

The Logarithm of the Riemann Zeta Function and a Variant of the Functional Equation

Darrell Cox ¹

¹*Department of Mathematics,
Grayson County College, United States*

Abstract

A variant of the functional equation of the Riemann zeta function is derived. This variant maps the square of the zeta function to the logarithm of the zeta function using an incomplete $\Pi(s)$ function. Similar functions map higher powers of the zeta function to the logarithm of the zeta function. A variant of the logarithm of the zeta function is derived.

Keywords: Riemann zeta function, logarithmic spirals, functional equations

1. INTRODUCTION

Equation (5) in section 1.6 of Edwards' [1] book is

$$\Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s) = \Pi\left(\frac{1-s}{2} - 1\right)\pi^{-(1-s)/2}\zeta(1-s). \quad (1)$$

This relationship between $\zeta(s)$ and $\zeta(1-s)$ is known as the functional equation of the zeta function.

The function $\Pi(s/2) - 1)\pi^{-s/2}\zeta(s)$ which occurs in the above equation has poles at $s = 0$ and $s = 1$. Riemann multiplies it by $s(s-1)/2$ and defines

$$\xi(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s). \quad (2)$$

(Equation (1) in section 1.8 of Edwards' book.) Then $\xi(s)$ is an analytic function of s which is defined for all values of s and the functional equation of the zeta function is equivalent to $\xi(s) = \xi(1-s)$.

Equation (4) in section 1.5 of Edwards' book is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (3)$$

This is Dirichlet's function for real values of s greater than 1.

2. A VARIANT OF $\zeta(s)$

Equation (3) in section 1.3 of Edwards' book is

$$\Pi(s) = \lim_{N \rightarrow \infty} \frac{1 \cdot 2 \cdots N}{(s+1)(s+2) \cdots (s+N)} (N+1)^s \quad (4)$$

This equation is valid for all s in the halfplane $\text{Re } s > -1$. Let $\Pi_1(s)$ denote

$$\lim_{N \rightarrow \infty} \frac{1 \cdot 2 \cdots N}{(\Re s + 1)(\Re s + 2) \cdots (\Re s + N)} (N+1)^s \quad (5)$$

Let $\zeta_1(s, N)$ denote $\Pi_1(s/2, N)\Pi_1(s, N)(s-1)\zeta(s, N)^2$. A plot of $\zeta_1(s, N)$ for the tenth non-trivial zeta function zero ($s = (0.5, 49.77383247767230\dots)$) and $N \leq 2000$ is

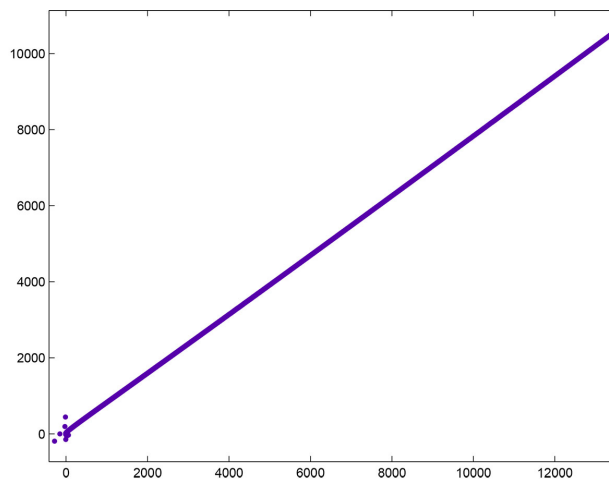


Figure 1

This curve is a line (neglecting the first few points) if and only if s is a zeta function zero. A plot of the curve when $s = (0.49, 49.77383247767230\dots)$ is

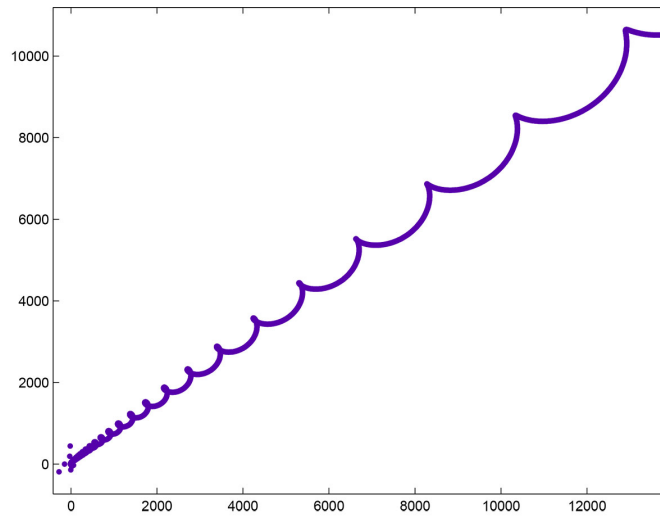


Figure 2

A plot of the curve when $s = (.50, 49.0)$ is

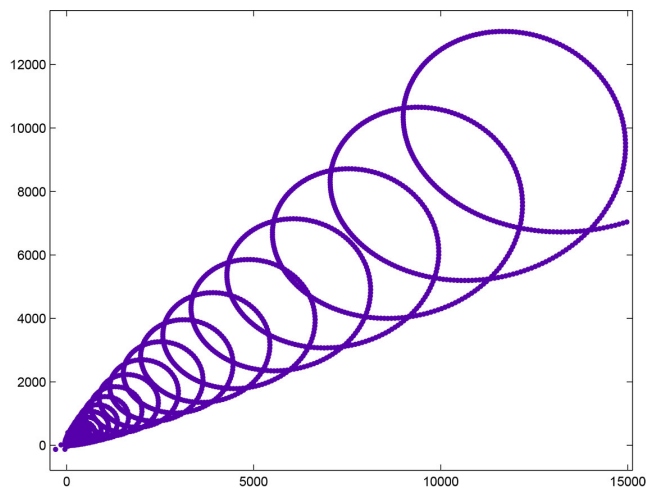


Figure 3

3. THE TENTH ZETA FUNCTION ZERO

Let $\tilde{\zeta}_2(s, N)$ denote $(\Pi_1(s/2, N) + N^{s/2}e^{-N})(\Pi_1(s, N) + N^s e^{-N})(s - 1)\zeta(s, N)^2$. Incomplete $\Pi(s)$ values are used. C code is given in the Methods section. A plot of

