all z in the critical strip.

For the real parts of $R(z,1,N)$, $R(z,2,N)$, $R(z,3,N)$,...,$R(z,15,N)$, $z = (0.0001, 600)$ for even n or $z = (0.0001, 500)$ for odd n, and $N \leq 10000000$, the respective maximum amplitudes are 10000.4199, 5000.4585, 1666.8744, 416.7288, 83.3474, 13.8915, 1.9845, 0.2481, 0.0276, 0.002756, 0.0002506, 0.00002088, 0.000001606, 0.0000001147, and 0.000000007649. These values divided by their sum approximately give a Poisson probablity distribution where $\lambda = 0.5$. A plot of the

Poisson probability distribution versus these values is

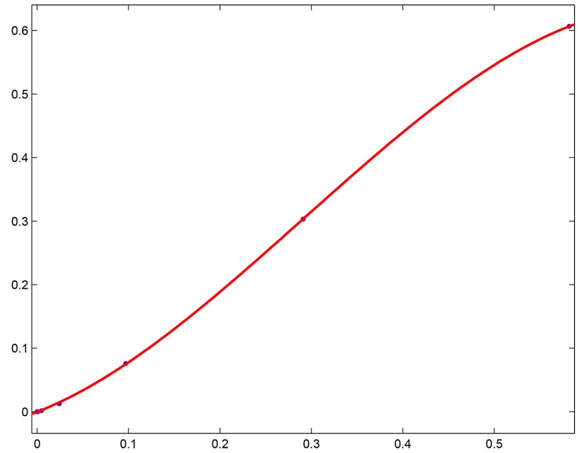

Figure 8

For a cubic least-squares fit of the curve, $p_1 = -2.862$ with a 95% confidence interval of $(-3.011, -2.653)$, $p_2 = 2.469$ with a 95% confidence interval of (2.322, 2.669), $p_3 = 0.5597$ with a 95% confidence interval of (0.5277, 0.5918), $p_4 = -0.0002606$ with a 95% confidence interval of $(-0.0007765, 0.0002552)$, SSE=$6.737 \cdot 10^{-6}$, R-squared=1, and RMSE=0.0007826.

For the real parts of $R(z, 1, N)$, $R(z, 2, N)$, $R(z, 3, N)$,...,$R(z, 15, N)$, $z = (0.50, 600)$ for even n or $z = (0.50, 500)$ for odd n, and $N \leq 10000000$, the respective maximum amplitudes are 3.54490703, 2.36327076, 0.94530797, 0.27008787, 0.06001949, 0.01091263, 0.00167886, 0.00022385, 0.00002634, 0.000002772, 0.0000002640, and 0.000000002295. The second value divided by the first is 2/3. These values divided by their sum approximately give a Poisson probablity distribution where $\lambda = 2/3$. A plot of the Poisson probability distribution versus these values is

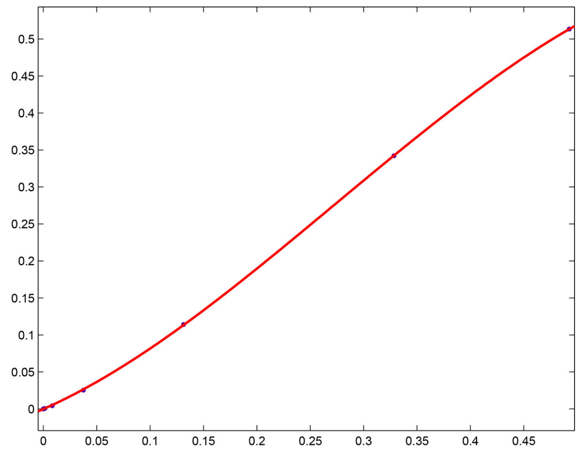

Figure 9

For a cubic least-squares fit of the curve, $p_1 = -2.531$ with a 95% confidence interval of $(-2.69, -2.371)$, $p_2 = 2.072$ with a 95% confidence interval of (1.956, 2.188), $p_3 = 0.6361$ with a 95% confidence interval of (0.6159, 0.6563), $p_4 = -0.0002058$ with a 95% confidence interval of (-0.0005653, 0.0001537), SSE=$3.205 \cdot 10^{-6}$, R-squared=1, and RMSE=0.0005398.

