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ANALYSIS OF FIXED POINT RESULTS AND SOME RELATED APPLICATIONS IN CONE *S*-METRIC SPACE

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Abstract

The aim of this paper is to establish various common fixed-point (CFP) theorems for self-maps in a complete cone *S*-metric space (C_S-MS). We extended and initialized the notion of contractive type of self-maps and proved related fixed-point (FP) results in the framework of C_S-MS. Consequently, we demonstrated certain applications for fixed and periodic-point for these contractive type maps. These results of FP theory extend the existing literature in C_S-MS.

MSC (2020): 47H10; 54H25; **37C25**.

Keywords: Common fixed point; Cone *S*-metric space; Self-maps; Partial ordering; Normal cone; Periodic-point.

1. INTRODUCTION

In 1922, Banach S. [3] introduced a remarkable idea as Banach FP-theorem {Banach contraction principle (BCP)}, that was initialized as one of finest research works of FP theory of mathematical sciences and nonlinear analysis in recent decades. In 2007, Huang and Z. [7] established the idea of cone metric space (CMS) as a generalization and unification of the metric spaces (MS) by replacing the set of real numbers ($\mathbb{R}$) by ordering Banach spaces. They obtained certain CFP results and later Jankovic *et al.*, [8] extended the FP results in CMS. G-MS as an extension of MS was presented by Mustafa and S. [10] and Sedghi *et al.*, [15] established the framework S-MS. Certainly, analogues to FP theory many researchers proposed certain assumptions to guarantee the existence of CFP theorems in MS and its generalized spaces viz: G-MS,

S-MS, fuzzy-MS, Q-MS, CMS and those outcomes are already existing in literature [9, 12, 15]. Abbas and R. [1] initially gave the fixed and periodic-points contractions maps in CMS. Moreover, in 2016 Rauf *et al.*, [13] pointed out a few CFP theorems for self-maps in CMS. In 2017, Dhamodharan and K. [5] by amalgamating CMS and S-MS proposed the concept of cone S-metric space (C_S-MS) and underwent related outcomes. Dubey *et al.*, [6] in 2014 and Vigaya *et al.*, [19] in 2016 pointed to CFP theorems using contractive type self-maps in CMS. Saluja [14], in 2020, by considering implicit relation presented CFP in C_S-MS. In 2025, Raji *et al.*, [11] established FP results in CMS for the contractive type maps with related application to fixed and periodic-point.

Influenced by the research work done by these eminent researchers, now we study novel CFP results for self-maps in a complete C_S-MS to obtain an existence and uniqueness of FP. We are extending the definition of contractive type self-mappings in C_S-MS and also demonstrate certain applications in the mentioned space.

2. PRELIMINARIES

This section begins with some pertinent definitions and related lemmas. In this paper $\mathbb{B}$ represents the 'real Banach space' and $\mathbb{N}$ represents the set of positive integers. θ is zero of $\mathbb{B}$.

Definition 2.1. [7] A subset $O \subset \mathbb{B}$ is defined as a cone iff:

(C_1) O is closed, non-empty and $O \neq \{\theta\}$,

(C_2) $\mathfrak{d}_1, \mathfrak{d}_2 \in [0, \infty)$ and $\varpi, w \in O$, then $\mathfrak{d}_1\varpi + \mathfrak{d}_2 w \in O$,

(C_3) $\varpi \in O$ and $-\varpi \in O$ then $\varpi = \theta$.

Given a cone $O \subset \mathbb{B}$, we define a partial ordering in $\mathbb{B}$ with respect to O as $\varpi \leq w$ iff $w - \varpi \in O$ and $\varpi < w$ indicate that $\varpi \leq w$ but $\varpi \neq w$, while $\varpi \ll w$ stands for $w - \varpi \in int(O)$, where: $int(O)$ denotes the interior of O.

In the present paper, all cones have a non-empty interior.

Note 2.1.1. [7] A cone O is said to be normal if for a given $Y > 0$ s.t. for all $\varpi, w \in \mathbb{B}$, $0 \leq \varpi \leq w$ implies $\|\varpi\| \leq Y\|w\|$.

Note 2.1.2. [7] A cone O is said to be regular if every monotone increasing sequence which is bounded from above is convergent, i.e., for every $m \in \mathbb{N}$ s.t. $p_m \leq p_{m+1} \leq w$; for some $w \in O$, then there is $\varpi \in O$ s.t. $\lim_{m\to\infty} \|p_m - \varpi\| = 0$. In Similar way, O is regular if every monotone decreasing sequence which is bounded from below is convergent.

Remark 2.1.3. [7] Every regular cone is a normal cone.

Definition 2.2. CMS [7] Let $\mathbb{B}$ stand for 'real Banach space', O is a cone in $\mathbb{B}$ (with $intO \neq \emptyset$) and '$\leq$' is a partial ordering w.r.t. cone O. Suppose $\tilde{\mathfrak{A}}(\neq \{\emptyset\})$ set and a map $C : \tilde{\mathfrak{A}}^2 \to \mathbb{B}$ on $\tilde{\mathfrak{A}}$ s.t. pair $(\tilde{\mathfrak{A}}, C)$ said to be a CMS, if it satisfies the following for all $\varpi, w, \xi \in \tilde{\mathfrak{A}}$:

(C_M 1) $0 < C(\varpi, w)$ and $\varpi = w$ iff $C(\varpi, w) = 0$,
(C_M 2) $C(\varpi, w) = C(w, \varpi)$,
(C_M 3) $C(\varpi, w) \leq C(\varpi, \xi) + C(\varpi, w)$.

Definition 2.3. S-MS [15]: The pair $(\breve{\mathfrak{A}}, \hat{S})$ is called an S-MS if $\breve{\mathfrak{A}}(\neq \{\emptyset\})$ and $\hat{S}: \breve{\mathfrak{A}}^3 \to [0, \infty)$ s.t. for any $\varpi, w, \xi, i_1 \in \breve{\mathfrak{A}}$ satisfies the following axioms:
(S_M 1) $\hat{S}(\varpi, w, \xi) \geq 0$,
(S_M 2) $\hat{S}(\varpi, w, \xi) = 0$ if and only if $\varpi = w = \xi$,
(S_M 3) $\hat{S}(\varpi, w, \xi) \leq \hat{S}(\varpi, \varpi, i_1) + \hat{S}(w, w, i_1) + \hat{S}(\xi, \xi, i_1)$.
Example 2.4. [16] Consider $\breve{\mathfrak{A}} = \mathbb{R}^n$ and $\|\cdot\|$ a norm on $\breve{\mathfrak{A}}$, then a map $\hat{S} : \breve{\mathfrak{A}}^3 \to [0, \infty)$ be defined as $\hat{S}(\varpi, w, \xi) = \|w + \xi - 2\varpi\| + \|w - \xi\|$ for any $\varpi, w, \xi \in \breve{\mathfrak{A}}$ is $\hat{S}$-metric on $\breve{\mathfrak{A}}$.
i.e., $(\breve{\mathfrak{A}}, \hat{S})$ is a S-MS.

Definition 2.5. C_S-MS [5]: Suppose that $\mathbb{B}$ is a real Banach space. O is a cone in $\mathbb{B}$ (with $intO \neq \emptyset$) and '$\leq$' is a partial ordering w.r.t. O. Let $\breve{\mathfrak{A}}(\neq \{\emptyset\})$ set and function $\breve{C} : \breve{\mathfrak{A}}^3 \to \mathbb{R}$, then the ordered pair $(\breve{\mathfrak{A}}, \breve{C})$ is said to be a cone S-MS (C_S-MS) if satisfy the following axioms for each $\varpi, w, \xi, a \in \breve{\mathfrak{A}}$:
(C_{SM}1) $\breve{C}(\varpi, w, \xi) \geq 0$,
(C_{SM}2) $\breve{C}(\varpi, w, \xi) = 0$ iff $\varpi = w = \xi$,
(C_{SM}3) $\breve{C}(\varpi, w, \xi) \leq \breve{C}(\varpi, \varpi, a) + \breve{C}(w, w, a) + \breve{C}(\xi, \xi, a)$.
Example 2.6. [5] Let $\mathbb{B} = \mathbb{R}^2$, $O = \{(\varpi, w) \in \mathbb{R}^2 : \varpi \geq 0, w \geq 0\}$. Then the map $\hat{S}: \breve{\mathfrak{A}}^3 \to \mathbb{B}$ be defined as $\breve{C}(\varpi, w, \xi) \leq \{|\varpi - w| + |\varpi - \xi|, k(|\varpi - w| + |\varpi - \xi|)\}$, where $k \geq 0$ is a C_S-MS on $\breve{\mathfrak{A}}$.
Lemma 2.7. [5] Let $(\breve{\mathfrak{A}}, \breve{C})$ is a C_S-MS. Then, $\breve{C}(\varpi, \varpi, w) = \breve{C}(w, w, , \varpi)$ for all $\varpi, w \in \breve{\mathfrak{A}}$.

Definition 2.8. [5] Let $(\breve{\mathfrak{A}}, \breve{C})$ is called a C_S-MS, then

(i) A sequence $\{p_m\}$ in $\breve{\mathfrak{A}}$ converges to ϖ iff $\breve{C}(p_m, p_m, \varpi) \to 0$ as $m \to \infty$, i.e., there exists $m_0 \in \mathbb{N}$ s.t. for any $m \geq m_0$, $\breve{C}(p_m, p_m, \varpi) \ll c \in \mathbb{B}$, $0 \ll c$. Notion of convergence is: $\lim_{m\to\infty} p_m = \varpi$ or $\lim_{m\to\infty} \breve{C}(p_m, p_m, \varpi) = 0$.

(ii) A sequence $\{p_m\}$ in $\breve{\mathfrak{A}}$ is called a Cauchy sequence if $\breve{C}(p_m, p_m, \varpi) \to 0$ as $m \to \infty$, i.e., there exists $m_0 \in \mathbb{N}$ s.t. for every $n, m \geq m_0$, $\breve{C}(p_m, p_m, p_n) \ll c$ for all $c \in \mathbb{B}$, $0 \ll c$.