$\zeta_2(s, N)$ for the tenth zeta function zero and $N \leq 2000$ is

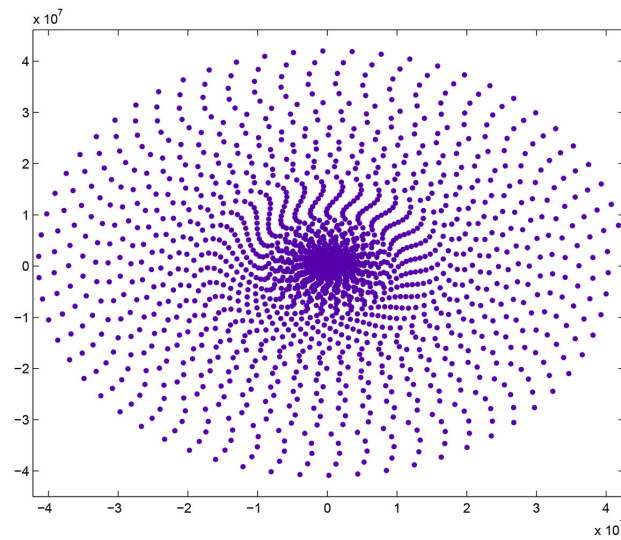


Figure 4

A plot of the real part versus N is

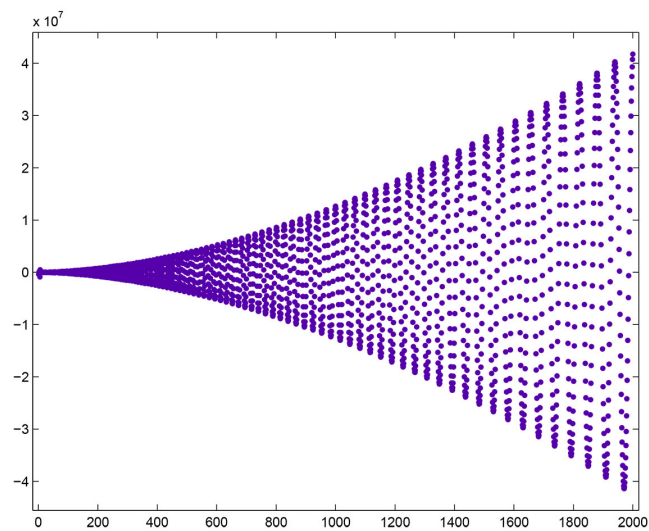


Figure 5

A plot of the logarithms of the N values where the absolute values of the amplitude crest is

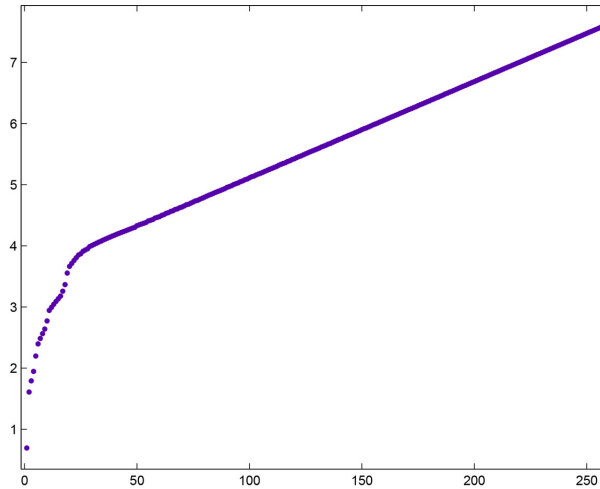


Figure 6

For a linear least-squares fit of the curve (disregarding the first fifty points), $p_1 = 0.01576$ with a 95% confidence interval of (0.01576, 0.01577), $p_2 = 3.594$ with a 95% confidence interval of (3.594, 3.595), SSE=0.0004629, R-squared=1, and RMSE=0.00151.

A plot of the logarithms of the absolute value of the amplitude at these N values is

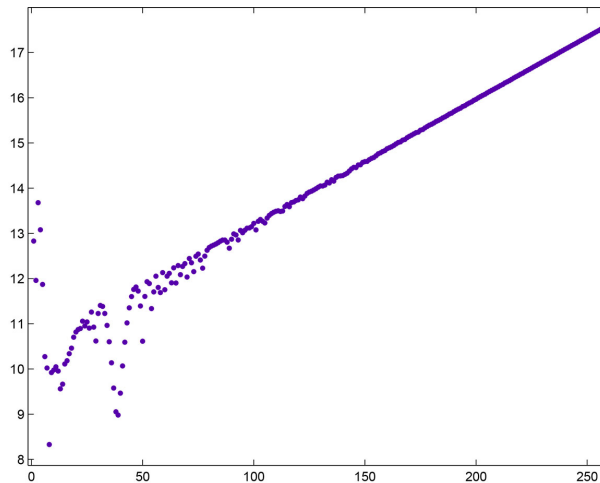


Figure 7

For a linear least-squares fit of the curve (disregarding the first one hundred and fifty points), $p_1 = 0.02755$ with a 95% confidence interval of (0.02752, 0.02757), $p_2 = 10.22$ with a 95% confidence interval of (10.21, 10.22), SSE=0.001406, R-squared=1, and RMSE=0.003695.

$\log \zeta(s) = \sum_p - \log(1 - p^{-s})$ where p is a prime. Here, partial sums of the logarithm are computed using Dirichlet's definition of the zeta function. C code is given in the

Methods section. A plot of the logarithm of the zeta function for the tenth zeta function zero and $N \leq 2000$ is

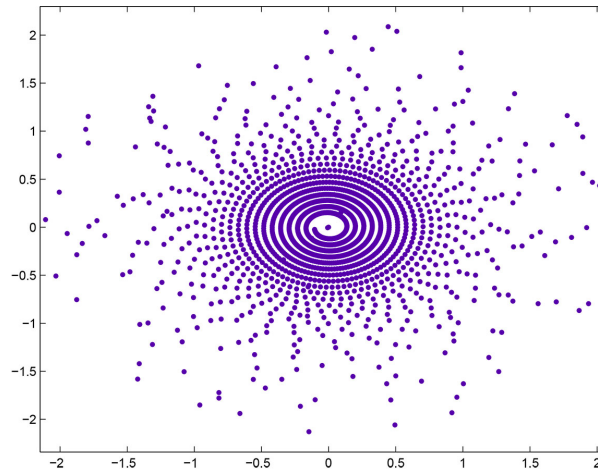


Figure 8

Unlike $\zeta_2(s, N)$, the curve converges. A plot of the logarithms of the N values where the absolute values of the amplitude crest for the real part is