Real parts of z that are rational numbers are easier to analyze. For real parts of 1/11, 2/11, 3/11, 4/11, 5/11, 6/11, 7/11, 8/11, 9/11, and 10/11, the ratio of the first two values of the maximum amplitudes of $R(z, 1, N)$, $R(z, 2, N)$, $R(z, 3, N)$,...,$R(z, 15, N)$ are 11/21, 11/20, 11/19, 11/18, 11/17, 11/16, 11/15, 11/14, 11/13, and 11/12. The sums of $R(z, 1, N)$, $R(z, 2, N)$, $R(z, 3, N)$,...,$R(z, 15, N)$ can then be computed and used to normalize the distributions. The Poisson probability distributions versus these values can then be computed. A plot of the p_1 parameters of the cubic least-squares fits are

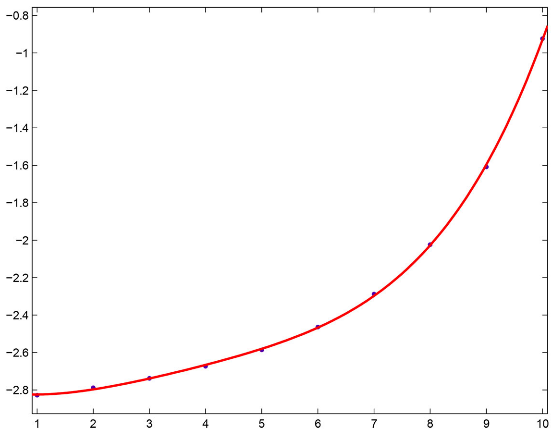

Figure 10

A plot of the p_2 parameters is

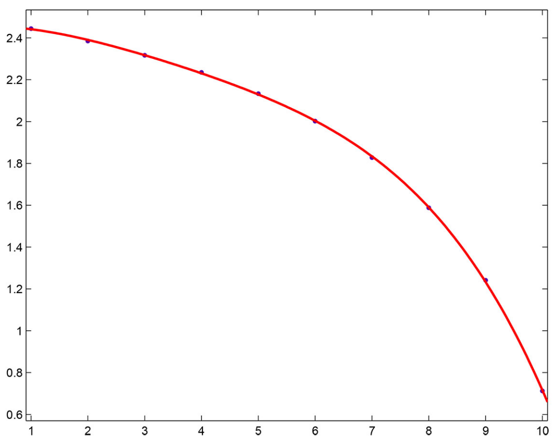

Figure 11

A plot of the p_3 parameters is

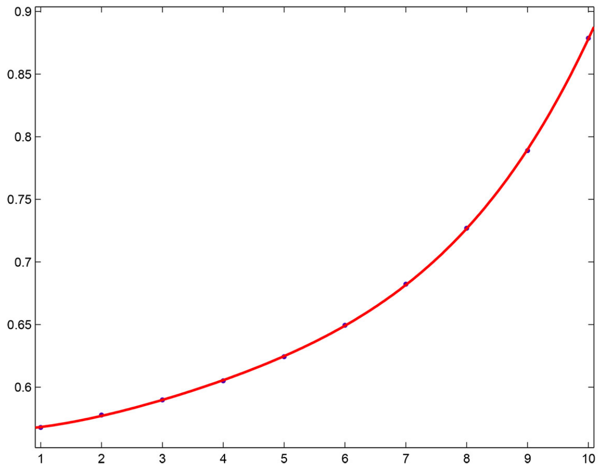

Figure 12

A plot of the p_4 parameters is

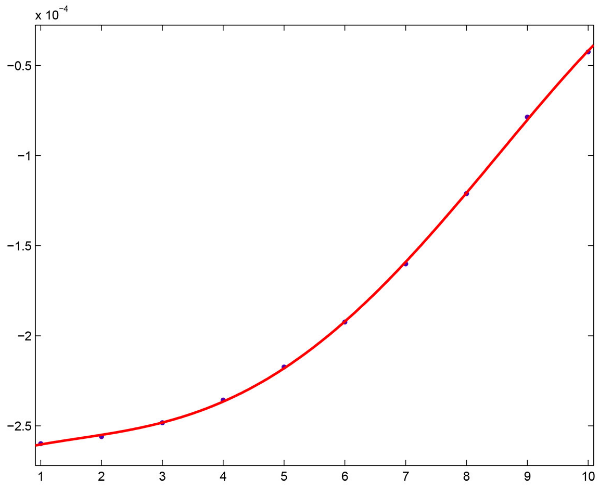

Figure 13

All the above curves are quartic.