(iii) The C_S-MS $(\breve{\mathfrak{A}}, \breve{C})$ is called complete if every Cauchy sequence is convergent.

Definition 2.9. BCP [3] Consider a self-map $\tilde{\wp}: \mathfrak{X} \to \mathfrak{X}$ of a MS $(\mathfrak{X}, \breve{d})$, then $\tilde{\wp}$ is said to be a contraction map if there exists a constant $0 \leq k < 1$ s.t. for any $\varpi, w \in \mathfrak{X}$ it satisfies:
$\breve{d}(\tilde{\wp}\varpi, \tilde{\wp}w) \leq k\breve{d}(\varpi, w)$.

Definitions 2.10. [4] Consider two self-maps $\check{f}, \tilde{\wp}: \mathfrak{X} \to \mathfrak{X}$ of a MS $(\mathfrak{X}, \breve{d})$, then $\tilde{\wp}$ is said to be $\check{f}$- contraction map if there exists a constant $0 \leq k < 1$ s.t. for all $\varpi, w \in \mathfrak{X}$ it satisfies:
$\breve{d}(\check{f}\tilde{\wp}\varpi, \check{f}\tilde{\wp}w) \leq k\breve{d}(\check{f}\varpi, \check{f}w)$.

Definition 2.11. [17] Consider $\mathfrak{X}\{\neq (\emptyset)\}$ and two self-maps $\tilde{\wp}, \check{f}: \mathfrak{X} \to \mathfrak{X}$ of a MS $(\mathfrak{X}, \breve{d})$. Then self-maps $\tilde{\wp}$ and $\check{f}$ have CFP if $\tilde{\wp}\varpi = \check{f}\varpi = \varpi$.

Definition 2.12. [18] Consider $\tilde{\wp}, \Delta : \mathfrak{X} \to \mathfrak{X}$ be two self-maps, then the pair $(\tilde{\wp}, \Delta)$ is said to be *IT*-commuting at $\varpi \in \mathfrak{X}$, if $\tilde{\wp}(\Delta(\varpi)) = \Delta(\tilde{\wp}(\varpi))$ with $\tilde{\wp}(\varpi) = \Delta(\varpi)$. The ordered pairs $(\tilde{\wp}, \Delta)$ and (f, g) are sharing the *IT*- commuting if $\tilde{\wp}[\Delta\{f(g(\varpi))\}] = \Delta[\tilde{\wp}\{g(f(\varpi))\}]$.

Definition 2.13.[11] Consider a normal cone O with Y be normal constant and $(\mathfrak{X}, \breve{c})$ as a complete CMS, then two self-maps $\tilde{\wp}, \check{f}: \mathfrak{X} \to \mathfrak{X}$ have type-*I* contraction, if for each $\varpi, w \in \mathfrak{X}$ with $\varpi \neq w$ and for $k_1, k_2, k_3, k_4 \geq 0$, $k_1 + 2k_2 + k_3 + k_4 < 1$ satisfies the following condition:

$$\breve{c}(\tilde{\wp}\varpi, \check{f}w) \leq k_1\{\breve{c}(\varpi, w)\} + k_2\{\breve{c}(\tilde{\wp}\varpi, \varpi) + \breve{c}(\check{f}w, w)\}$$
$$+k_3\left\{\frac{\breve{c}(\tilde{\wp}\varpi, \varpi)\breve{c}(\check{f}w, \varpi) + \breve{c}(\tilde{\wp}\varpi, w)\breve{c}(\check{f}w, w)}{\breve{c}(\tilde{\wp}\varpi, w) + \breve{c}(\check{f}w, \varpi)}\right\}$$
$$+k_4\left\{\frac{\breve{c}(\tilde{\wp}\varpi, \varpi)\breve{c}(\check{f}w, w)}{\breve{c}(\varpi, w) + \breve{c}(\tilde{\wp}\varpi, w) + \breve{c}(\check{f}w, \varpi)}\right\}.$$

Definition 2.14. [11] Consider a normal cone O with Y be normal constant and $(\mathfrak{X}, \breve{c})$ as a complete CMS, then two self-maps $\tilde{\wp}, \check{f}: \mathfrak{X} \to \mathfrak{X}$ have type-*II* contraction, if for each $\varpi, w \in \mathfrak{X}$ with $\varpi \neq w$ and for $k_1, k_2, k_3, k_4 \geq 0$, $k_1 + 2k_2 + k_3 + k_4 < 1$ satisfies the following condition:

$$\breve{c}(\tilde{\wp}\varpi, \check{f}w) \leq k_1\{\breve{c}(\varpi, w)\} + k_2\{\breve{c}(\tilde{\wp}\varpi, w) + \breve{c}(\check{f}w, \varpi)\}$$
$$+k_3\left\{\frac{\breve{c}(\tilde{\wp}\varpi, \varpi)\breve{c}(\check{f}w, \varpi) + \breve{c}(\tilde{\wp}\varpi, w)\breve{c}(\check{f}w, w)}{\breve{c}(\tilde{\wp}\varpi, w) + \breve{c}(\check{f}w, \varpi)}\right\}$$
$$+k_4\left\{\frac{\breve{c}(\tilde{\wp}\varpi, \varpi)\breve{c}(\check{f}w, w)}{\breve{c}(\varpi, w) + \breve{c}(\tilde{\wp}\varpi, w) + \breve{c}(\check{f}w, \varpi)}\right\}.$$

3. CFP RESULTS FOR FIVE SELF-MAPS IN COMPLETE C_S-MS

In the present section, we prove some CFP theorems in complete C_S-MS for five self-maps.

Theorem 3.1. Consider the five continuous self-maps $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ and $\tilde{\wp}$ of a complete C_5-MS $(\tilde{\mathfrak{A}}, \breve{\mathcal{C}})$ s.t. if following holds:

($A^{3.1.1}$) $\zeta_2(\tilde{\mathfrak{A}}) \subset \zeta_3(\tilde{\mathfrak{A}}), \zeta_1(\tilde{\mathfrak{A}}) \subset \zeta_4(\tilde{\mathfrak{A}})$ and $\zeta_1(\tilde{\mathfrak{A}}) = \zeta_2(\tilde{\mathfrak{A}})$;

($A^{3.1.2}$) Map $\tilde{\wp}$ is an injective and normal cone O with Y be normal constant;

($A^{3.1.3}$) $\zeta_1(\tilde{\mathfrak{A}})$ or $\zeta_2(\tilde{\mathfrak{A}})$ is a complete subspace of $\tilde{\mathfrak{A}}$, for any $\varpi, w \in \tilde{\mathfrak{A}}$ and $k_1, k_2, k_3 \geq 0$,

$k_1 + 2k_2 + 2k_3 < 1$ fulfils the following condition:

$$\breve{\mathcal{C}}(\tilde{\wp}\zeta_1\varpi, \tilde{\wp}\zeta_1\varpi, \tilde{\wp}\zeta_2 w) \leq k_1\{\breve{\mathcal{C}}(\tilde{\wp}\zeta_3\varpi, \tilde{\wp}\zeta_3\varpi, \tilde{\wp}\zeta_4 w)\}$$
$$+k_2\{\breve{\mathcal{C}}(\tilde{\wp}\zeta_1\varpi, \tilde{\wp}\zeta_1\varpi, \tilde{\wp}\zeta_3\varpi) + \breve{\mathcal{C}}(\tilde{\wp}\zeta_2 w, \tilde{\wp}\zeta_2 w, \tilde{\wp}\zeta_4 w)\}$$
$$+k_3\{\breve{\mathcal{C}}(\tilde{\wp}\zeta_1\varpi, \tilde{\wp}\zeta_1\varpi, \tilde{\wp}\zeta_4 w) + \breve{\mathcal{C}}(\tilde{\wp}\zeta_2 w, \tilde{\wp}\zeta_2 w, \tilde{\wp}\zeta_3\varpi)\}.$$
(3.1)

Then, the maps $(\zeta_1, \zeta_3, \tilde{\wp})$ and $(\zeta_2, \zeta_4, \tilde{\wp})$ have a coincidence point in $\tilde{\mathfrak{A}}$, along with self-map $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ and $\tilde{\wp}$ have a unique CFP.

Proof: Suppose an arbitrary point $p_0 \in \tilde{\mathfrak{A}}$, moreover $\zeta_1(\tilde{\mathfrak{A}}) \subset \zeta_4(\tilde{\mathfrak{A}})$, implies that there occurs $p_1 \in \tilde{\mathfrak{A}}$ s.t. $\zeta_1 p_0 = \zeta_4 p_1$ and similarly as $\zeta_2(\tilde{\mathfrak{A}}) \subset \zeta_3(\tilde{\mathfrak{A}})$, there is $p_2 \in \tilde{\mathfrak{A}}$ s.t. $\zeta_2 p_1 = \zeta_3 p_2$.

By iteration for the formation of sequences, one can have

$q_{2m} = \tilde{\wp}\zeta_1 p_{2m} = \tilde{\wp}\zeta_4 p_{2m+1}$ and

$q_{2m+1} = \tilde{\wp}\zeta_2 p_{2m+1} = \tilde{\wp}\zeta_3 p_{2m+2}$.

