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Centre and Radius Corresponding to Fermi's Measure of Entropy and its Generalization

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Abstract

In this paper we introduced a new measure of information centre and radius corresponding to Fermi's measure of entropy and it has been generalized. The generalized measure is interpreted as the minimum value of the given set of probability distributions from any distribution.

Key Words: Entropy, directed divergence, information radius/centre and radius of probability distribution.

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Introduction

Let $P=(p_1,p_2,...,p_n)$ and $Q=(q_1,q_2,...,q_n)$ be two probability distributions. Shannon's [4] measure of entropy for probability distribution P is given by

$$H(P)=-\sum_{i=1}^{n} p_i \ln p_i \tag{1}$$

Kapur [2] has introduced the concept of information radius by using Shannon's measure of entropy, it is given by

$$R(P:Q)=H\left(\frac{P+Q}{2}\right)-\frac{H(P)+H(Q)}{2} \tag{2}$$

and equation (2) is called information radius (Sibson[5]) or Jensen-Shannon directed divergence measure(Burbea and Rao[1]). Om Prakash and Verma[3] have introduced the information radius on Kapur's entropy and its generalization.

In this paper, we study a new measure of information centre and radius corresp-onding to Fermi's measure of entropy and its generalization.

New probabilistic measure of information radius

Fermi Dirac's measure of entropy defined as

$$F(P) = -\sum_{i=1}^{n} p_i \ln p_i - \sum_{i=1}^{n} (1-p_i)\ln(1-p_i) \tag{3}$$

We now introduce a new probabilistic measure of information radius given by

$$R_1(P:Q) = F\left(\frac{P+Q}{2}\right) - \frac{F(P)+F(Q)}{2}$$

$$= -\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\ln\left(\frac{p_i+q_i}{2}\right) - \sum_{i=1}^{n}\left(1-\frac{p_i+q_i}{2}\right)\ln\left(1-\frac{p_i+q_i}{2}\right)$$

$$-\frac{1}{2}\left[-\sum_{i=1}^{n} p_i \ln p_i - \sum_{i=1}^{n}(1-p_i)\ln(1-p_i) - \sum_{i=1}^{n} q_i \ln q_i - \sum_{i=1}^{n}(1-q_i)\ln(1-q_i)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} p_i \ln\frac{p_i}{\frac{p_i+q_i}{2}} + \sum_{i=1}^{n}(1-p_i)\ln\frac{(1-p_i)}{\left(1-\frac{p_i+q_i}{2}\right)} + \sum_{i=1}^{n} q_i \ln\frac{q_i}{\frac{p_i+q_i}{2}} + \sum_{i=1}^{n}(1-q_i)\ln\frac{(1-q_i)}{\left(1-\frac{p_i+q_i}{2}\right)}\right]$$

$$= \frac{1}{2}\left[D_f\left(P:\frac{P+Q}{2}\right) + D_f\left(Q:\frac{P+Q}{2}\right)\right] \tag{4}$$

where

$$D_f(P:Q) = \sum_{i=1}^{n} p_i \ln\frac{p_i}{q_i} + \sum_{i=1}^{n}(1-p_i)\ln\frac{(1-p_i)}{(1-q_i)} \tag{5}$$

which is a directed divergence corresponding to Fermi-Dirac's measure of entropy $F(P)$. $D_f(P:Q)$ has following properties:

(i) $D_f(P:Q) \ge 0$,

(ii) $D_f(P:Q) = 0$ *iff* $P = Q$,

(iii) $D_f(P:Q)$ is a convex function of $P = (p_1, p_2, ..., p_n)$ and $Q = (q_1, q_2, ..., q_n)$.

From (4) it is obvious that

$R_1(P:Q) \ge 0$ (being sum of two directed divergences)

and $R_1(P:Q) = 0$ *iff* $P = Q = \frac{P+Q}{2}$.

$R_1(P:Q)$ is a convex function of $P = (p_1, p_2, ..., p_n)$ and $Q = (q_1, q_2, ..., q_n)$.

Thus $R_1(P:Q)$ satisfies all the essential properties of a directed divergence. It is symmetric also in the sense $R_1(P:Q)=R_1(Q:P)$ but not in the sense $D(P:Q)=D(Q:P)$.

Generalized information radius

We define

$$D_\lambda(P:Q)=F(\lambda P+(1-\lambda)Q)-\lambda F(P)-(1-\lambda)F(Q), 0\le\lambda\le 1$$
$$=\lambda D(P:\lambda P+(1-\lambda)Q)+(1-\lambda)D(Q:\lambda P+(1-\lambda)Q) \quad (6)$$

We generalize it further to define

$$R(P_1,P_2,...,P_m)=\sum_{j=1}^{m}\lambda_j D\left(P_j:\sum_{j=1}^{m}\lambda_j P_j\right) \quad (7)$$

where $$\lambda=(\lambda_1,\lambda_2,...,\lambda_m),\lambda_j\ge 0,\sum_{j=1}^{n}\lambda_j=1 \quad (8)$$

This is also ≥ 0 and vanishes *iff*

$$P_1=P_2=...........=P_m=\sum_{j=1}^{m}\lambda_j P_j \quad (9)$$

i.e. when all the probability distributions are coincide.