A plot of the logarithm of the sums of $R(z,1,N)$, $R(z,2,N)$, $R(z,3,N)$,...,$R(z,15,N)$, is

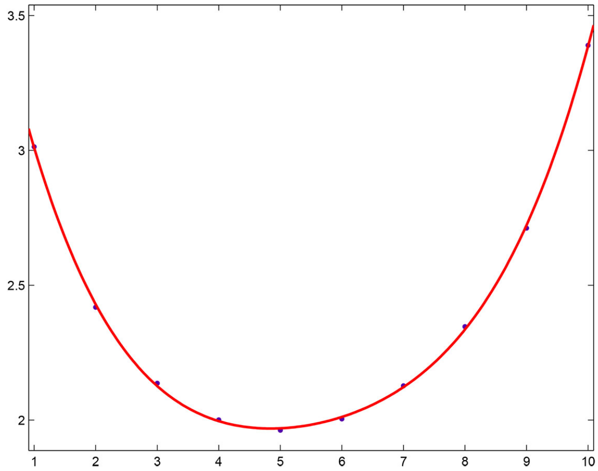

Figure 14

For a quartic least-squares fit of the curve, $p_1 = 0.001451$ with a 95% confidence interval of (0.001226, 0.001677), $p_2 = -0.3194$ with a 95% confidence interval of (-0.03693, -0.02695), $p_3 = 0.2942$ with a 95% confidence interval of (0.2569, 0.3316), $p_4 = -1.262$ with a 95% confidence interval of (-1.37, -1.154), $p_5 = 4.008$ with a 95% confidence interval of (3.913, 4.104), SSE=0.0006336, R-squared=0.9997, and RMSE=0.01126.

3. THE CUMULATIVE PROBABILITY DISTRIBUTION OF THE POISSON PROBABILITY DISTRIBUTION AND THE INCOMPLETE GAMMA FUNCTION

The cumulative distribution function of the Poisson probability distribution is $\frac{\Gamma(\lfloor k+1 \rfloor, \lambda)}{[k]!}$. The incomplete gamma function is used here. The incomplete gamma function is $\Gamma(z+1,x) = z\Gamma(z,x) + x^z e^{-x}$. Let $C(z,x)$ denote $z\Pi_1(z,x) + x^z e^{-x}$ where $z-1$ is substituted for z (to convert from $\Gamma(z)$ to $\Pi(z)$). C code for computing $C(z,x)$ is given in the Methods section. A plot of the logarithm of the modulus of $C(z,x)$ for the first zeta function zero and $x \leq 120$ is