Consequently, this means that

$$\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m+1}) = \breve{\mathcal{C}}(\tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_2 p_{2m+1})$$
$$\leq k_1\{\breve{\mathcal{C}}(\tilde{\wp}\zeta_3 p_{2m}, \tilde{\wp}\zeta_3 p_{2m}, \tilde{\wp}\zeta_4 p_{2m+1})\}$$
$$+k_2\{\breve{\mathcal{C}}(\tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_3 p_{2m}) + \breve{\mathcal{C}}(\tilde{\wp}\zeta_2 p_{2m+1}, \tilde{\wp}\zeta_2 p_{2m+1}, \tilde{\wp}\zeta_4 p_{2m+1})\}$$
$$+k_2\{\breve{\mathcal{C}}(\tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_4 p_{2m+1}) + \breve{\mathcal{C}}(\tilde{\wp}\zeta_2 p_{2m+1}, \tilde{\wp}\zeta_2 p_{2m+1}, \tilde{\wp}\zeta_3 p_{2m})\}$$
$$= k_1\{\breve{\mathcal{C}}(q_{2m-1}, q_{2m-1}, q_{2m})\} + k_2\{\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m-1}) + \breve{\mathcal{C}}(q_{2m+1}, q_{2m+1}, q_{2m})\}$$
$$+k_3\{\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m}) + \breve{\mathcal{C}}(q_{2m+1}, q_{2m+1}, q_{2m-1})\}$$
$$\leq k_1\{\breve{\mathcal{C}}(q_{2m-1}, q_{2m-1}, q_{2m})\} + k_2\{\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m-1})\} + k_2\{\breve{\mathcal{C}}(q_{2m+1}, q_{2m+1}, q_{2m})\}$$
$$+k_3\{\breve{\mathcal{C}}(q_{2m+1}, q_{2m+1}, q_{2m})\} + k_3\{\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m-1})\}$$
$$= (k_1 + k_2 + k_3)\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m-1}) + (k_2 + k_3)\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m+1}).$$

Simplifying further, we imply that

$$(1 - k_2 - k_3)\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m+1}) \leq (k_1 + k_2 + k_3)\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m-1})$$
and $$\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m-1}) \leq \left(\frac{k_1+k_2+k_3}{1-k_2-k_3}\right)\{\breve{\mathcal{C}}(q_{2m-1}, q_{2m-1}, q_{2m})\}.$$

Assume that $\alpha = \left(\frac{k_1+k_2+k_3}{1-k_2-k_3}\right)$ and for every instance of $\alpha < 1$, it follows that

$$\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m+1}) \leq \alpha\{\breve{\mathcal{C}}(q_{2m-1}, q_{2m-1}, q_{2m})\}. \tag{3.2}$$

Likewise, $\breve{\mathcal{C}}(q_{2m}, q_{2m}, q_{2m-1}) \leq \alpha^{2m}\{\breve{\mathcal{C}}(q_0, q_1)\}$.

Now, for choice of $n, m \in \mathbb{N}$, $n > m$, we deduce that

$$\breve{C}(\mathfrak{q}_{2m}, \mathfrak{q}_{2m}, \mathfrak{q}_{2n}) \leq \breve{C}(\mathfrak{q}_{2m}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}) + \breve{C}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+2})$$
$$+\breve{C}(\mathfrak{q}_{2m+2}, \mathfrak{q}_{2m+2}, \mathfrak{q}_{2m+3}) + \cdots + \breve{C}(\mathfrak{q}_{2n-1}, \mathfrak{q}_{2n-1}, \mathfrak{q}_{2n})$$
$$\leq \alpha^{2m}\{\breve{C}(\mathfrak{q}_0, \mathfrak{q}_0, \mathfrak{q}_1)\} + \alpha^{2m+1}\{\breve{C}(\mathfrak{q}_0, \mathfrak{q}_0, \mathfrak{q}_1)\} + \cdots + \alpha^{2n-1}\{\breve{C}(\mathfrak{q}_0, \mathfrak{q}_0, \mathfrak{q}_1)\}$$
$$= (\alpha^{2m} + \alpha^{2m+1} + \cdots + \alpha^{2n-1})\{\breve{C}(\mathfrak{q}_0, \mathfrak{q}_0, \mathfrak{q}_1)\}$$
$$= \alpha^{2m}(1 + \alpha + \alpha^2 + \cdots + \alpha^{2n-2m-1})\{\breve{C}(\mathfrak{q}_0, \mathfrak{q}_0, \mathfrak{q}_1)\}$$
$$= \frac{\alpha^{2m}}{1-\alpha}\{\breve{C}(\mathfrak{q}_0, \mathfrak{q}_0, \mathfrak{q}_1)\}.$$

On the other hand, on functionality of normality of cone, imply that

$$\|\breve{C}(\mathfrak{q}_{2m}, \mathfrak{q}_{2m}, \mathfrak{q}_{2n})\| \leq Y\left|\frac{\alpha^{2m}}{1-\alpha}\right| \|\breve{C}(\mathfrak{q}_0, \mathfrak{q}_0, \mathfrak{q}_1)\|,$$

i.e., $\breve{C}(\mathfrak{q}_{2m}, \mathfrak{q}_{2m}, \mathfrak{q}_{2n})$ tending to 0, as n, m approaching to ∞.

This means that, $\{\mathfrak{q}_{2m}\}$ is a Cauchy sequence in $\zeta_2(\widetilde{\mathfrak{A}})$. Equivantenly, $\zeta_2(\widetilde{\mathfrak{A}})$ is a complete $C_{\mathcal{S}}$-MS, therefore the sequence $\{\mathfrak{q}_{2m}\}$ converges to some point $\xi \in \zeta_2(\widetilde{\mathfrak{A}})$.

Meanwhile, $\zeta_2(\widetilde{\mathfrak{A}}) \subset \zeta_3(\widetilde{\mathfrak{A}})$, then there exist $\xi_1 \in \widetilde{\mathfrak{A}}$ s.t. $\xi = \zeta_3\xi_1$ and $\widetilde{\wp}\xi = \widetilde{\wp}\zeta_3\xi_1$.

Now, we settled to initially show that $\widetilde{\wp}\zeta_1\xi_1 = \widetilde{\wp}\xi$.

In this way, we deduce that

$$\breve{C}(\widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\xi) \leq \breve{C}(\widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}) + \breve{C}(\widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\xi)$$
$$\leq k_1\{\breve{C}(\widetilde{\wp}\zeta_3\xi_1, \widetilde{\wp}\zeta_3\xi_1, \widetilde{\wp}\zeta_4\mathfrak{q}_{2m-1})\}$$
$$+k_2\{\breve{C}(\widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_3\xi_1) + \breve{C}(\widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\zeta_4\mathfrak{q}_{2m-1})\}$$
$$+k_3\begin{Bmatrix}\breve{C}(\widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_4\mathfrak{q}_{2m-1}) + \breve{C}(\widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\zeta_3\xi_1) \\ +\breve{C}(\widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\xi)\end{Bmatrix}$$
$$= k_1\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\zeta_4\mathfrak{q}_{2m-1})\}$$
$$+k_2\{\breve{C}(\widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\xi)\} + k_2\{\breve{C}(\widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\zeta_4\mathfrak{q}_{2m-1})\}$$
$$+k_3\begin{Bmatrix}\breve{C}(\widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_4\mathfrak{q}_{2m-1}) + \breve{C}(\widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\xi) \\ +\breve{C}(\widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\zeta_2\mathfrak{q}_{2m-1}, \widetilde{\wp}\xi)\end{Bmatrix}$$
$$= (k_2 + k_3)\{\breve{C}(\widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\xi)\} + k_2\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\}$$
$$+k_3\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\} + k_3\{(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\}$$
$$\leq (k_1 + k_2 + k_3)\{\breve{C}(\widetilde{\wp}\xi_1, \widetilde{\wp}\xi_1, \widetilde{\wp}\xi)\} < \breve{C}(\widetilde{\wp}\zeta_1, \widetilde{\wp}\zeta_1, \widetilde{\wp}\xi)$$

which contradict to our given hypothesis.

Indeed, we implies that

$$\breve{C}(\widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\xi) = 0 \text{ iff } \widetilde{\wp}\zeta_1\xi_1 = \widetilde{\wp}\xi. \tag{3.3}$$

Although, $\zeta_1(\widetilde{\mathfrak{A}}) \subset \zeta_4(\widetilde{\mathfrak{A}})$, then there exist $\xi_2 \in \widetilde{\mathfrak{A}}$ in such a way that $\widetilde{\wp}\zeta_4\xi_2 = \widetilde{\wp}\xi$.

Moreover, we claim that $\widetilde{\wp}\zeta_2\xi_2 = \widetilde{\wp}\xi$. Since,

$$\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\zeta_2\xi_2) = \breve{C}(\widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_2\xi_2)$$
$$\leq k_1(\widetilde{\wp}\zeta_3\xi_1, \widetilde{\wp}\zeta_3\xi_1, \widetilde{\wp}\zeta_4\xi_2)$$
$$+ k_2\{\breve{C}(\widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_1\xi_1, \widetilde{\wp}\zeta_3\xi_1) + \breve{C}(\widetilde{\wp}\zeta_2\xi_2, \widetilde{\wp}\zeta_2\xi_2, \widetilde{\wp}\zeta_4\xi_2)\}$$

$+k_3\{\breve{C}(\widetilde{\wp}\zeta_1\xi_1,\widetilde{\wp}\zeta_1\xi_1,\widetilde{\wp}\zeta_4\xi_2)+\breve{C}(\widetilde{\wp}\zeta_1\xi_1,\widetilde{\wp}\zeta_1\xi_1,\widetilde{\wp}\zeta_3\xi_1)\}$
$=k_1\{\breve{C}(\widetilde{\wp}\xi,\widetilde{\wp}\xi,\widetilde{\wp}\xi)\}+k_2\{\breve{C}(\widetilde{\wp}\xi,\widetilde{\wp}\xi,\widetilde{\wp}\xi)\}+k_2\{\breve{C}(\widetilde{\wp}\zeta_2\xi_2,\widetilde{\wp}\zeta_2\xi_2,\widetilde{\wp}\xi)\}$
$+k_3\{\breve{C}(\widetilde{\wp}\xi,\widetilde{\wp}\xi,\widetilde{\wp}\xi)\}+k_3\{\breve{C}(\widetilde{\wp}\xi,\widetilde{\wp}\xi,\widetilde{\wp}\xi)\}$
$=k_2\{\breve{C}(\widetilde{\wp}\zeta_2\xi_2,\widetilde{\wp}\zeta_2\xi_2,\widetilde{\wp}\xi)\}.$

