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Isomorphisms and Structural Equivalence in Intuitionistic Fuzzy Fields

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Abstract

Field homomorphisms constitute essential mappings between fields that preserve the underlying algebraic operations of addition and multiplication. This study develops theoretical framework for intuitionistic fuzzy field homomorphisms, focusing on their structural properties and behavioral characteristics. Furthermore, the paper explores the notion of isomorphism within the context of intuitionistic fuzzy fields, providing insights into their equivalence and interrelationships.

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1. INTRODUCTION

The theory of fuzzy sets, introduced by Lotfi A. Zadeh [8] in 1965, has profoundly enriched the mathematical apparatus for representing and analyzing uncertainty, vagueness and imprecision. By allowing elements to assume degrees of membership, fuzzy set theory has fostered the development of generalized frameworks in diverse branches of mathematics, including logic, topology, and algebra. Within the algebraic

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domain, the integration of fuzzy concepts into classical structures has given rise to a wide range of fuzzy algebraic systems.

Atanassov [1] subsequently extended Zadeh's framework by introducing intuitionistic fuzzy sets, which incorporate both membership and non-membership functions, thereby providing a more expressive means of modeling uncertainty, vagueness and hesitation. Furthermore, theoretical properties of intuitionistic fuzzy sets allow the extension of classical mathematical structures into the intuitionistic fuzzy setting, bridging fundamental theory and practical applications.

Building upon these developments, Santhosh and Ramakrishnan [7] introduced the concept of intuitionistic fuzzy fields, extending the idea of fuzzy fields proposed by Gu Wenxiang and Lu Tu [3]. The present paper aims to further this line of investigation by introducing and studying the notions of homomorphism and isomorphism between intuitionistic fuzzy fields. The paper is organized as follows. Section 2 presents the necessary preliminaries and a concise overview of intuitionistic fuzzy fields. Section 3 introduces the concept of homomorphism of intuitionistic fuzzy fields and establishes several fundamental properties. Section 4 defines isomorphisms between intuitionistic fuzzy fields and explores some key structural properties. It is established that the relation "being isomorphic" is an equivalence relation on the class of all intuitionistic fuzzy fields.

2. PRELIMINARIES

This section gives necessary preliminaries regarding intuitionistic fuzzy sets and brief overview of intuitionistic fuzzy fields.

Definition 2.1. [1] An *intuitionistic fuzzy set* A in a nonempty set X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and the degree of nonmembership of the element $x \in X$ to A, respectively, and for every $x \in X : 0 \le \mu_A(x) + \nu_A(x) \le 1$.

Definition 2.2. [1] A *set of* $(\alpha, \beta) - level,$ generated by an intuitionistic fuzzy set A in X, where $\alpha, \beta \in [0,1]$ are fixed numbers such that $\alpha + \beta \le 1$, is defined as $N_{\alpha,\beta}(A) = \{x \in X : \mu_A(x) \ge \alpha, \nu_A(x) \le \beta\}$.

Definition 2.3. [4] Let f be a mapping of a set X to a set Y. If B is an intuitionistic fuzzy set in Y, then the *preimage* of B under f, denoted by $f^{-1}(B)$, is the intuitionistic fuzzy set in X defined by $f^{-1}(B) = \{\langle x, \mu_B(f(x)), \nu_B(f(x))\rangle : x \in X\}$. The *image* of an intuitionistic fuzzy set A in X under f, denoted by $f(A)$, is the intuitionistic fuzzy set in Y defined by $f(A) = \left\{ \left\langle y, \mu_{f(A)}(y), \nu_{f(A)}(y) \right\rangle : y \in Y \right\}$, where

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

$$\nu_{f(A)}(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.4. [7] If F is an intuitionistic fuzzy set in a field X satisfying the following conditions: for all $a, b \in X$,

(i) $\mu_F(a+b) \geq \min\{\mu_F(a), \mu_F(b)\}, \nu_F(a+b) \leq \max\{\nu_F(a), \nu_F(b)\}$

(ii) $\mu_F(-a) = \mu_F(a), \nu_F(-a) = \nu_F(a)$

(iii) $\mu_F(ab) \geq \min\{\mu_F(a), \mu_F(b)\}, \nu_F(ab) \leq \max\{\nu_F(a), \nu_F(b)\}$

(iv) $\mu_F(a^{-1}) = \mu_F(a), \nu_F(a^{-1}) = \nu_F(a)$ (if $a \neq 0$),

then (F, X) is called an *intuitionistic fuzzy field of* X.