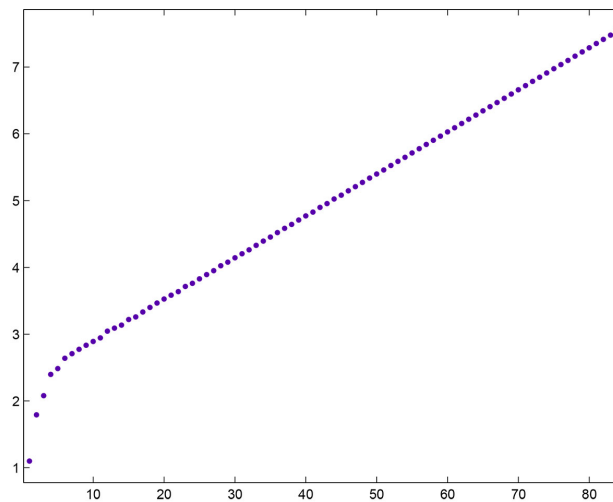


Figure 9

For a linear least-squares fit of the curve (neglecting the first seventeen points), $p_1 = 0.06285$ with a 95% confidence interval of (0.0628, .0629), $p_2 = 2.259$ with a 95% confidence interval of (2.256, 2.261), $SSE=0.001037$, $R\text{-squared}=1$, and $RMSE=0.003965$. One fourth of this slope (0.0157125) almost equals the corresponding slope for $\zeta_2(s, N)$ (0.01576).

A plot of the absolute values of the amplitudes at these N values is

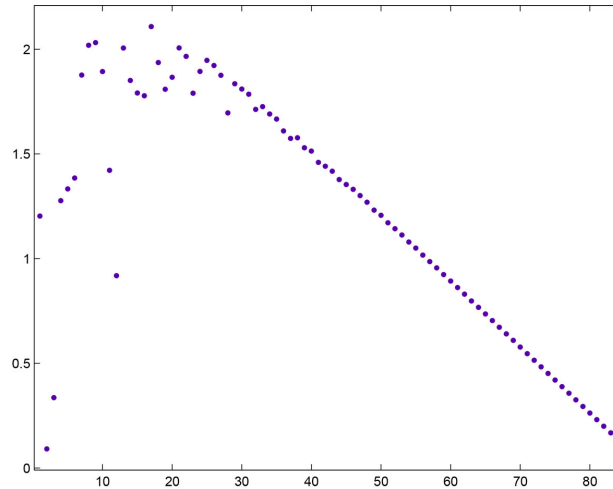


Figure 10

For a linear least-squares fit of the curve (neglecting the first forty-five points), $p_1 = -0.03137$ with a 95% confidence interval of $(-0.03143, -0.0313)$, $p_2 = 2.774$ with a 95% confidence interval of $(2.769, 2.778)$, $SSE=0.0001308$, $R\text{-squared}=1$, and $RMSE=0.001962$.

4. THE THIRTYTH ZETA FUNCTION ZERO

A plot of $\xi_2(s, N)$ for the thirtyth zeta function zero ($s = (0.5, 101.31785100573138\dots)$) and $N \leq 2000$ is

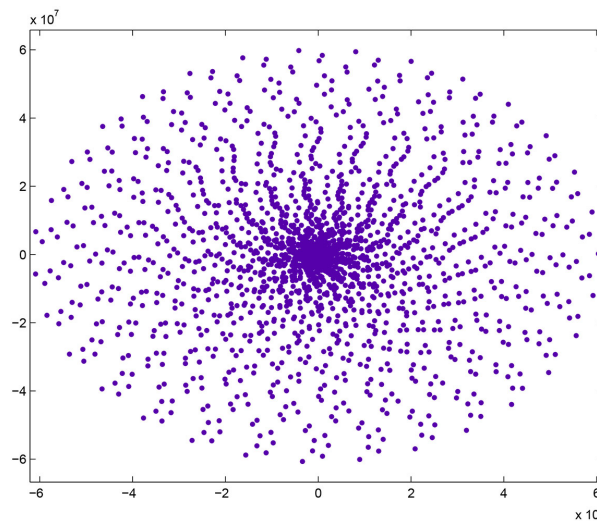


Figure 11

A plot of the logarithms of the N values where the absolute values of the amplitude crest for the real part is

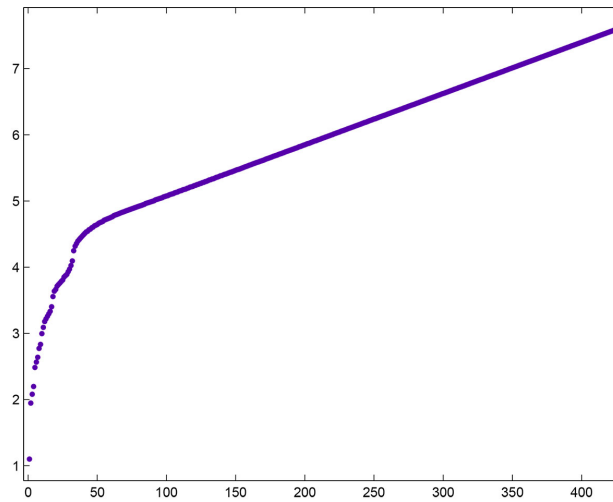


Figure 12

For a linear least-squares fit of the curve (neglecting the first seventy-five points), $p_1 = 0.00774$ with a 95% confidence interval of (0.007745, 0.007747), $p_2 = 4.301$ with a 95% confidence interval of (4.301, 4.301), SSE=0.0002868, R-squared=1, and RMSE=0.0009052.