Thus, one can have
$(1-k_2)\{\breve{C}(\widetilde{\wp}\xi,\widetilde{\wp}\xi,\widetilde{\wp}\zeta_2\xi_2)\}\leq 0,$
again, which is a contradiction. Therefore, we get

$$\breve{C}(\widetilde{\wp}\xi,\widetilde{\wp}\xi,\widetilde{\wp}\zeta_2\xi_2)=0 \text{ iff } \widetilde{\wp}\xi=\widetilde{\wp}\zeta_4\xi_2. \quad (3.4)$$

Through equations (3.3) and (3.4), we deduce that

$$\widetilde{\wp}\xi=\widetilde{\wp}\zeta_1\xi_1=\widetilde{\wp}\zeta_3\xi_1=\widetilde{\wp}\zeta_2\xi_2=\widetilde{\wp}\zeta_4\xi_2.$$

Meanwhile, pairs (ζ_1,ζ_3) and (ζ_2,ζ_4) are (IT)-commuting,
$\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\widetilde{\wp}\xi_1)=\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_3\widetilde{\wp}\xi_1)$
$=\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_2\widetilde{\wp}\xi_2)\leq k_1\{\breve{C}(\zeta_3\zeta_1\widetilde{\wp}\zeta_1,\zeta_3\zeta_1\widetilde{\wp}\zeta_1,\zeta_4\widetilde{\wp}\xi_2)\}$
$+k_2\{\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\zeta_1,\zeta_1\zeta_1\widetilde{\wp}\zeta_1,\zeta_3\zeta_1\widetilde{\wp}\zeta_1)+\breve{C}(\zeta_2\widetilde{\wp}\xi_2,\zeta_2\widetilde{\wp}\xi_2,\zeta_4\widetilde{\wp}\xi_2)\}$
$+k_3\{\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\zeta_1,\zeta_1\zeta_1\widetilde{\wp}\zeta_1,\zeta_4\widetilde{\wp}\xi_2)+\breve{C}(\zeta_2\widetilde{\wp}\xi_2,\zeta_2\widetilde{\wp}\xi_2,\zeta_3\zeta_1\widetilde{\wp}\zeta_1)\}$
$=k_1\{\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\widetilde{\wp}\xi_1)\}$
$+k_2\{\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1)+\breve{C}(\widetilde{\wp}\zeta_1\xi_1,\widetilde{\wp}\zeta_1\xi_1,\widetilde{\wp}\zeta_1\xi_1)\}$
$+k_3\{\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\widetilde{\wp}\xi_1)+\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\widetilde{\wp}\xi_1)\}$
$=(k_1+2k_3)\{\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\widetilde{\wp}\xi_1)\}$

i.e., $(1-k_1-2k_3)\{\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1,\widetilde{\wp}\zeta_1\xi_1)\}\leq 0,$

again, we arrive at contradiction. This means that

$$\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\widetilde{\wp}\xi_1)=0 \text{ and } \zeta_1\zeta_1\widetilde{\wp}\xi_1=\zeta_1\widetilde{\wp}\xi_1=\widetilde{\wp}\xi$$

(3.5)

Thus, $\zeta_1\widetilde{\wp}\xi_1=\widetilde{\wp}\xi$ is a CFP for self-maps ζ_1 and ζ_3.

Equivantenly,
$\breve{C}(\zeta_2\zeta_2\widetilde{\wp}\xi_2,\zeta_2\zeta_2\widetilde{\wp}\xi_2,\zeta_2\widetilde{\wp}\xi_2)=\breve{C}(\zeta_2\zeta_2\widetilde{\wp}\xi_2,\zeta_2\zeta_2\widetilde{\wp}\xi_2,\zeta_4\widetilde{\wp}\xi_2)$
$=\breve{C}(\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_1\zeta_1\widetilde{\wp}\xi_1,\zeta_2\zeta_2\widetilde{\wp}\xi_2)\leq k_1\{\breve{C}(\zeta_3\zeta_1\widetilde{\wp}\xi_1,\zeta_3\zeta_1\widetilde{\wp}\xi_1,\widetilde{\wp}\zeta_4\zeta_2\xi_2)\}$
$+k_2\{\breve{C}(\widetilde{\wp}\zeta_1\zeta_1\xi_1,\widetilde{\wp}\zeta_1\zeta_1\xi_1,\widetilde{\wp}\zeta_3\zeta_1\xi_1)+\breve{C}(\widetilde{\wp}\zeta_2\zeta_2\xi_2,\widetilde{\wp}\zeta_2\zeta_2\xi_2,\widetilde{\wp}\zeta_4\zeta_2\xi_2)\}$
$+k_3\{\breve{C}(\widetilde{\wp}\zeta_1\zeta_1\xi_1,\widetilde{\wp}\zeta_1\zeta_1\xi_1,\widetilde{\wp}\zeta_4\zeta_2\xi_2)+\breve{C}(\widetilde{\wp}\zeta_2\zeta_2\xi_2,\widetilde{\wp}\zeta_2\zeta_2\xi_2,\widetilde{\wp}\zeta_3\zeta_1\xi_2)\}$
$=k_1\{\breve{C}(\zeta_3\widetilde{\wp}\zeta_1\xi_1,\zeta_3\widetilde{\wp}\zeta_1\xi_1,\widetilde{\wp}\zeta_2\zeta_2\xi_2)\}$
$+k_2\{\breve{C}(\widetilde{\wp}\zeta_1\zeta_1\xi_1,\widetilde{\wp}\zeta_1\zeta_1\xi_1,\widetilde{\wp}\zeta_1\zeta_1\xi_1)+\breve{C}(\widetilde{\wp}\zeta_2\zeta_2\xi_2,\widetilde{\wp}\zeta_2\zeta_2\xi_2,\widetilde{\wp}\zeta_2\zeta_2\xi_2)\}$
$+k_3\{\breve{C}(\widetilde{\wp}\zeta_1\zeta_1\xi_1,\widetilde{\wp}\zeta_1\zeta_1\xi_1,\widetilde{\wp}\zeta_2\zeta_2\xi_2)+\breve{C}(\widetilde{\wp}\zeta_1\zeta_1\xi_2,\widetilde{\wp}\zeta_1\zeta_1\xi_2,\widetilde{\wp}\zeta_1\zeta_1\xi_1)\}$

$=\{1-(k_1+2k_3)\}\{\breve{C}(\widetilde{\wp}\zeta_1\zeta_1\xi_1,\widetilde{\wp}\zeta_1\zeta_1\xi_1,\widetilde{\wp}\zeta_2\zeta_2\xi_2)\}\leq 0,$

it is also a contradiction. Therefore,

$$\breve{C}(\zeta_2\zeta_2\widetilde{\wp}\xi_2,\zeta_2\zeta_2\widetilde{\wp}\xi_2,\zeta_2\widetilde{\wp}\xi_2)=0 \text{ iff } \zeta_2\zeta_2\widetilde{\wp}\xi_2=\zeta_2\widetilde{\wp}\xi_2. \quad (3.6)$$

Implies that, $\widetilde{\wp}\xi$ is a CFP of self-maps ζ_2 and ζ_4.

Along with, we have

$\breve{C}(\zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\widetilde{\wp}\xi_2) \leq \breve{C}(\zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\zeta_4\widetilde{\wp}\xi_2, q_{2m}) + \breve{C}(q_{2m}, q_{2m}, \zeta_4\widetilde{\wp}\xi_2)$
$= \breve{C}(\zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\zeta_4\widetilde{\wp}\xi_2, q_{2m}) + \breve{C}(\widetilde{\wp}\zeta_1 p_{2m}, \widetilde{\wp}\zeta_1 p_{2m}, \zeta_2\widetilde{\wp}\xi_2)$
$\leq \breve{C}(\zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\zeta_4\widetilde{\wp}\xi_2, q_{2m}) + k_1\{\breve{C}(\zeta_3\widetilde{\wp}p_{2m}, \zeta_3\widetilde{\wp}p_{2m}, \zeta_4\widetilde{\wp}\xi_2)\}$
$+k_2\{\breve{C}(\widetilde{\wp}\zeta_1 p_{2m}, \widetilde{\wp}\zeta_1 p_{2m}, \zeta_3\widetilde{\wp}p_{2m}) + \breve{C}(\zeta_2\widetilde{\wp}\xi_2, \zeta_2\widetilde{\wp}\xi_2, \zeta_4\widetilde{\wp}\xi_2)\}$
$+k_3\{\breve{C}(\widetilde{\wp}\zeta_1 p_{2m}, \widetilde{\wp}\zeta_1 p_{2m}, \zeta_4\widetilde{\wp}\xi_2) + \breve{C}(\zeta_2\widetilde{\wp}\xi_2, \zeta_2\widetilde{\wp}\xi_2, \zeta_3\widetilde{\wp}p_{2m})\}$
$= \breve{C}(\zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\zeta_4\widetilde{\wp}\xi_2, \widetilde{\wp}\xi) + k_1\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\} + k_2\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\}$
$\quad +k_2\{(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\} + k_3\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\} + k_3\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\}.$

On using normality condition, one can get

$$\|\breve{C}(\zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\widetilde{\wp}\xi_2)\|$$
$$\leq Y\|\breve{C}(\zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\zeta_4\widetilde{\wp}\xi_2, \widetilde{\wp}\xi)\| = Y\|\breve{C}(\zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\widetilde{\wp}\xi_2)\|.$$

So, that

$(1-Y)\{\breve{C}(\zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\widetilde{\wp}\xi_2)\} \leq 0.$

Thus,

$$\breve{C}(\zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\zeta_4\widetilde{\wp}\xi_2, \zeta_4\zeta_5\widetilde{\wp}\xi_2) = 0 \text{ and } \zeta_4\zeta_4\widetilde{\wp}\xi_2 = \zeta_4\widetilde{\wp}\xi_2. \qquad (3.7)$$

Hence, $\widetilde{\wp}\xi$ is a FP of $\zeta_4\widetilde{\wp}\xi_2$.