Proposition 2.1. [7] *If* (F, X) *is an intuitionistic fuzzy field of* X*, then*

(i) $\mu_F(0) \geq \mu_F(a)$, $\nu_F(0) \leq \nu_F(a)$ *for all* $a \in X$

(ii) $\mu_F(1) \geq \mu_F(a)$, $\nu_F(1) \leq \nu_F(a)$ *for all* $a(\neq 0) \in X$

(iii) $\mu_F(0) \geq \mu_F(1)$, $\nu_F(0) \leq \nu_F(1)$.

The following theorem is an (α, β)-level set characterization of intuitionistic fuzzy fields.

Theorem 2.1. [7] *Let* X *be a field and* F *be an intuitionistic fuzzy set in* X. *Then* (F, X) *is an intuitionistic fuzzy field of* X *if and only if for any* $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \leq 1,\ N_{\alpha,\beta}(F) \neq \emptyset$ *implies* $N_{\alpha,\beta}(F)$ *is a subfield of* X.

3. HOMOMORPHISM BETWEEN INTUITIONISTIC FUZZY FIELDS

This section is to formally introduce the notion of homomorphism between intuitionistic fuzzy fields along with an analysis of some essential structural properties.

Consider a function $f : X \to Y$, an intuitionistic fuzzy set A in X and an element $a \in X$. It follows from Definition 2.3 that the membership and nonmembership functions of the intuitionistic fuzzy set $f(A)$ in Y satisfies $\mu_{f(A)}(f(a)) \geq \mu_A(a')$ and $\nu_{f(A)}(f(a)) \leq \nu_A(a')$ for all $a' \in A$ for which $f(a') = f(a)$. In particular, $\mu_{f(A)}(f(a)) \geq \mu_A(a)$ and $\nu_{f(A)}(f(a)) \leq \nu_A(a)$. Further, if the function f is injective, then $\mu_{f(A)}(f(a)) = \mu_A(a)$ and $\nu_{f(A)}(f(a)) = \nu_A(a)$.

A field homomorphism is either injective or maps every element to 0 [2]. Therefore, when considering field homomorphisms in the context of intuitionistic fuzzy fields, the following result is obtained.

Proposition 3.1. *Let X_1 and X_2 be two fields, $f : X_1 \to X_2$ be a homomorphism and F be an intuitionistic fuzzy field in X_1. Then, for all $a \in X_1$,*

$$\mu_{f(F)}(f(a)) = \begin{cases} \mu_F(a), & \text{if } f \text{ is injective} \\ \mu_F(0), & \text{otherwise} \end{cases}$$

and

$$\nu_{f(F)}(f(a)) = \begin{cases} \nu_F(a), & \text{if } f \text{ is injective} \\ \nu_F(0), & \text{otherwise.} \end{cases}$$

Proof. If f is not injective, then $f(a) = 0$ for all $a \in X_1$. In addition to this, $\mu_F(0) \geq \mu_F(a')$ and $\nu_F(0) \leq \nu_F(a')$ for all $a' \in X_1$. Hence the proof follows.

Theorem 3.1. *Every homomorphism of a field X_1 into another field X_2 together with an intuitionistic fuzzy field in X_2 induces an intuitionistic fuzzy field in X_1*

Proof. Let $f : X_1 \to X_2$ be a homomorphism and (F, X_2) be an intuitionistic fuzzy field of X_2.
Consider the composite functions $\mu_F \circ f : X_1 \to [0,1]$ and $\nu_F \circ f : X_1 \to [0,1]$.
Let $a, b \in X_1$.

$$\text{(i)} \qquad \begin{aligned} (\nu_F \circ f)(a+b) &= \nu_F(f(a+b)) = \nu_F(f(a) + f(b)) \\ &\leq \max\{\nu_F(f(a)), \nu_F(f(b))\} \\ &= \max\{(\nu_F \circ f)(a), (\nu_F \circ f)(b)\} \end{aligned}$$

$$\text{(ii)} \qquad \begin{aligned} (\nu_F \circ f)(-a) &= \nu_F(f(-a)) = \nu_F(-f(a)) \\ &= \nu_F(f(a)) = (\nu_F \circ f)(a) \end{aligned}$$

$$\text{(iii)} \qquad \begin{aligned}(\nu_F \circ f)(ab) &= \nu_F(f(ab)) = \nu_F(f(a)f(b)) \\ &\leq \max\{\nu_F(f(a)), \nu_F(f(b))\} \\ &= \max\{(\nu_F \circ f)(a), (\nu_F \circ f)(b)\}\end{aligned}$$

(iv) Let $a(\neq 0) \in X_1$.