A plot of the logarithms of the absolute value of the amplitude at these N values is

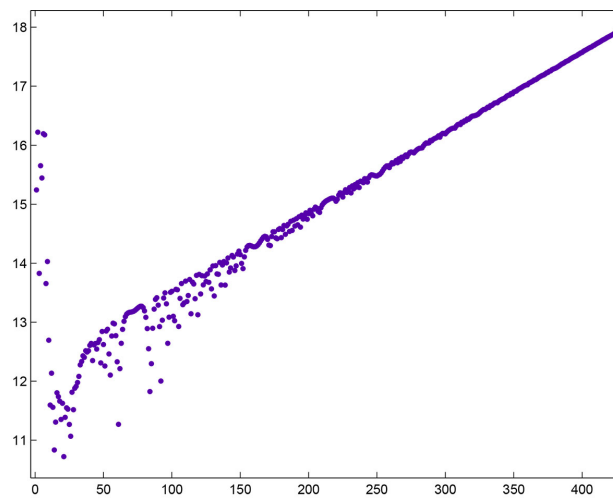


Figure 13

For a linear least-squares fit of the curve (neglecting the first three hundred points), $p_1 = 0.01361$ with a 95% confidence interval of (0.01358, 0.01364), $p_2 = 12.13$ with a 95% confidence interval of (12.12, 12.14), SSE=0.004358, R-squared=0.0000, and RMSE=0.005905.

A plot of the logarithm of the thirtyth zeta function zero and $N \leq 2000$ is

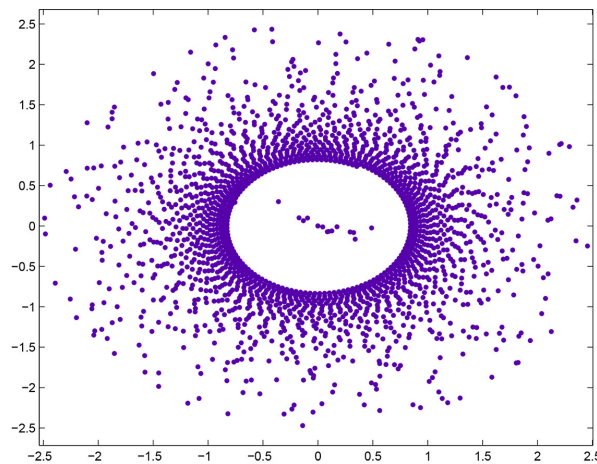


Figure 14

A plot of the logarithm of the N values where the absolute value of the amplitude crest for the real part is

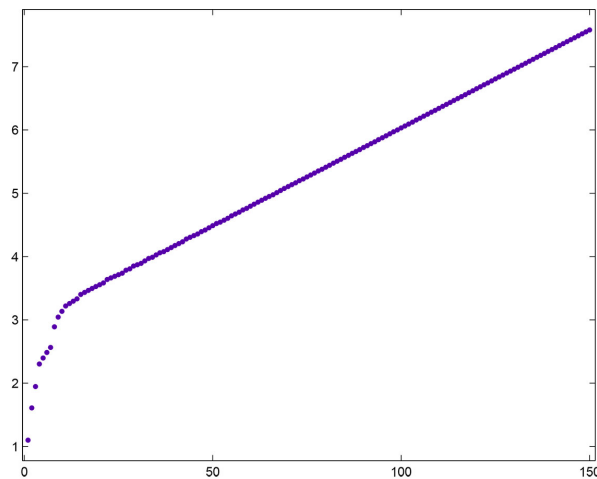


Figure 15

For a linear least-squares fit of the curve (neglecting the first thirty-nine points), $p_1 = 0.03095$ with a 95% confidence interval of (0.03094, 0.03097), $p_2 =$

2.937 with a 95% confidence interval of (2.936, 2.938), SSE=0.0004, R-squared=1, and RMSE=0.001939. One fourth of this slope (0.0077375) almost equals the corresponding slope for $\zeta_2(s, N)$ (0.007746) (the same as for the tenth zeta function zero).

A plot of the absolute values of the amplitudes at these N values is

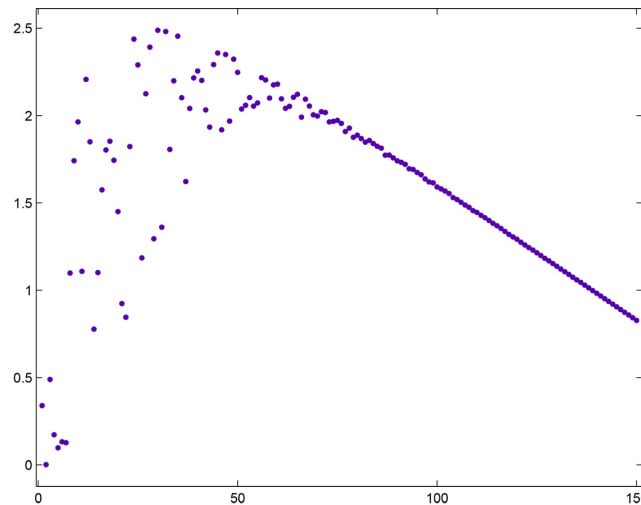


Figure 16

For a linear least-squares fit of the curve (neglecting the first hundred points), $p_1 = -0.01537$ with a 95% confidence interval of $(-0.0154, -0.01534)$, $p_2 = 3.134$ with a 95% confidence interval of $(3.13, 3.139)$, SSE=0.000149, R-squared=0.9999, and RMSE=0.001744. This slope divided by the corresponding slope for the tenth zeta function zero (-0.03137) is about 0.4900. The imaginary part of the tenth zeta function zero divided by the imaginary part of the thirtyth zeta function zero is about 0.4913.

5. LINEAR LEAST-SQUARES FITS VERSUS THE SQUARE ROOTS OF IMAGINARY PARTS OF ZETA FUNCTION ZEROS

The following curve pertains to the logarithms of the N values where the absolute values of the amplitude of $\zeta_2(s, N)$ crest. The slopes of the linear least-squares fits for the first twenty zeta function zeros are plotted against the square roots of the imaginary part of the zeta function zeros.

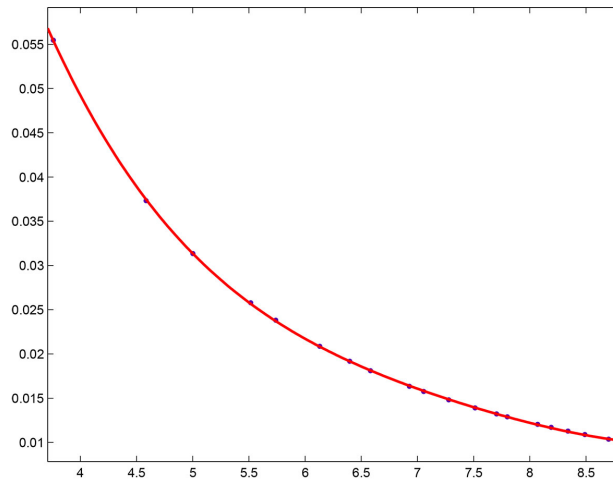


Figure 17

For a quartic least-squares fit of the curve, $p_1 = 8.439 \cdot 10^{-5}$ with a 95% confidence interval of $(7.458 \cdot 10^{-5}, 9.42 \cdot 10^{-5})$, $p_2 = -0.002563$ with a 95% confidence interval of $(-0.002812, -0.002311)$, $p_3 = 0.02982$ with a 95% confidence interval of $(0.02751, 0.03213)$, $p_4 = -0.1611$ with a 95% confidence interval of $(-0.1704, -0.1517)$, $p_5 = 0.3588$ with a 95% confidence interval of $(0.3451, 0.3724)$, $SSE=7.878 \cdot 10^{-8}$, $R\text{-squared}=1$, and $RMSE=7.247 \cdot 10^{-5}$.