Although, we have

$\breve{C}(\zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\widetilde{\wp}\xi_1)$
$\qquad \leq \breve{C}(\zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\zeta_3\widetilde{\wp}\xi_1, q_{2m+1}) + \breve{C}(q_{2m+1}, q_{2m+1}, \zeta_3\widetilde{\wp}\xi_1)$
$= \breve{C}(\zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\zeta_3\widetilde{\wp}\xi_1, q_{2m+1}) + \breve{C}(\widetilde{\wp}\zeta_2 q_{2m+1}, \widetilde{\wp}\zeta_2 q_{2m+1}, \zeta_1\widetilde{\wp}\xi_1)$
$= \breve{C}(\zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\zeta_3\widetilde{\wp}\xi_1, q_{2m+1}) + \breve{C}(\zeta_1\widetilde{\wp}\xi_1, \zeta_1\widetilde{\wp}\xi_1, \widetilde{\wp}\zeta_2 p_{2m+1})$
$\leq \breve{C}(\zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\zeta_3\widetilde{\wp}\xi_1, q_{2m+1}) + k_1\{\breve{C}(\zeta_3\widetilde{\wp}\xi_1, \zeta_3\widetilde{\wp}\xi_1, \zeta_4\widetilde{\wp}p_{2m+1})\}$
$+k_2\{\breve{C}(\zeta_1\widetilde{\wp}\xi_1, \zeta_1\widetilde{\wp}\xi_1, \zeta_3\widetilde{\wp}\xi_1) + \breve{C}(\widetilde{\wp}\zeta_2 p_{2m+1}, \widetilde{\wp}\zeta_2 p_{2m+1}, \zeta_4\widetilde{\wp}p_{2m+1})\}$
$+k_3\{\breve{C}(\zeta_1\widetilde{\wp}\xi_1, \zeta_1\widetilde{\wp}\xi_1, \zeta_4\widetilde{\wp}p_{2m+1}) + \breve{C}(\widetilde{\wp}\zeta_2 p_{2m+1}, \widetilde{\wp}\zeta_2 p_{2m+1}, \zeta_3\widetilde{\wp}\xi_1)\}$
$= \breve{C}(\zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\zeta_3\widetilde{\wp}\xi_1, \widetilde{\wp}\xi) + k_1\{\breve{C}(\zeta_3\widetilde{\wp}\xi_1, \zeta_3\widetilde{\wp}\xi_1, \widetilde{\wp}\xi)\} + k_2\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\}$
$+k_2\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\} + k_3\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\} + k_3\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\}$
$= \breve{C}(\zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\widetilde{\wp}\xi_1) + k_1\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\}$
$\quad +2k_2\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\} + 2k_3\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\}.$

Simplifying further, we have

$\breve{C}(\zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\widetilde{\wp}\xi_1) \leq Y\|\breve{C}(\zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\widetilde{\wp}\xi_1)\|$

i.e., $(1-Y)\{\breve{C}(\zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\widetilde{\wp}\xi_1)\} \leq 0.$

Implies that,

$$\breve{C}(\zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\zeta_3\widetilde{\wp}\xi_1, \zeta_3\widetilde{\wp}\xi_1) = 0 \text{ iff } \zeta_3\zeta_3\widetilde{\wp}\xi_1 = \zeta_3\widetilde{\wp}\xi_1. \qquad (3.8)$$

Therefore, $\widetilde{\wp}\xi$ is a FP of $\zeta_3\widetilde{\wp}\xi_1$. Moreover, as

$\breve{C}(\widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\xi) = \breve{C}(\widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\widetilde{\wp}\xi, q_{2m}) + \breve{C}(q_{2m}, q_{2m}, \widetilde{\wp}\xi)$
$+\breve{C}(\widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\widetilde{\wp}\xi, q_{2m}) + \breve{C}(\widetilde{\wp}\zeta_1 p_{2m}, \widetilde{\wp}\zeta_1 p_{2m}, \zeta_2\widetilde{\wp}\xi_2)$
$\leq \breve{C}(\widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\widetilde{\wp}\xi, q_{2m}) + k_1\{\breve{C}(\widetilde{\wp}\zeta_3 p_{2m}, \widetilde{\wp}\zeta_3 p_{2m}, \zeta_4\widetilde{\wp}\xi_2)\}$

$+k_2\{\breve{C}(\widetilde{\wp}\zeta_1 p_{2m}, \widetilde{\wp}\zeta_1 p_{2m}, \zeta_3\widetilde{\wp}p_{2m}) + \breve{C}(\zeta_2\widetilde{\wp}\xi_2, \zeta_2\widetilde{\wp}\xi_2, \zeta_4\widetilde{\wp}\xi_2)\}$

$+k_3\{\breve{C}(\widetilde{\wp}\zeta_1 p_{2m}, \widetilde{\wp}\zeta_1 p_{2m}, \zeta_4\widetilde{\wp}\xi_2) + \breve{C}(\zeta_2\widetilde{\wp}\xi_2, \zeta_2\widetilde{\wp}\xi_2, \zeta_3\widetilde{\wp}p_{2m})\}.$

Thus,

$$\|\breve{C}(\widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\xi)\|$$
$$\leq Y\|\breve{C}(\widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\xi) + (k_1 + 2k_2 + 2k_3)\{\breve{C}(\widetilde{\wp}\xi, \widetilde{\wp}\xi, \widetilde{\wp}\xi)\}\|$$

i.e., $(1 - Y)\breve{C}(\widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\xi) \leq 0$,

which is a contradiction. Implies that

$$\breve{C}(\widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\widetilde{\wp}\xi, \widetilde{\wp}\xi) = 0 \text{ iff } \widetilde{\wp}\widetilde{\wp}\xi = \widetilde{\wp}\xi \qquad (3.9)$$

Therefore, $\widetilde{\wp}\xi$ is FP of $\widetilde{\wp}\widetilde{\wp}\xi$.

Hence, we conclude from equations (3.5) to (3.9) that $\widetilde{\wp}\xi = \xi$ i.e., ξ is the CFP of five self- maps ζ_1, ζ_2, ζ_3, ζ_4 and $\widetilde{\wp}$.

Uniqueness: To prove it, assume that u_σ be another CFP of $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ and $\widetilde{\wp}$. So, we have

$\breve{C}(\xi, u_\sigma) = \breve{C}(\zeta_1\xi_1, \zeta_1\xi_1, \zeta_2 u_\sigma)$

$\leq k_1\{\breve{C}(\zeta_3\xi_1, \zeta_3\xi_1, \zeta_4 u_\sigma)\} + k_2\{\breve{C}(\zeta_1\xi_1, \zeta_1\xi_1, \zeta_3\xi_1)\} + k_2\{\breve{C}(\zeta_2 u_\sigma, \zeta_2 u_\sigma, \zeta_4 u_\sigma)\}$

$+k_3\{\breve{C}(\zeta_1\xi_1, \zeta_1\xi_1, \zeta_4 u_\sigma)\} + k_3\{\breve{C}(\zeta_2 u_\sigma, \zeta_2 u_\sigma, \zeta_3\xi_1)\}$

$= k_1\{\breve{C}(\xi, \xi, u_\sigma)\} + k_2\{\breve{C}(\xi, \xi, \xi)\} + k_2\{\breve{C}(u_\sigma, u_\sigma, u_\sigma)\}$

$+k_3\{(\xi, \xi, u_\sigma)\} + k_3\{\breve{C}(\xi, \xi, u_\sigma)\} + k_3\{\breve{C}(u_\sigma, u_\sigma, \xi)\}$

$\leq (k_1 + 2k_3)\{\breve{C}(\xi, \xi, u_\sigma)\} = (k_1 + 2k_2 + 2k_3)\{\breve{C}(\xi, \xi, u_\sigma)\} < \breve{C}(u_\sigma, u_\sigma, u_\sigma)$

Obviously, a contradiction to u_σ is another FP.

Hence, $\xi = u_\sigma$, i.e., uniqueness is established.

□

Theorem 3.2. Consider the five continuous self-maps $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ and $\widetilde{\wp}$ of a complete C_S-MS $(\mathfrak{U}, \breve{C})$ s.t. following holds:

($A^{3.2.1}$) $\zeta_2(\mathfrak{U}) \subset \zeta_3(\mathfrak{U})$, $\zeta_1(\mathfrak{U}) \subset \zeta_4(\mathfrak{U})$ and $\zeta_1(\mathfrak{U}) = \zeta_2(\mathfrak{U})$;

($A^{3.2.2}$) Map $\widetilde{\wp}$ is an injective and normal cone O with Y be normal constant;

($A^{3.2.3}$) $\zeta_1(\mathfrak{U})$ and $\zeta_2(\mathfrak{U})$ is a complete subspace of $\mathfrak{U}$, for every $\varpi, w \in \mathfrak{U}$ and for $\sum_{i=1}^{5} k_i < 1$,

$k_i \geq 0$ fulfils the following:

$\breve{C}(\widetilde{\wp}\zeta_1\varpi, \widetilde{\wp}\zeta_1\varpi, \widetilde{\wp}\zeta_2 w) \leq k_1\{\breve{C}(\widetilde{\wp}\zeta_3\varpi, \widetilde{\wp}\zeta_3\varpi, \widetilde{\wp}\zeta_4 w)\}$

$+k_2\{\breve{C}(\widetilde{\wp}\zeta_1\varpi, \widetilde{\wp}\zeta_1\varpi, \widetilde{\wp}\zeta_3 w)\} + k_3\{\breve{C}(\widetilde{\wp}\zeta_2 w, \widetilde{\wp}\zeta_2 w, \widetilde{\wp}\zeta_4 w)\}$

$+k_4\{\breve{C}(\widetilde{\wp}\zeta_1\varpi, \widetilde{\wp}\zeta_1\varpi, \widetilde{\wp}\zeta_4 w)\} + k_5\{\breve{C}(\widetilde{\wp}\zeta_2 w, \widetilde{\wp}\zeta_2 w, \widetilde{\wp}\zeta_3\varpi)\}.$

If (ζ_1, ζ_3) and (ζ_2, ξ_1) are Banach pairs, then self-maps $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ and $\widetilde{\wp}$ have a unique CFP.