A Minimization problem

We find the distribution P for which

$$\sum_{j=1}^{m}\lambda_j D(P_j:P)=\sum_{j=1}^{m}\lambda_j\left[\sum_{i=1}^{n}p_{ji}\ln\frac{p_{ji}}{p_i}+\sum_{i=1}^{n}(1-p_{ji})\ln\frac{(1-p_{ji})}{(1-p_i)}\right] \quad (10)$$

is minimum, subject to

$$\sum_{i=1}^{n}p_i=1 \quad (11)$$

Since p_{ji}'s are given to us, we have to choose $p_1,p_2,...,p_n$ to minimize

$$-\sum_{j=1}^{m}\left[\sum_{i=1}^{n}p_{ji}\ln p_i+\sum_{i=1}^{n}(1-p_{ji})\ln(1-p_i)\right] \quad (12)$$

subject to (11), we get that

$$\sum_{j=1}^{m}\lambda_j\frac{p_{ji}}{p_i}-\sum_{j=1}^{m}\lambda_j\frac{(1-p_{ji})}{(1-p_i)} \quad (13)$$

should be independent of i, so that using (9)

$$\frac{p_1}{\sum_{j=1}^{m}\lambda_j p_{j1}} = \frac{p_2}{\sum_{j=1}^{m}\lambda_j p_{j2}} = ... = \frac{p_n}{\sum_{j=1}^{m}\lambda_j p_{jn}} = \frac{1}{\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_j p_{ji}} \tag{14}$$

Let $$S = \sum_{j=1}^{m}\lambda_j P_j \tag{15}$$

so that $s_i = ith$ component of $S = \sum_{j=1}^{m}\lambda_j p_{ji}$ (16)

and $$\sum_{i=1}^{n} s_i = \sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_j p_{ji} = 1 \tag{17}$$

so that $p_i = s_i, \forall i = 1,2,...,n$.

Thus $\sum_{j=1}^{m}\lambda_j D\left(P_j : P\right)$ is minimum when

$$P = \sum_{j=1}^{m}\lambda_j P_j \tag{18}$$

Thus the weighted sum of directed divergences of $P_1, P_2, ..., P_m$ from any distribution P is minimum when P is weighted sum of the m given distributions and the minimum value of the weighted sum is

$$\sum_{j=1}^{m}\lambda_j D\left(P_j : \sum_{j=1}^{m}\lambda_j P_j\right) \tag{19}$$

We shall call the distribution P (18) as the centre and (19) as the radius of the set of distribution $P_1, P_2, ..., P_m$ with weights $\lambda_1, \lambda_2, ..., \lambda_n$.

We shall explain the reason for this terminology below. For the present we note the special case,

$m = 2$, $P_1 = P$, $P_2 = Q$

The centre is $\left(\frac{P+Q}{2}\right)$ and the radius is

$$\frac{1}{2}\left[D\left(P : \frac{P+Q}{2}\right) + D\left(Q : \frac{P+Q}{2}\right)\right]$$

which was called information radius by Sibson.

Centre and radius of a set of probability distributions

We have defined the centre of a set of probability distribution $P_1, P_2, ..., P_m$ as that distribution P, for which $\sum_{j=1}^{m} \lambda_j D\left(P_j : P\right)$ is minimum, is $P = \sum_{j=1}^{m} \lambda_j P_j$. Also the radius of the set of distribution is the minimum value *i.e.*(19).

Now we have two distinct divergences between two probability distributions. as such for a given set of probability distributions, there will be two centers and two radii.

Another minimization problem, the second centre and radius

Here we seek to choose P to minimize.

$$\sum_{j=1}^{m} \lambda_j D\left(P : P_j\right) = \sum_{j=1}^{m} \lambda_j \left[\sum_{i=1}^{n} p_i \ln \frac{p_i}{p_{ji}} + \sum_{i=1}^{n} \left(1 - p_i\right) \ln \frac{\left(1 - p_i\right)}{\left(1 - p_{ji}\right)} \right] \tag{20}$$

Minimizing it subject to (11), we get

$$\sum_{j=1}^{m} \lambda_j \ln \frac{p_i \big/ \left(1 - p_i\right)}{p_{ji} \big/ \left(1 - p_{ji}\right)} = \ln \frac{p_i \big/ \left(1 - p_i\right)}{\prod_{j=1}^{n} \left(p_{ji} \big/ \left(1 - p_{ji}\right) \right)^{\lambda_j}} \tag{21}$$

is independent of i, so that the i th component of P is proportional to the weighted geometrical mean of the i th component of $P_1, P_2, ..., P_m$.

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