A plot of the corresponding y -intercepts versus the imaginary parts of the zeta function zeros is

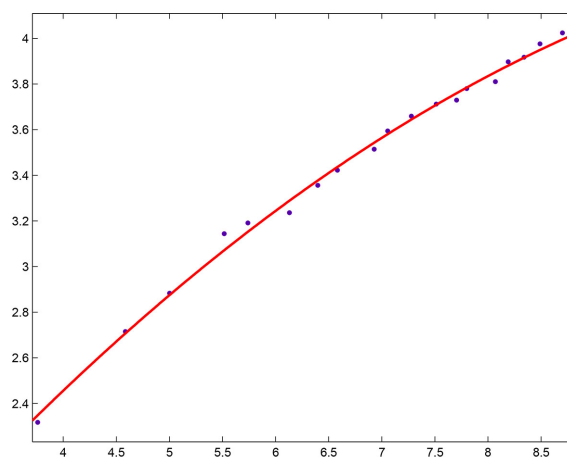


Figure 18

The curve is roughly quadratic.

A plot of the corresponding amplitudes versus the imaginary parts of the zeta function

zeros is

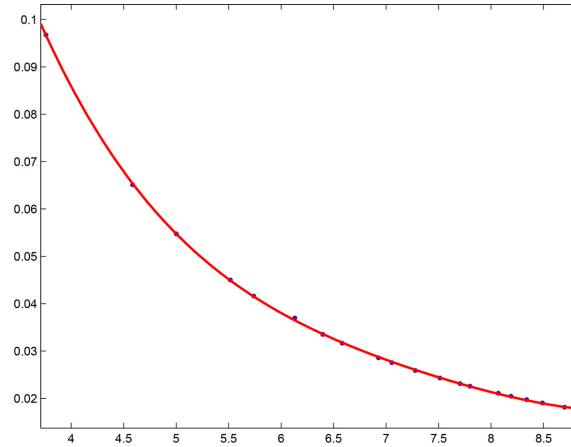


Figure 19

For a quartic least-squares fit of the curve, $p_1 = 0.0001552$ with a 95% confidence interval of $(0.0001298, 0.0001807)$, $p_2 = -0.004669$ with a 95% confidence interval of $(-0.005315, -0.004023)$, $p_3 = 0.05378$ with a 95% confidence interval of $(0.04777, 0.05978)$, $p_4 = -0.2876$ with a 95% confidence interval of $(-0.3117, -0.2634)$, $p_5 = 0.6348$ with a 95% confidence interval of $(0.5994, 0.6703)$, $SSE=5.317 \cdot 10^{-7}$, $R\text{-squared}=0.9999$, and $RMSE=0.0001883$.

The following curve pertains to the logarithm of the N values where the absolute values of the amplitude of the logarithm of the zeta function crest. The slopes of the linear least-squares fits for the first twenty zeta function zeros are plotted against the square roots of the imaginary part of the zeta function zeros.

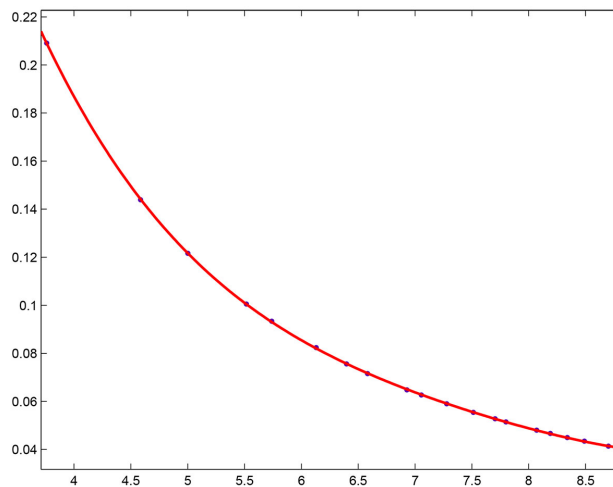


Figure 20

For a quartic least-squares fit of the curve, $p_1 = 0.0002736$ with a 95% confidence interval of (0.0002485, 0.0002986), $p_2 = -0.008414$ with a 95% confidence interval of (-0.009049, -0.007778), $p_3 = 0.09935$ with a 95% confidence interval of (0.09344, 0.1053), $p_4 = -0.547$ with a 95% confidence interval of (-0.5708, -0.05232), $p_5 = 1.254$ with a 95% confidence interval of (1.219, 1.289), $SSE=5.144 \cdot 10^{-7}$, $SSE=1$, and $RMSE=0.0001852$.

6. A VARIANT OF THE LOGARITHM OF THE ZETA FUNCTION

$\Pi(z) = \Gamma(z + 1) = z\Gamma(z) = \int_0^\infty e^{-t}t^z dt$ so that $\Pi(n) = n!$ for every non-negative integer n . The reflection formula for the $\Pi(s)$ function is $\Pi(z) = z\Pi(z - 1)$. A plot of $\Pi_1(z) - z\Pi_1(z - 1)$ (denoted by $R(z)$ or $R(z, N)$) for the first zeta function zero and $N \leq 2000$ is

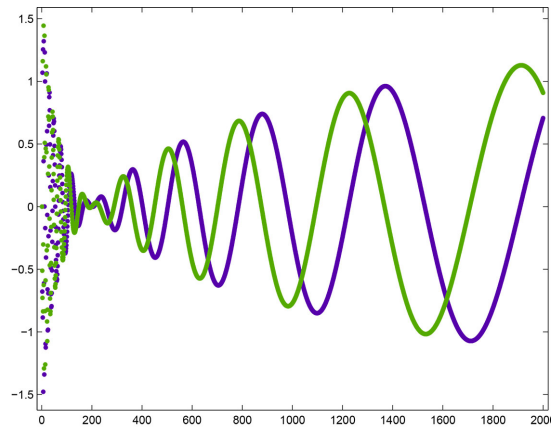


Figure 21

A plot of the logarithm of the zeta function for the first zeta function zero and $N \leq 2000$ is