Proof: Let $p_0 \in \tilde{\mathfrak{A}}$ an arbitrary point, although $\zeta_2(\tilde{\mathfrak{A}}) \subset \zeta_3(\tilde{\mathfrak{A}})$ and $\zeta_1(\tilde{\mathfrak{A}}) \subset \zeta_4(\tilde{\mathfrak{A}})$, then there survives $p_1, p_2 \in \tilde{\mathfrak{A}}$ s.t. $\zeta_1 p_0 = \zeta_4 p_1$ and $\zeta_2 p_1 = \zeta_3 p_2$ respectively. By iteration, we generate

$q_{2m} = \tilde{\wp}\zeta_1 p_{2m} = \tilde{\wp}\zeta_4 p_{2m+1}$

$q_{2m+1} = \tilde{\wp}\zeta_2 p_{2m+1} = \tilde{\wp}\zeta_3 p_{2m+2}$.

Obviously, which implies that

$$\breve{C}(q_{2m}, q_{2m}, q_{2m+1}) = \breve{C}(\tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_2 p_{2m+1})$$
$$\leq k_1\{\breve{C}(\tilde{\wp}\zeta_3 p_{2m}, \tilde{\wp}\zeta_3 p_{2m}, \tilde{\wp}\zeta_4 p_{2m+1})\}$$
$$+k_2\{\breve{C}(\tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_3 p_{2m})\} + k_3\{\breve{C}(\tilde{\wp}\zeta_2 p_{2m+1}, \tilde{\wp}\zeta_2 p_{2m+1}, \tilde{\wp}\zeta_4 p_{2m+1})\}$$
$$+k_4\{\breve{C}(\tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_1 p_{2m}, \tilde{\wp}\zeta_3 p_{2m+1})\} + k_5\{\breve{C}(\tilde{\wp}\zeta_2 p_{2m+1}, \tilde{\wp}\zeta_2 p_{2m+1}, \tilde{\wp}\zeta_4 p_{2m-1})\}$$
$$\leq k_1\{\breve{C}(q_{2m-1}, q_{2m-1}, q_{2m})\} + k_2\{\breve{C}(q_{2m}, q_{2m}, q_{2m-1})\}$$
$$+k_3\{\breve{C}(q_{2m+1}, q_{2m+1}, q_{2m})\} + k_4\{\breve{C}(q_{2m}, q_{2m}, q_{2m})\} + k_5\{\breve{C}(q_{2m+1}, q_{2m+1}, q_{2m})\}$$
$$= (k_1 + k_2 + k_5)\{\breve{C}(q_{2m}, q_{2m}, q_{2m-1})\} + (k_3 + k_5)\{\breve{C}(q_{2m+1}, q_{2m+1}, q_{2m})\}.$$

Simplifying further, we have

$$\breve{C}(q_{2m}, q_{2m}, q_{2m+1}) \leq \left(\frac{k_1+k_2+k_5}{k_3+k_5}\right)\{\breve{C}(q_{2m-1}, q_{2m-1}, q_{2m})\}.$$

Assume that, $\alpha = \frac{k_1+k_2+k_5}{k_3+k_5}$ and for every instance of $\alpha < 1$, it follows that

$$\breve{C}(q_{2m}, q_{2m}, q_{2m+1}) \leq \alpha\{\breve{C}(q_{2m-1}, q_{2m-1}, q_{2m}) \tag{3.10}$$

On comparing equation (3.10) with (3.2), one can conclude the proof from Theorem 3.1. □

Now, we illustrate some related corollaries induced from above theorems in this section:

Corollary 3.3. Consider the four continuous self-maps $\zeta_1, \zeta_2, \zeta_3$ and ζ_4 of a complete C_S-MS $(\tilde{\mathfrak{A}}, \breve{C})$ s.t. following holds:

($A^{3.3.1}$) $\zeta_2(\tilde{\mathfrak{A}}) \subset \zeta_3(\tilde{\mathfrak{A}})$, $\zeta_1(\tilde{\mathfrak{A}}) \subset \zeta_4(\tilde{\mathfrak{A}})$ and $\zeta_1(\tilde{\mathfrak{A}}) = \zeta_2(\tilde{\mathfrak{A}})$;

($A^{3.3.2}$) O be normal cone with Y be normal constant;

($A^{3.3.3}$) $\zeta_1(\tilde{\mathfrak{A}})$ and ζ_2 $(\tilde{\mathfrak{A}})$ is a complete subspace of $\tilde{\mathfrak{A}}$, for every $\varpi, w \in \tilde{\mathfrak{A}}$ and for $\sum_{i=1}^{5} k_i < 1$,

$k_i \geq 0$ fulfils the following:

$$\breve{C}(\zeta_1\varpi, \zeta_1\varpi, \zeta_2 w) \leq k_1\{\breve{C}(\zeta_3\varpi, \zeta_3\varpi, \zeta_4 w)\} + k_2\{\breve{C}(\zeta_1\varpi, \zeta_1\varpi, \zeta_3 w)\}$$
$$+k_3\{\breve{C}(\zeta_2 w, \zeta_2 w, \zeta_4 w)\} + k_4\{\breve{C}(\zeta_1\varpi, \zeta_4 w)\} + k_5\{\breve{C}(\zeta_2 w, \zeta_3\varpi)\}.$$

If (ζ_1, ζ_3) and (ζ_2, ξ_1) are Banach pairs, then $\zeta_1, \zeta_2, \zeta_3$ and ζ_4 have a unique CFP.

Proof: The proof can be done if $\tilde{\wp}$ is an identity map in Theorem 3.2, i.e., $\tilde{\wp} = I_{\varpi}$. □

Corollary 3.4. Consider the three continuous self-maps ζ_1, ζ_2 and $\tilde{\wp}$ of a complete C_S-MS $(\tilde{\mathfrak{A}}, \breve{C})$ s.t. following holds:

$(A^{3.4.1})$ $\tilde{\wp}$ is an injective map and normal cone O with Y be normal constant;
$(A^{3.4.2})$ $\zeta_1(\mathfrak{X})$ and $\zeta_2(\mathfrak{X})$ is a complete subspace of $\mathfrak{X}$, for every $\varpi, w \in \mathfrak{X}$ and for $\sum_{i=1}^{5} k_i < 1$,
$k_i \geq 0$ fulfils the following:

$$\breve{c}(\tilde{\wp}\zeta_1\varpi, \tilde{\wp}\zeta_1\varpi, \tilde{\wp}\zeta_2 w) \leq k_1\{\breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, \tilde{\wp}w)\} + k_2\{\breve{c}(\tilde{\wp}\zeta_1\varpi, \tilde{\wp}\zeta_1\varpi, \tilde{\wp}\varpi)\}$$

$$+k_3\{\breve{c}(\tilde{\wp}\zeta_2 w, \tilde{\wp}\zeta_2 w, \tilde{\wp}w)\} + k_4\{\breve{c}(\tilde{\wp}\zeta_1\varpi, \tilde{\wp}\zeta_1\varpi, \tilde{\wp}w)\} + k_5\{\breve{c}(\tilde{\wp}\zeta_2 w, \tilde{\wp}\zeta_2 w, \tilde{\wp}w)\}$$

.

If $(\zeta_1, \tilde{\wp})$ and $(\zeta_2, \tilde{\wp})$ are Banach pairs, then ζ_1, ζ_2 and $\tilde{\wp}$ have a unique CFP.

Proof: One can conclude the proof by Theorem 3.2 by setting $\zeta_1 = \zeta_3$ and $\zeta_2 = \zeta_4$. □

4. CFP THEOREMS IN COMPLETE C_S-MS USING CONTRACTIONS MAPS

In this section, firstly we defined the notion of contraction type-I & II maps in C_S-MS. Further, we prove the existence and uniqueness of CFP results for these contractive maps in C_S-MS.