If f is injective, then

$$\begin{aligned}(\nu_F \circ f)(a^{-1}) &= \nu_F\left(f(a^{-1})\right) = \nu_F\left((f(a))^{-1}\right) \\ &= \nu_F(f(a)) = (\nu_F \circ f)(a).\end{aligned}$$

If f is not injective, then $f(a^{-1}) = 0 = f(a)$ from which

$$(\nu_F \circ f)(a^{-1}) = (\nu_F \circ f)(a).$$

Analogously,

$$\begin{aligned}(\mu_F \circ f)(a+b) &\geq \min\{(\mu_F \circ f)(a), (\mu_F \circ f)(b)\} \\ (\mu_F \circ f)(-a) &= (\mu_F \circ f)(a) \\ (\mu_F \circ f)(ab) &\geq \min\{(\mu_F \circ f)(a), (\mu_F \circ f)(b)\} \\ \text{and } (\mu_F \circ f)(a^{-1}) &= (\mu_F \circ f)(a) \text{ if } a \neq 0.\end{aligned}$$

Consequently, the intuitionistic fuzzy set in X_1 with membership function $\mu_F \circ f$ and nonmembership function $\nu_F \circ f$ is an intuitionistic fuzzy field in X_1.

Definition 3.1. Two intuitionistic fuzzy fields (F_1, X_1) and (F_2, X_2) are said to be homomorphic if there exists a field homomorphism $f : X_1 \to X_2$ such that $\mu_{F_1} = \mu_{F_2} \circ f$ and $\nu_{F_1} = \nu_{F_2} \circ f$. In such a case, $(f, (F_1, X_1), (F_2, X_2))$ is referred to as an intuitionistic fuzzy field homomorphism.

Proposition 3.2. *If two intuitionistic fuzzy fields (F_1, X_1) and (F_2, X_2) are homomorphic, then*

(i) $\mu_{F_1}(0) = \mu_{F_2}(0)$ *and* $\nu_{F_1}(0) = \nu_{F_2}(0)$

(ii)

$$\mu_{F_1}(1) = \begin{cases} \mu_{F_2}(1), & \textit{if the homomorphism involved is injective} \\ \mu_{F_2}(0), & \textit{otherwise} \end{cases}$$

and

$$\nu_{F_1}(1) = \begin{cases} \nu_{F_2}(1), & \textit{if the homomorphism involved is injective} \\ \nu_{F_2}(0), & \textit{otherwise,} \end{cases}$$

where 0 *and* 1 *on left sides of the equalities are respectively the zero and unity in the field* X_1 *and those on right sides are the zero and unity in the field* X_2*.*

Proof. Since (F_1, X_1) and (F_2, X_2) are homomorphic, there exists a homomorphism $f : X_1 \to X_2$ such that $\mu_{F_1} = \mu_{F_2} \circ f$ and $\nu_{F_1} = \nu_{F_2} \circ f$. As a result,

(i) $\nu_{F_1}(0) = (\nu_{F_2} \circ f)(0) = \nu_{F_2}(f(0)) = \nu_{F_2}(0)$.

(ii) If f is injective, then $\nu_{F_1}(1) = (\nu_{F_2} \circ f)(1) = \nu_{F_2}(f(1)) = \nu_{F_2}(1)$.
If f is not injective, then $\nu_{F_1}(1) = (\nu_{F_2} \circ f)(1) = \nu_{F_2}(0))$.

Likewise, $\mu_{F_1}(0) = \mu_{F_2}(0)$ and $\mu_{F_1}(1) = \mu_{F_2}(1)$ or $\mu_{F_1}(1) = \mu_{F_2}(0)$ according as f is injective or f is not injective.

Proposition 3.3. *If* $(f, (F_1, X_1), (F_2, X_2))$ *is an intuitionistic fuzzy field homomorphism, then*

(i) $f(F_1) \subseteq F_2$

(ii) $f^{-1}(F_2) = F_1$.