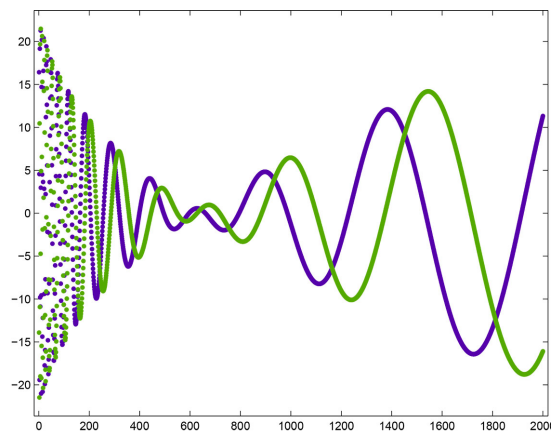


Figure 22

The logarithms of the wavelengths are almost the same. $R(z, N)$ is then a variant of the logarithm of the zeta function. See Cox [2] for details.

7. HIGHER POWERS OF THE ZETA FUNCTION

Let $\tilde{\zeta}_3(s, N)$ denote $\Pi_1(s/3, N)\Pi_1(s/2, N)\Pi_1(s, N)(s-1)\zeta(s, N)^3$. A plot of this function for $s = (0.50, 49.7735)$ (almost equal to the tenth zeta function zero) and $N \leq 2000$ is

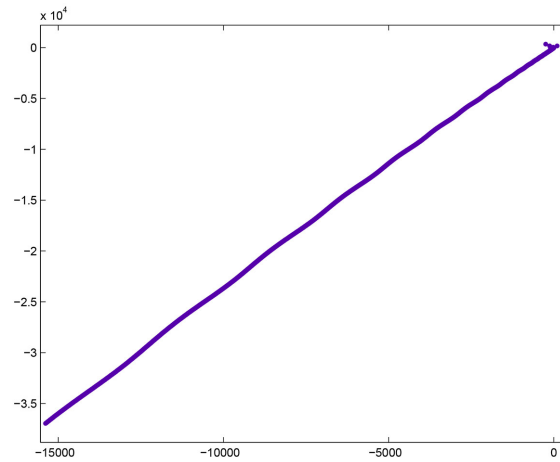


Figure 23

Let $\tilde{\zeta}_4(s, N)$ denote $\Pi_1(s/4, N)\Pi_1(s/3, N)\Pi_1(s/2, N)\Pi_1(s, N)(s-1)\zeta(s, N)^4$. This function is more accurate than $\tilde{\zeta}_3(s, N)$.

Let $\tilde{\zeta}_5(s, N)$ denote $(\Pi_1(s/4, N) + N^{s/4}e^{-N})(\Pi_1(s/3, N) + N^{s/3}e^{-N})(\Pi_1(s/2, N) + N^{s/2}e^{-N})(\Pi_1(s, N) + N^s e^{-N})(s-1)\zeta(s, N)^4$. A plot of $\tilde{\zeta}_5(s, N)$ versus N for the thirtyth zeta function zero and $N \leq 2000$ is

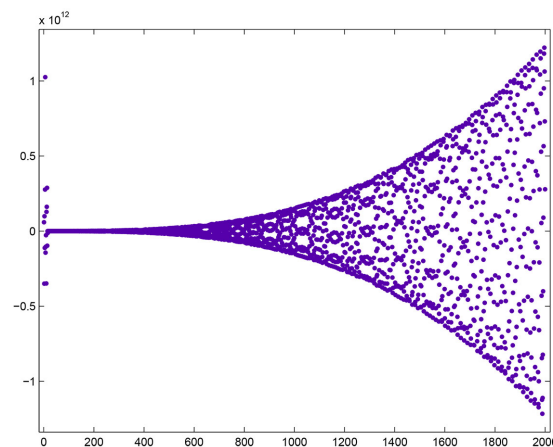


Figure 24

A plot of the logarithms of the N values where the absolute values of the amplitude crest for the real part is

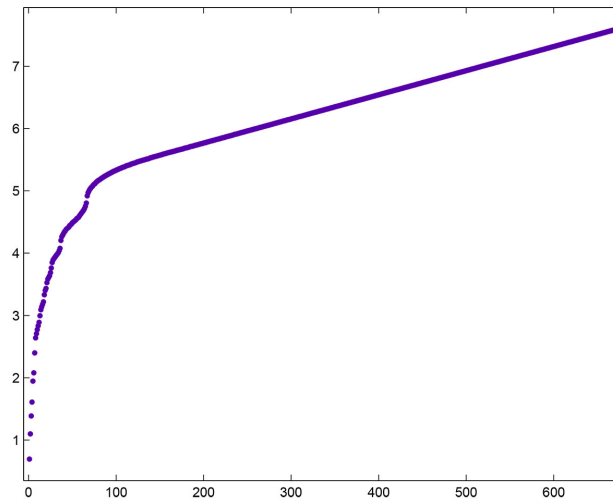


Figure 25

For a linear least-squares fit of the curve (disregarding the first hundred and fifty points), $p_1 = 0.003874$ with a 95% confidence interval of (0.003874, 0.003874), $p_2 = 4.993$ with a 95% confidence interval of (4.993, 4.993), SSE=0.000131, R-squared=1, and RMSE=0.000559. The slope is about half (0.500129) that of $\zeta_2(s, N)$ (0.007746).