Definition 4.1. Consider a normal cone O with Y be a normal constant and $(\mathfrak{X}, \breve{c})$ as a complete C_S-MS. Then two self-maps $\tilde{\wp}, \check{f}: \mathfrak{X} \to \mathfrak{X}$ have type-I contraction, if for every $\varpi, w \in \mathfrak{X}$ with $\varpi \neq w$ and for $k_1, k_2, k_3, k_4 \geq 0$, $k_1 + 2k_2 + k_3 + k_4 < 1$ satisfies the following condition:

$$\breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, \check{f}w) \leq k_1\{\breve{c}(\varpi, \varpi, w)\} + k_2\{\breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, \varpi) + \breve{c}(\check{f}w, \check{f}w, w)\}$$
$$+k_3\left\{\frac{\breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, \varpi)\breve{c}(\check{f}w, \check{f}w, \varpi) + \breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, w)\breve{c}(\check{f}w, \check{f}w, w)}{\breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, w) + \breve{c}(\check{f}w, \check{f}w, \varpi)}\right\}$$
$$+k_4\left\{\frac{\breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, \varpi)\breve{c}(\check{f}w, \check{f}w, w)}{\breve{c}(\varpi, \varpi, w) + \breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, w) + \breve{c}(\check{f}w, \check{f}w, \varpi)}\right\}. \tag{4.1}$$

Definition 4.2. Consider a normal cone O with Y be a normal constant and $(\mathfrak{X}, \breve{c})$ as a complete C_S-MS. Then two self-maps $\tilde{\wp}, \check{f}: \mathfrak{X} \to \mathfrak{X}$ have type-II contraction, if for every $\varpi, w \in \mathfrak{X}$ with $\varpi \neq w$ and for $k_1, k_2, k_3, k_4 \geq 0$, $k_1 + 2k_2 + k_3 + k_4 < 1$ satisfies the following condition:

$$\breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, \check{f}w) \leq k_1\{\breve{c}(\varpi, \varpi, w)\} + k_2\{\breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, w) + \breve{c}(\check{f}w, \check{f}w, \varpi)\}$$
$$+k_3\left\{\frac{\breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, \varpi)\breve{c}(\check{f}w, \check{f}w, \varpi) + \breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, w)\breve{c}(\check{f}w, \check{f}w, w)}{\breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, w) + \breve{c}(\check{f}w, \check{f}w, \varpi)}\right\}$$
$$+k_4\left\{\frac{\breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, \varpi)\breve{c}(\check{f}w, \check{f}w, w)}{\breve{c}(\varpi, \varpi, w) + \breve{c}(\tilde{\wp}\varpi, \tilde{\wp}\varpi, w) + \breve{c}(\check{f}w, \check{f}w, \varpi)}\right\} \tag{4.2}$$

Theorem 4.3. Consider a normal cone O with Y be a normal constant and $(\tilde{\mathfrak{A}}, \breve{c})$ as a complete C_S-MS with two self-maps $\tilde{\wp}, \check{f}: \tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$ having type-I contraction. Then, self-maps $\tilde{\wp}$ and $\check{f}$ have a unique CFP in $\tilde{\mathfrak{A}}$. Moreover, any FP of $\tilde{\wp}$ admits a FP of $\check{f}$ and vice-versa.

Proof: Suppose an arbitrary element $p_0 \in \tilde{\mathfrak{A}}$. Let $\{p_m\}$ be a sequence from $\tilde{\mathfrak{A}}$ in such a way that by induction process for any $m \geq 0$: $p_{2m+1} = \tilde{\wp} p_{2m}$ and $p_{2m+2} = \check{f} p_{2m+1}$.

Then from equation (4.1), we imply that

$$\breve{c}(p_{2m+1}, p_{2m+1}, p_{2m+2}) = \breve{c}(\tilde{\wp} p_{2m}, \tilde{\wp} p_{2m}, \check{f} p_{2m+1})$$

$$\leq k_1\{\breve{c}(p_{2m}, p_{2m}, p_{2m+1})\} + k_2\{\breve{c}(\tilde{\wp} p_{2m}, \tilde{\wp} p_{2m}, p_{2m}) + \breve{c}(\check{f} p_{2m+1}, \check{f} p_{2m+1}, p_{2m+1})\}$$

$$+k_3\left\{\frac{\breve{c}(\tilde{\wp} p_{2m}, \tilde{\wp} p_{2m}, p_{2m})\breve{c}(\check{f} p_{2m+1}, \check{f} p_{2m+1}, p_{2m}) + \breve{c}(\tilde{\wp} p_{2m}, \tilde{\wp} p_{2m}, p_{2m+1})\breve{c}(\check{f} p_{2m+1}, \check{f} p_{2m+1}, p_{2m+1})}{\breve{c}(\tilde{\wp} p_{2m}, \tilde{\wp} p_{2m}, p_{2m+1}) + \breve{c}(\check{f} p_{2m+1}, \check{f} p_{2m+1}, p_{2m})}\right\}$$

$$+k_4\left\{\frac{\breve{c}(\tilde{\wp} p_{2m}, \tilde{\wp} p_{2m}, p_{2m})\breve{c}(\check{f} p_{2m+1}, \check{f} p_{2m+1}, p_{2m+1})}{\breve{c}(p_{2m}, p_{2m}, p_{2m+1}) + \breve{c}(\tilde{\wp} p_{2m}, \tilde{\wp} p_{2m}, p_{2m+1}) + \breve{c}(\check{f} p_{2m+1}, \check{f} p_{2m+1}, p_{2m})}\right\}$$

$$= (k_1 + k_2 + k_3 + k_4)\{\breve{c}(p_{2m}, p_{2m}, p_{2m+1})\} + k_2\{\breve{c}(p_{2m+1}, p_{2m+1}, p_{2m+2})\}$$

i.e., $\breve{c}(p_{2m+1}, p_{2m+1}, p_{2m+2}) \leq \alpha\{\breve{c}(p_{2m}, p_{2m}, p_{2m+1})\}$

where $\alpha = \frac{k_1+k_2+k_3+k_4}{1-k_2} < 1$.

Repeating the above procedure, one can get

$\breve{c}(p_{2m+2}, p_{2m+2}, p_{2m+3}) \leq \alpha\{\breve{c}(p_{2m+1}, p_{2m+1}, p_{2m+2})\}$.

Moreover, for all m, we deduce that

$\breve{c}(p_{m+1}, p_{m+1}, p_{m+2}) \leq \alpha\{\breve{c}(p_m, p_m, p_{m+1})\} \leq \cdots \leq \alpha^{m+1}\{\breve{c}(p_0, p_0, p_1)\}$.

Now, for any $n > m$, where $m, n \in \mathbb{N}$, we can have

$$\breve{c}(p_m, p_m, p_n) \leq \breve{c}(p_n, p_n, p_{n+1}) + \breve{c}(p_{n+1}, p_{n+1}, p_{n+2}) + \cdots + \breve{c}(p_{m-1}, p_{m-1}, p_m)$$

$$\leq \{\alpha^{m+1} + \alpha^{m+2} + \cdots + \alpha^{n-1}\}\{\breve{c}(p_0, p_0, p_1)\}$$

$$\leq \left\{\frac{\alpha^m}{1-\alpha}\right\}\{\breve{c}(p_0, p_0, p_1)\}. \qquad (4.3)$$

Obviously, by using equation (4.3) and normality together, imply that

$$\|\breve{c}(p_m, p_m, p_n)\| \leq Y\left\{\frac{\alpha^m}{1-\alpha}\right\}\|\breve{c}(p_0, p_0, p_1)\|$$

i.e., $\breve{c}(p_m, p_m, p_n)$ approaches to 0 as n, m tends to ∞.

Finally, $\{p_m\}$ is a Cauchy sequence in $\tilde{\mathfrak{A}}$ and since it is complete C_S-MS, therefore $p_m \to \varpi_*$ as $m \to \infty$ for some $\varpi_* \in \tilde{\mathfrak{A}}$.

This means from equation (4.1) and lemma 2.7, we can imply that

$\breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*) = \breve{C}(\varpi_*,\varpi_*,\check{f}\varpi_*) \leq \breve{C}(\varpi_*,\varpi_*,p_{2m+1}) + \breve{C}(p_{2m+1},p_{2m+1},\check{f}\varpi_*)$
$= \breve{C}(\varpi_*,\varpi_*,p_{2m+1}) + \breve{C}(\tilde{\wp}p_{2m},\tilde{\wp}p_{2m},\check{f}\varpi_*)$
$\leq \breve{C}(\varpi_*,\varpi_*,p_{2m+1}) + k_1\{\breve{C}(p_{2m},p_{2m},\varpi_*)\} + k_2\begin{Bmatrix}\breve{C}(p_{2m+1},p_{2m+1},p_{2m}) \\ +\breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)\end{Bmatrix}$
$+k_3\left\{\frac{\breve{C}(p_{2m+1},p_{2m+1},p_{2m})\breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,p_{2m}) + \breve{C}(p_{2m+1},p_{2m+1},\varpi_*)\breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}{\breve{C}(p_{2m+1},p_{2m+1},\varpi_*) + \breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,p_{2m})}\right\}$
$+k_4\left\{\frac{\breve{C}(p_{2m+1},p_{2m+1},p_{2m})\breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}{\breve{C}(p_{2m},p_{2m},\varpi_*) + \breve{C}(p_{2m+1},p_{2m+1},\varpi_*) + \breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,p_{2m})}\right\}.$

Simplifying further, we have

$$\breve{C}(\varpi_*,\varpi_*,\check{f}\varpi_*) \leq \left\{\frac{1}{1-k_2}\right\}\begin{bmatrix}\breve{C}(\varpi_*,\varpi_*,p_{2m+1}) + k_1\{\breve{C}(p_{2m},p_{2m},\varpi_*)\} \\ +k_2\{\breve{C}(p_{2m+1},p_{2m+1},p_{2m})\} \\ +k_3\{\breve{C}(p_{2m+1},p_{2m+1},p_{2m})\} \\ +k_4\{\breve{C}(p_{2m+1},p_{2m+1},p_{2m})\}\end{bmatrix}.$$

(4.4)

Therefore, by using equation (4.4) and normality, we deduce that

$$\|\breve{C}(\varpi_*,\varpi_*,\check{f}\varpi_*)\| \leq Y\left\{\frac{1}{1-k_2}\right\}\begin{Bmatrix}\|\breve{C}(\varpi_*,\varpi_*,p_{m+1})\| \\ +k_1\|\breve{C}(p_{m+1},p_{m+1},\varpi_*)\| + k_2\|\breve{C}(p_m,p_m,p_{m+1})\| \\ +k_3\|\breve{C}(p_m,p_m,p_{m+1})\| + k_4\|\breve{C}(p_m,p_m,p_{m+1})\|\end{Bmatrix}.$$

(4.5)

By letting as $m \to \infty$ in equation (4.5) and lemma 2.7, we get
$\|\breve{C}(\varpi_*,\varpi_*,\check{f}\varpi_*)\| = 0 = \|\breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)\|$.

Thus,
$\check{f}\varpi_* = \varpi_*$.