Proof. For all $a \in X_1$,
$\mu_{F_1}(a) = (\mu_{F_2} \circ f)(a) = \mu_{F_2}(f(a))$ and $\nu_{F_1}(a) = (\nu_{F_2} \circ f)(a) = \nu_{F_2}(f(a))$.

(i) *Case 1*: f is injective.
Let $b \in X_2$.
If $f^{-1}(b) \neq \phi$, then there exists unique $a \in X_1$ such that $f(a) = b$. Therefore $\mu_{f(F_1)}(b) = \mu_{f(F_1)}(f(a)) = \mu_{F_1}(a) = \mu_{F_2}(f(a)) = \mu_{F_2}(b)$ and similarly, $\nu_{f(F_1)}(b) = \nu_{F_2}(b)$.
If $f^{-1}(b) = \phi$, then $\mu_{f(F_1)}(b) = 0 \leq \mu_{F_2}(b)$ and $\nu_{f(F_1)}(b) = 1 \geq \nu_{F_2}(b)$.

Case 2: f is not injective.
In this case, $f(a) = 0$ for all $a \in X_1$. Hence if $b \in X_2$ and $b \neq 0$, then $f^{-1}(b) = \phi$. So $\mu_{f(F_1)}(b) = 0 \leq \mu_{F_2}(b)$ and $\nu_{f(F_1)}(b) = 1 \geq \nu_{F_2}(b)$.
If $b = 0$, then

$$\begin{aligned}
\mu_{f(F_1)}(b) &= \mu_{f(F_1)}(0) = \sup\left\{\mu_{F_1}(a) : a \in X_1, f(a) = 0\right\} \\
&= \mu_{F_1}(0), \text{ since } f(0) = 0 \text{ and } \mu_{F_1}(0) \geq \mu_{F_1}(a) \\
&= \mu_{F_2}(f(0)) = \mu_{F_2}(b)
\end{aligned}$$

and

$$\begin{aligned}\nu_{f(F_1)}(b) &= \nu_{f(F_1)}(0) = \inf\left\{\nu_{F_1}(a) : a \in X_1, f(a) = 0\right\} \\ &= \nu_{F_1}(0), \text{ since } f(0) = 0 \text{ and } \nu_{F_1}(0) \leq \nu_{F_1}(a) \\ &= \nu_{F_2}(f(0)) = \nu_{F_2}(b).\end{aligned}$$

Thus, in either case, $\mu_{f(F_1)}(b) \leq \mu_{F_2}(b)$ and $\nu_{f(F_1)}(b) \geq \nu_{F_2}(b)$ for all $b \in X_2$. Hence $f(F_1) \subseteq F_2$.

(ii) For all $a \in X_1$, $\mu_{f^{-1}(F_2)}(a) = \mu_{F_2}(f(a))) = \mu_{F_1}(a)$ and also $\nu_{f^{-1}(F_2)}(a) = \nu_{F_1}(a)$.

The result below stems from the preceding proof.

Corollary 3.1. *If* $(f, (F_1, X_1), (F_2, X_2))$ *is an intuitionistic fuzzy field homomorphism, then*

$$\mu_{f(F_1)}(b) = \begin{cases}\mu_{F_2}(b), & \textit{if } b \in Range(f) \\ 0, & \textit{otherwise}\end{cases}$$

and

$$\nu_{f(F_1)}(b) = \begin{cases}\nu_{F_2}(b), & \textit{if } b \in Range(f) \\ 1, & \textit{otherwise}\end{cases}.$$

Corollary 3.2. *If* $(f, (F_1, X_1), (F_2, X_2))$ *is an intuitionistic fuzzy field homomorphism, then for all* $\alpha, \beta \in [0, 1]$,

(i) $N_{\alpha,\beta}(f(F_1)) \subseteq N_{\alpha,\beta}(F_2)$

(ii) $N_{\alpha,\beta}(f^{-1}(F_2)) = N_{\alpha,\beta}(F_1)$.

Proof. Immediate from the notion of (α, β)-level sets and from proposition 3.3.

4. ISOMORPHISM OF INTUITIONISTIC FUZZY FIELDS

Definition 4.1. If $(f, (F_1, X_1), (F_2, X_2))$ is an intuitionistic fuzzy field homomorphism and if f is a bijection, then the intuitionistic fuzzy fields (F_1, X_1) and (F_2, X_2) are said to be isomorphic and we write $(F_1, X_1) \cong (F_2, X_2)$. In this case, $(f, (F_1, X_1), (F_2, X_2))$ is called an intuitionistic fuzzy field isomorphism.