A plot of $R(z, N)$ for the thirtyth zeta function zero and $N \leq 2000$ is

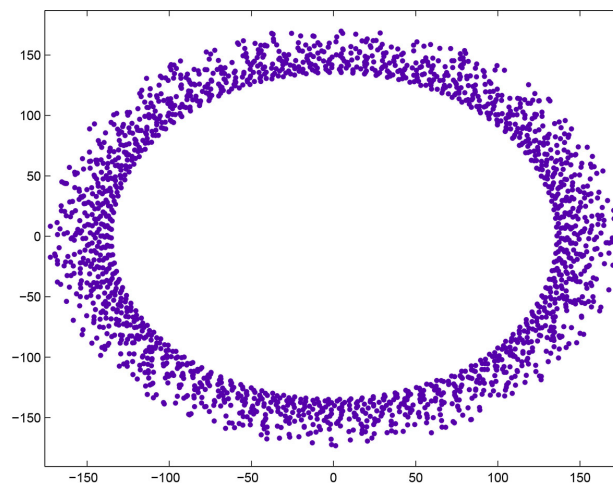


Figure 26

A plot of the logarithms of the absolute value of the amplitude where the peaks crest is

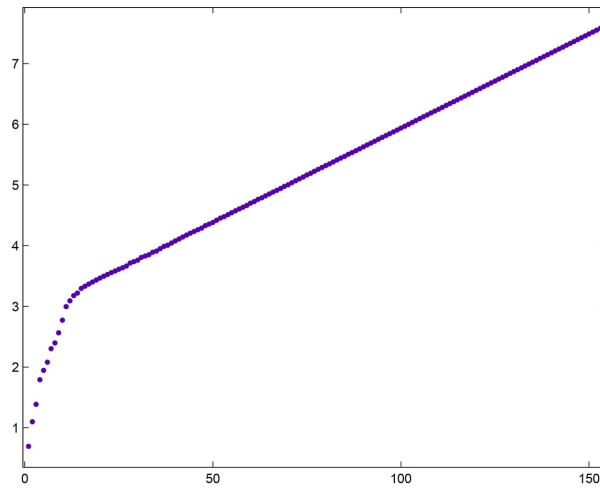


Figure 27

For a linear least-squares fit of the curve (neglecting the first twenty points), $p_1 = 0.031$ with a 95% confidence interval of (0.03099, 0.03102), $p_2 = 2.837$ with a 95% confidence interval of (2.836, 2.839), $SSE=0.001137$, $R\text{-squared}=1$, and $RMSE=0.002935$. The slope is about eight (8.00207) times that of $\zeta_5(s, N)$. For the tenth zeta function zero, the ratio is about 8.016 (0.06319/0.009883).

A plot of the N values where the absolute values of the amplitude crest is

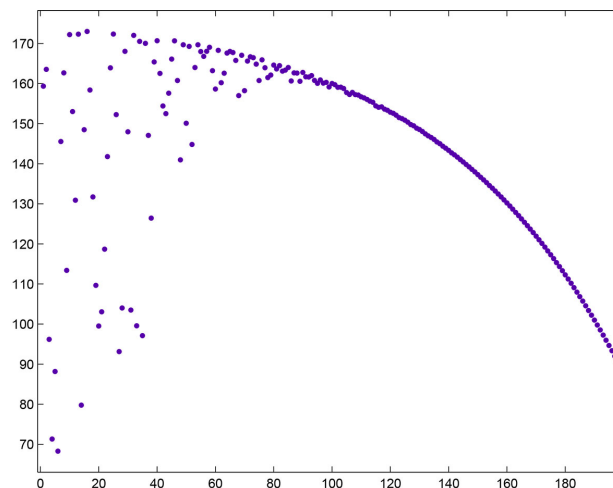


Figure 28

A plot of the curve disregarding the first one hundred and twenty points is

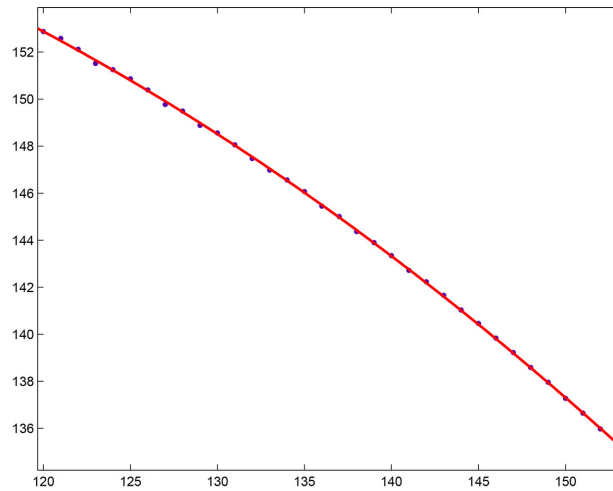


Figure 29

For a quadratic least-squares fit of the curve, $p_1 = -0.004141$ with a 95% confidence interval of $(-0.004376, -0.0039087)$, $p_2 = 0.5991$ with a 95% confidence interval of $(0.535, 0.6633)$, $p_3 = 140.6$ with a 95% confidence interval of $(136.3, 145)$, $SSE=0.1032$, $R\text{-squared}=0.9999$, and $RMSE=0.5771$.

8. METHODS

```
#include <math.h>
#include <stdio.h>
// xi function, another variant
// Pi(s/2)*Pi(s)*(s-1)*zeta(s)*zeta(s)
unsigned int max=2000;
double s=0.50;
//double t=49.7;
double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
```

```

//double t=52.97032147771446;
//double t=56.44624769706339;
//double t=59.34704400260235;
//double t=60.83177852460981;
//double t=65.11254404808160;
//double t=67.07981052949417;
//double t=69.54640171117399;
//double t=72.06715767448191;
//double t=75.70469069908393;
//double t=77.14484006887480;
unsigned int out=1; // 2 for top, 3 for left, 4 for bottom, 5 for right
unsigned int mod=1; // set for incomplete
double pi=3.14159265359;
void main() {
unsigned int temp,x;
double temp1,temps,temp1,prods,a,b,c,d,olds,oldt,sums,sumt;
double newprod,newtemp1,newtemps,newtempt,newa,newb;
double g,h,j,k,temp2,oldolds,oldoldt,tsums,tsumt,s1;
FILE *Outfp;
Outfp = fopen("transe2y.dat","w");
oldolds=0.0;
olds=0.0;
oldoldt=0.0;
oldt=0.0;
prods=1.0;
newprod=1.0;
sums=0.0;
sumt=0.0;
for (x=1; x<=max; x++) {
    temp=x;
    s1=s/2.0;
    prods=prods*(double)temp/((double)temp+s1);
    if (s1>=0.0)
        temp1=pow((double)(x+1),s1);
    else {
        temp1=pow((double)(x+1),-s1);
        temp1=1.0/temp1;
    }
}