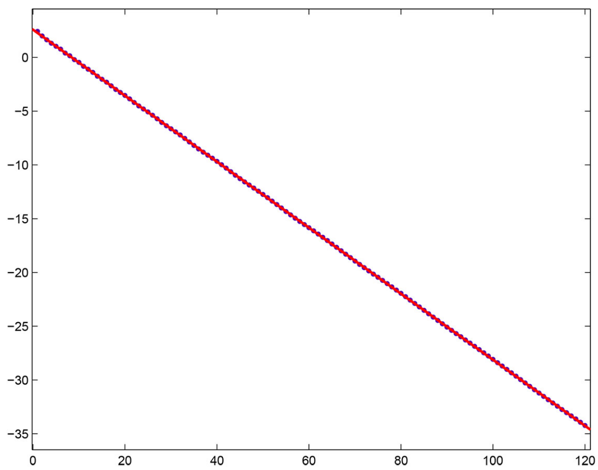

Figure 15

For a linear least-squares fit of the curve, $p_1 = -0.307$ with a 95% confidence interval of $(-0.3072, -0.3069)$, $p_2 = 2.58$ with a 95% confidence interval of (2.569, 2.59), SSE=0.09705, R-squared=1, and RMSE=0.02865. For the first twenty zeta function zeros, the slope is -0.307 and the y-intercepts are 2.58, 2.975, 3.15, 3.345, 3.427, 3.557, 3.641, 3.699, 3.801, 3.84, 3.902, 3.965, 4.015, 4.04, 4.107, 4.137, 4.173, 4.21, 4.258, and 4.278. A plot of the imaginary parts of the corresponding zeta function zeros versus these values is

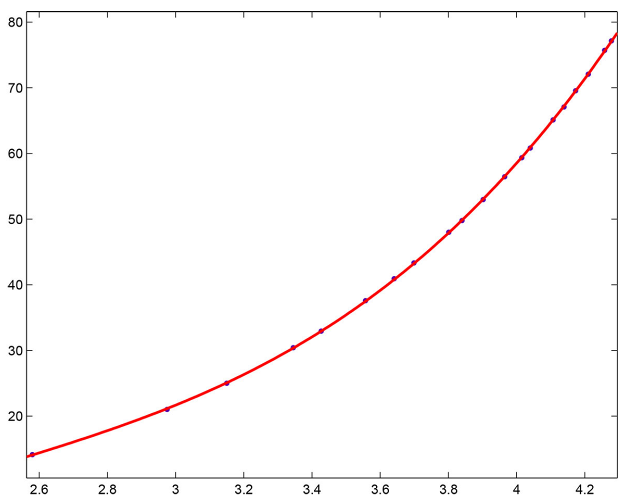

Figure 16

For a cubic least-squares fit of the curve, $p_1 = 5.966$ with a 95% confidence interval of (5.552, 6.342), $p_2 = -43.93$ with a 95% confidence interval of $(-48.24, -39.61)$,

$p_3 = 123.6$ with a 95% confidence interval of (108.8, 138.4), $p_4 = -114.9$ with a 95% confidence interval of (-131.5, -98.19), SSE=0.1316, R-squared=1, and RMSE=0.0907.