Proposition 4.1. *If* $(f, (F_1, X_1), (F_2, X_2))$ *is an intuitionistic fuzzy field isomorphism, then* $f(F_1) = F_2$.

Proof. For each $b \in X_2$, there is a unique $a \in X_1$ such that $f(a) = b$, whence $\mu_{f(F_1)}(b) = \mu_{f(F_1)}(f(a)) = \mu_{F_1}(a) = \mu_{F_2}(f(a)) = \mu_{F_2}(b)$ and, in the same way, $\nu_{f(F_1)}(b) = \nu_{F_2}(b)$.

Corollary 4.1. *If $(f, (F_1, X_1), (F_2, X_2))$ is an intuitionistic fuzzy field isomorphism, then $N_{\alpha,\beta}(f(F_1)) = N_{\alpha,\beta}(F_2)$.*

Theorem 4.1. *The relation "being isomorphic" is an equivalence relation on the class of all intuitionistic fuzzy fields.*

Proof. *Reflexivity*. Any intuitionistic fuzzy field (F, X) is isomorphic to itself because the identity mapping $I : X \to X$ serves as an isomorphism satisfying $\mu_F \circ I = \mu_F$ and $\nu_F \circ I = \nu_F$.

Symmetry. Let $(F_1, X_1) \cong (F_2, X_2)$. Then there exists an isomorphism $f : X_1 \to X_2$ satisfying $\mu_{F_2} \circ f = \mu_{F_1}$ and $\nu_{F_2} \circ f = \nu_{F_1}$. Its inverse $f^{-1} : X_2 \to X_1$ is an isomorphism. Further, corresponding to every $b \in X_2$, there exists unique $a \in X_1$ such that $f^{-1}(b) = a$. As a result, $\left(\nu_{F_1} \circ f^{-1}\right)(b) = \nu_{F_1}(f^{-1}(b)) = \nu_{F_1}(a) = \nu_{F_2}(f(a)) = \nu_{F_2}(b)$ and similarly, $\left(\mu_{F_1} \circ f^{-1}\right)(b) = \nu_{F_2}(b)$, which means that $\mu_{F_2} = \mu_{F_1} \circ f^{-1}$ and $\nu_{F_2} = \nu_{F_1} \circ f^{-1}$. Thus $(F_2, X_2) \cong (F_1, X_1)$.

Transitivity. If $(F_1, X_1) \cong (F_2, X_2)$ and $(F_2, X_2) \cong (F_3, X_3)$, then there exists an isomorphism $f : X_1 \to X_2$ such that $\mu_{F_2} \circ f = \mu_{F_1}$, $\nu_{F_2} \circ f = \nu_{F_1}$ and there exists an isomorphism $g : X_2 \to X_3$ such that $\mu_{F_3} \circ g = \mu_{F_2}$, $\nu_{F_3} \circ g = \nu_{F_2}$. These yield the composite isomorphism $g \circ f : X_1 \to X_3$ for which

$$\begin{aligned} \left(\nu_{F_3} \circ (g \circ f)\right)(a) &= \left((\nu_{F_3} \circ g) \circ f\right)(a) = (\nu_{F_3} \circ g)(f(a)) \\ &= \nu_{F_2}(f(a)) = \nu_{F_1}(a) \end{aligned}$$

and analogously, $\left(\mu_{F_3} \circ (g \circ f)\right)(a) = \mu_{F_1}(a)$ for all $a \in X_1$.
Hence $\mu_{F_1} = \mu_{F_3} \circ (g \circ f)$ and $\nu_{F_1} = \nu_{F_3} \circ (g \circ f)$ so that $(F_1, X_1) \cong (F_3, X_3)$.

Corollary 4.2. *The relation of isomorphism on the class of all intuitionistic fuzzy fields of a field is an equivalence relation.*

5. CONCLUSION

This study, by integrating the notions of membership and non-membership functions, explores the concept of homomorphism and isomorphism between intuitionistic fuzzy fields, extending classical algebraic homomorphisms and isomorphisms to the intuitionistic fuzzy framework. The results demonstrate that such isomorphisms preserve both the algebraic operations and the intuitionistic fuzziness inherent in the field elements, ensuring consistency between intuitionistic fuzzy structures and their crisp counterparts. This research thus provides a theoretical foundation for further mathematical development and real-world implementation of intuitionistic fuzzy algebraic structures.

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