```

```

temps=temp1*(cos(t*log(x+1)));
tempt=temp1*(sin(t*log(x+1)));
a=prods*temps-tempt;
b=prods*tempt+temps;
if (mod!=0) {
    j=a*s1+b*t;
    k=a*t-b*s1;
    if (s1>=0.0)
        temp1=pow((double)x,s1);
    else {
        temp1=pow((double)x,-s1);
        temp1=1.0/temp1;
    }
    g=temp1*(cos(t*log(x)));
    h=temp1*(sin(t*log(x)));
    temp2=1.0/exp((double)x);
    temps=g*temp2;
    tempt=h*temp2;
    a=temps+j;
    b=tempt+k;
}
newprod=newprod*(double)temp/((double)temp+s);
if (s>=0.0)
    newtemp1=pow((double)(x+1),s);
else {
    newtemp1=pow((double)(x+1),-s);
    newtemp1=1.0/newtemp1;
}
newtemps=newtemp1*(cos(t*log(x+1)));
newtempt=newtemp1*(sin(t*log(x+1)));
newa=newprod*newtemps-newtempt;
newb=newprod*newtempt+newtemps;
if (mod!=0) {
    j=newa*s+newb*t;
    k=newa*t-newb*s;
    if (s>=0.0)
        temp1=pow((double)x,s);
    else {

```

```

        temp1=pow((double)x,-s);
        temp1=1.0/temp1;
    }
    g=temp1*(cos(t*log(x)));
    h=temp1*(sin(t*log(x)));
    temp2=1.0/exp((double)x);
    temps=g*temp2;
    tempt=h*temp2;
    newa=temps+j;
    newb=tempt+k;
}
if (s>=0.0)
    temp1=pow((double)x,s);
else {
    temp1=pow((double)x,-s);
    temp1=1.0/temp1;
}
temps=temp1*(cos(t*log(x)));
tempt=temp1*(sin(t*log(x)));
temp1=temps*temps+tempt*tempt;
c=temps/temp1;
d=tempt/temp1;
sums=sums+c;
sumt=sumt-d;
tsums=sums;
tsumt=sumt;
temp2=tsums*tsums-tsumt*tsumt;
tsumt=2.0*tsums*tsumt;
tsums=temp2;
c=a*tsums-b*tsumt;
d=a*tsumt+b*tsums;
temps=c*(s-1.0)-d*t;
tempt=c*t+d*(s-1.0);
a=temps*newa-tempt*newb;
b=temps*newb+tempt*newa;
temps=a;
tempt=b;
if (out==1)

```

```

    fprintf(Outfp," %.10lf %.10lf \n",temps,tempt);
    if ((out==2)&&((olds>0.0)&&(temps<0.0)))
        fprintf(Outfp," %.10lf \n",log(x));
    if ((out==3)&&((oldt>0.0)&&(tempt<0.0)))
        fprintf(Outfp," %.10lf \n",log(x));
    if ((out==4)&&((olds<0.0)&&(temps>0.0)))
        fprintf(Outfp," %.10lf \n",log(x));
    if ((out==5)&&((oldt<0.0)&&(tempt>0.0)))
        fprintf(Outfp," %.10lf \n",log(x));
    if ((out==6)&&(oldolds<olds)&&(olds>temps)&&(olds>0.0))
        fprintf(Outfp," %.16lf, %d \n",olds,x);
    if ((out==6)&&((oldolds>olds)&&(olds<temps)&&(olds<0.0)))
        fprintf(Outfp," %.16lf, %d \n",-olds,x);
    if ((out==7)&&((oldoldt<oldt)&&(oldt>tempt)))
        fprintf(Outfp," %.16lf, %d \n",oldt,x);
    if ((out==7)&&((oldoldt>oldt)&&(oldt<tempt)&&(oldt<0.0)))
        fprintf(Outfp," %.16lf, %d \n",-oldt,x);
    oldolds=olds;
    olds=temps;
    oldoldt=oldt;
    oldt=tempt;
}
fclose(Outfp);
return;
}

```

```

#include <math.h>
#include <stdio.h>
// log of zeta function
unsigned int max=2000;
double a=0.50;
double b=14.13472514173470;
//double b=21.02203963877156;
//double b=25.01085758014569;
//double b=30.42487612585951;
//double b=32.93506158773919;
//double b=37.58617815882568;

```

```

//double b=40.91871901214750;
//double b=43.32707328091500;
//double b=48.00515088116716;
//double b=49.77383247767230;
unsigned int out=6; // usually 1
void main() {
unsigned int x;
double sumr,sumi,R,I,temp1,olds,oldt,temps,tempt,oldolds,oldoldt;
double r,theta;
FILE *Outfp;
Outfp = fopen("spiral2.dat","w");
sumr=0.0;
sumi=0.0;
oldolds=0.0;
oldoldt=0.0;
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    temp1=pow((double)x,a);
    R=temp1*cos(b*log((double)x));
    I=temp1*sin(b*log((double)x));
    temp1=R*R+I*I;
    sumr=sumr+R/temp1;
    sumi=sumi-I/temp1;
    r=sqrt(sumr*sumr+sumi*sumi);
    theta=atan2(sumi,sumr);
    temps=log(r)*cos(theta);
    tempt=log(r)*sin(theta);
    if (out==1)
        fprintf(Outfp," %.16lf, %.16lf \n",temps,tempt);
    if ((out==6)&&(oldolds<olds)&&(olds>temps)&&(olds>0.0))
        fprintf(Outfp," %.16lf, %d \n",olds,x);
    if ((out==6)&&((oldolds>olds)&&(olds<temps)&&(olds<0.0)))
        fprintf(Outfp," %.16lf, %d \n",-olds,x);
    if ((out==7)&&((oldoldt<oldt)&&(oldt>tempt)))
        fprintf(Outfp," %.16lf, %d \n",oldt,x);
    if ((out==7)&&((oldoldt>oldt)&&(oldt<tempt)&&(oldt<0.0)))
        fprintf(Outfp," %.16lf, %d \n",-oldt,x);
}
}

```

```
    oldolds=olds;
    olds=temps;
    oldoldt=oldt;
    oldt=tempt;
  }
fclose(Outfp);
return;
}
```

REFERENCES

- [1] H. M. Edwards, *Riemann's Zeta Function*, Dover, (1974)
- [2] D. K. Cox, A Variant of the Gamma Function and the Riemann Hypothesis, *Global Journal of Pure and Applied Mathematics*, ISSN 0973-1768, Volume 21, Number 4 (2025), pp. 715-739