4. METHODS

```
#include <math.h>
#include <stdio.h>
// G(z-n)
unsigned int max=10000000;
double s=0.0001;
double t=600.0;
//double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
//double t=52.97032147771446;
//double t=56.44624769706339;
//double t=59.34704400260235;
//double t=60.83177852460981;
//double t=65.11254404808160;
//double t=67.07981052949417;
//double t=69.54640171117399;
//double t=72.06715767448191;
//double t=75.70469069908393;
//double t=77.14484006887480;
unsigned int out=6; // out=7 for imaginary components
unsigned int n=1;
void main() {
unsigned int x,i,savex;
double temp1,temps,tempt,prods,prods1,prods2,a,b,c,d,e,f,olds,oldt,s1;
double oldolds,oldoldt,temp,del,savolds,savoldt;
```

```
FILE *Outfp;
Outfp = fopen(”test3yc.dat”,”w”);
for (i=1; i<=15; i++) {
    if (n==(n/2)*2)
        t=600.0;
    else
        t=500.0;
    del=0.0;
    olds=0.0;
    oldolds=0.0;
    oldt=0.0;
    oldoldt=0.0;
    prods=1.0;
    prods1=1.0;
    prods2=1.0;
    for (x=1; x<=max; x++) {
        s1=s-(double)n-1.0;
        prods=prods*(double)x/((double)x+s1);
        if (s>=0.0)
            temp1=pow((double)(x+1),s1);
        else {
            temp1=pow((double)(x+1),-s1);
            temp1=1.0/temp1;
            }
        temps=temp1*(cos(t*log(x+1)));
        tempt=temp1*(sin(t*log(x+1)));
        a=prods*temps-tempt;
        b=prods*tempt+temps;
        s1=-(s-1.0);
        prods1=prods1*(double)x/((double)x+s1);
        if (s1>=0.0)
            temp1=pow((double)(x+1),s1);
        else {
            temp1=pow((double)(x+1),-s1);
            temp1=1.0/temp1;
            }
        temps=temp1*(cos(t*log(x+1)));
        tempt=temp1*(sin(t*log(x+1)));
```

```
c=prods1*temps-tempt;
d=prods1*tempt+temps;
s1=(double)n+1.0,-(s-1.0);
prods2=prods2*(double)x/((double)x+s1);
if (s1>=0.0)
    temp1=pow((double)(x+1),s1);
else {
    temp1=pow((double)(x+1),-s1);
    temp1=1.0/temp1;
    }
temps=temp1*(cos(t*log(x+1)));
tempt=temp1*(sin(t*log(x+1)));
e=prods2*temps-tempt;
f=prods2*tempt+temps;
temp=e*e+f*f;
temps=(c*e+d*f)/temp;
tempt=(c*f-d*e)/temp;
tempt=-tempt;
if (n!=(n/2)*2) {
    temps=a-temps;
    tempt=b-temps;
    }
else {
    temps=a+temps;
    tempt=b+temps;
    }
if ((out==6)&&((oldolds<olds)&&(olds>temps))) {
    savolds=olds;
    savex=x;
    }
if ((out==6)&&((oldolds>olds)&&(olds<temps)&&(olds<0.0))) {
    olds=-olds;
    savolds=olds;
    savex=x;
    }
if ((out==7)&&((oldoldt<oldt)&&(oldt>tempt))) {
    savoldt=oldt;
    savex=x;
```

```
            }
        if ((out==7)&&((oldoldt>oldt)&&(oldt<tempt)&&(oldt<0.0))) {
            oldt=-oldt;
            savoldt=oldt;
            savex=x;
            }
        oldolds=olds;
        olds=temps;
        oldoldt=oldt;
        oldt=tempt;
        }
    if (out==6) {
        printf(" %.8llf %d \n",savolds,savex);
        fprintf(Outfp," %.16llf,\n",savolds);
        }
    if (out==7) {
        printf(" %.8llf %d \n",savoldt,savex);
        fprintf(Outfp," %.16llf, \n",savoldt);
        }
    n=n+1;
    }
fclose(Outfp);
return;

#include <math.h>
#include <stdio.h>
// incomplete Pi function
unsigned int max=120;
double s=0.50;
double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
```

```
//double t=48.00515088116716;
//double t=49.77383247767230;
//double t=52.97032147771446;
//double t=56.44624769706339;
//double t=59.34704400260235;
//double t=60.83177852460981;
//double t=65.11254404808160;
//double t=67.07981052949417;
//double t=69.54640171117399;
//double t=72.06715767448191;
//double t=75.70469069908393;
//double t=77.14484006887480;
unsigned int dec=1;
void main() {
unsigned int x,n;
double temp1,temps,tempt,prods,a,b,c,d,g,h,s1,temp;
FILE *Outfp;
Outfp = fopen("test3yd.dat","w");
prods=1.0;
n=1;
for (x=1; x<=max; x++) {
    s1=s-(double)x-1.0;
    prods=prods*(double)x/((double)x+s1);
    if (s>=0.0)
        temp1=pow((double)(x+1),s1);
    else {
        temp1=pow((double)(x+1),-s1);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x+1)));
    tempt=temp1*(sin(t*log(x+1)));
    a=prods*temps-tempt;
    b=prods*tempt+temps;
    c=a*(s-1.0)+b*t;
    d=a*t+b*(s-1.0);
    if (x==(x/dec)*dec) {
        s1=s-1.0;
        if (s1>=0.0)
```

```
            temp1=pow((double)n,s1);
        else {
            temp1=pow((double)n,-s1);
            temp1=1.0/temp1;
            }
        g=temp1*(cos(t*log(x)));
        h=temp1*(sin(t*log(x)));
        temp=1.0/exp((double)n);
        temps=g*temp;
        tempt=h*temp;
        temps=temps+c;
        tempt=tempt+d;
        fprintf(Outfp," %.16llf %.16llf \n",temps,tempt);
        n=n+1;
        }
    }
fclose(Outfp);
return;
}
```

REFERENCES

[1] H. M. Edwards, *Riemann's Zeta Function*, Dover, (1974)