Consequently, one can have
$\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*) = \breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\check{f}\varpi_*)$
$\leq k_1\{\breve{C}(\varpi_*,\varpi_*,\varpi_*)\} + k_2\{\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*) + \breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)\}$
$+k_3\left\{\frac{\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*) + \breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}{\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*) + \breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}\right\}$
$+k_4\left\{\frac{\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}{\breve{C}(\varpi_*,\varpi_*,\varpi_*) + \breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*) + \breve{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}\right\}.$

Simplifying further, we arrive at
$\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*) \leq k_2\{\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\}$.

By using the property of cone O and partial ordering on $\mathbb{B}$, we conclude that
$\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*) = 0$, i.e., $\tilde{\wp}\varpi_* = \varpi_*$.

Uniqueness: Assume that $\mathfrak{u}_\sigma$ be another CFP of maps $\tilde{\wp}$ and $\check{f}$. Then, by equation (4.1), we get

$$\breve{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma) = \breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\check{f}\mathrm{u}_\sigma)$$
$$\leq k_1\{\breve{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma)\} + k_2\{\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*) + \breve{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\mathrm{u}_\sigma)\}$$
$$+k_3\left\{\frac{\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\breve{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\varpi_*) + \breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\mathrm{u}_\sigma)\breve{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\mathrm{u}_\sigma)}{\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\mathrm{u}_\sigma) + \breve{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\varpi_*)}\right\}$$
$$+k_4\left\{\frac{\breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\breve{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\mathrm{u}_\sigma)}{\breve{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma) + \breve{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\mathrm{u}_\sigma) + \breve{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\varpi_*)}\right\}$$

i.e., $\breve{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma) \leq k_1\{\breve{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma)\}$.

Thus,

$\breve{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma) = 0$, i.e., $\varpi_* = \mathrm{u}_\sigma$.

Hence, the uniqueness of CFP is established.

□

Theorem 4.4. Consider a normal cone O with Y be a normal constant and $(\mathfrak{A},\breve{C})$ as a complete C_S-MS with two self-maps $\tilde{\wp},\check{f}:\mathfrak{A}\to\mathfrak{A}$ having type-II contraction. Then, self-maps $\tilde{\wp}$ and $\check{f}$ have a unique CFP in $\mathfrak{A}$. Along with, any FP of $\tilde{\wp}$ admits a FP of $\check{f}$ and vice-versa.

Proof: Assume an arbitrary element $p_0 \in \mathfrak{A}$. Consider $\{p_m\}$ be a sequence from $\mathfrak{A}$ s.t. for any $m \geq 0$: $p_{2m+1} = \tilde{\wp}p_{2m}$ and $p_{2m+2} = \check{f}p_{2m+1}$.Then, from equation (4.2), we have

$$\breve{C}(p_{2m+1},p_{2m+1},p_{2m+2}) = \breve{C}(\tilde{\wp}p_{2m},\tilde{\wp}p_{2m},\check{f}p_{2m+1})$$
$$\leq k_1\{\breve{C}(p_{2m},p_{2m},p_{2m+1})\}$$
$$+ k_2\{\breve{C}(\tilde{\wp}p_{2m},\tilde{\wp}p_{2m},p_{2m+1}) + \breve{C}(\check{f}p_{2m+1},\check{f}p_{2m+1},p_{2m})\}$$
$$+k_3\left\{\frac{\breve{C}(\tilde{\wp}p_{2m},\tilde{\wp}p_{2m},p_{2m})\breve{C}(\check{f}p_{2m+1},\check{f}p_{2m+1},p_{2m}) + \breve{C}(\tilde{\wp}p_{2m},\tilde{\wp}p_{2m},p_{2m+1})\breve{C}(\check{f}p_{2m+1},\check{f}p_{2m+1},p_{2m+1})}{\breve{C}(\tilde{\wp}p_{2m},\tilde{\wp}p_{2m},p_{2m+1}) + \breve{C}(\check{f}p_{2m+1},\check{f}p_{2m+1},p_{2m})}\right\}$$
$$+k_4\left\{\frac{\breve{C}(\tilde{\wp}p_{2m},\tilde{\wp}p_{2m},p_{2m})\breve{C}(\check{f}p_{2m+1},\check{f}p_{2m+1},p_{2m+1})}{\breve{C}(p_{2m},p_{2m},p_{2m+1}) + \breve{C}(\tilde{\wp}p_{2m},\tilde{\wp}p_{2m},p_{2m+1}) + \breve{C}(\check{f}p_{2m+1},\check{f}p_{2m+1},p_{2m})}\right\}$$
$$= (k_1+k_2+k_3+k_4)\{\breve{C}(p_{2m},p_{2m},p_{2m+1})\} + k_2\{\breve{C}(p_{2m+1},p_{2m+1},p_{2m+2})\}$$

i.e., $\breve{C}(p_{2m+1},p_{2m+1},p_{2m+2}) \leq \alpha\{\breve{C}(p_{2m},p_{2m},p_{2m+1})\}$

where $\alpha = \frac{k_1+k_2+k_3+k_4}{1-k_2} < 1$. Likewise, one can deduce that

$$\breve{C}(p_{2m+2},p_{2m+2},p_{2m+3}) \leq \alpha\{\breve{C}(p_{2m+1},p_{2m+1},p_{2m+2})\}$$

Moreover, for any m, one can have

$$\breve{C}(p_{m+1},p_{m+1},p_{m+2}) \leq \alpha\{\breve{C}(p_m,p_m,p_{m+1})\} \leq \cdots \leq \alpha^{m+1}\{\breve{C}(p_0,p_0,p_1)\}.$$

Now, for all $n > m$, where m, n are positive integers, we get

$$\breve{C}(p_m, p_m, p_n) \leq \breve{C}(p_n, p_n, p_{n+1}) + \breve{C}(p_{n+1}, p_{n+1}, p_{n+2}) + \cdots + \breve{C}(p_{m-1}, p_{m-1}, p_m)$$
$$\leq \{\alpha^{m+1} + \alpha^{m+2} + \cdots + \alpha^{n-1}\}\{\breve{C}(p_0, p_0, p_1)\}$$
$$\leq \left\{\frac{\alpha^m}{1-\alpha}\right\}\{\breve{C}(p_0, p_0, p_1)\}. \quad (4.6)$$

Clearly, by using equation (4.6) and normality together, imply that

$$\|\breve{C}(p_m, p_m, p_n)\| \leq Y\left\{\frac{\alpha^m}{1-\alpha}\right\}\|\breve{C}(p_0, p_0, p_1)\|,$$

i.e., $\breve{C}(p_m, p_m, p_n)$ approaches to 0 as m, n tends to ∞.

Therefore, $\{p_m\}$ is a Cauchy sequence in $\mathfrak{U}$ and since it is complete C_S-MS, therefore $p_m \to \varpi_*$ as $m \to \infty$, for some $\varpi_* \in \breve{\mathfrak{U}}$.

This means from equation (4.2) and lemma 2.7, we can imply that

$$\breve{C}(\check{f}\varpi_*, \check{f}\varpi_*, \varpi_*) = \breve{C}(\varpi_*, \varpi_*, \check{f}\varpi_*) \leq \breve{C}(\varpi_*, \varpi_*, p_{2m+1}) + \breve{C}(p_{2m+1}, p_{2m+1}, \check{f}\varpi_*)$$
$$= \breve{C}(\varpi_*, \varpi_*, p_{2m+1}) + \breve{C}(\tilde{\wp}p_{2m}, \tilde{\wp}p_{2m}, \check{f}\varpi_*)$$
$$\leq \breve{C}(\varpi_*, \varpi_*, p_{2m+1}) + k_1\{\breve{C}(p_{2m}, p_{2m}, \varpi_*)\} + k_2\{\breve{C}(p_{2m+1}, p_{2m+1}, \varpi_*) +$$
$$\breve{C}(\check{f}\varpi_*, \check{f}\varpi_*, p_{2m})\}$$
$$+k_3\left\{\frac{\breve{C}(p_{2m+1}, p_{2m+1}, p_{2m})\breve{C}(\check{f}\varpi_*, \check{f}\varpi_*, p_{2m}) + \breve{C}(p_{2m+1}, p_{2m+1}, \varpi_*)\breve{C}(\check{f}\varpi_*, \check{f}\varpi_*, \varpi_*)}{\breve{C}(p_{2m+1}, p_{2m+1}, \varpi_*) + \breve{C}(\check{f}\varpi_*, \check{f}\varpi_*, p_{2m})}\right\}$$
$$+k_4\left\{\frac{\breve{C}(p_{2m+1}, p_{2m+1}, p_{2m})\breve{C}(\check{f}\varpi_*, \check{f}\varpi_*, \varpi_*)}{\breve{C}(p_{2m}, p_{2m}, \varpi_*) + \breve{C}(p_{2m+1}, p_{2m+1}, \varpi_*) + \breve{C}(\check{f}\varpi_*, \check{f}\varpi_*, p_{2m})}\right\}.$$

Simplyfying further, we have

$$\breve{C}(\varpi_*, \varpi_*, \check{f}\varpi_*) \leq \left\{\frac{1}{1-k_2}\right\}\begin{bmatrix} \breve{C}(\varpi_*, \varpi_*, p_{2m+1}) + k_1\{\breve{C}(p_{2m}, p_{2m}, \varpi_*)\} \\ +k_2\{\breve{C}(\varpi_*, \varpi_*, p_{2m})\} + k_2\{\breve{C}(p_{2m+1}, p_{2m+1}, \varpi_*)\} \\ +k_3\{\breve{C}(p_{2m+1}, p_{2m+1}, p_{2m})\} \\ +k_4\{\breve{C}(p_{2m+1}, p_{2m+1}, p_{2m})\} \end{bmatrix}$$

(4.7)

Therefore, by using equation (4.7) and normality, we deduce that

$$\|\breve{C}(\varpi_*, \varpi_*, \check{f}\varpi_*)\| \leq Y\left\{\frac{1}{1-k_2}\right\}\left\{\begin{matrix} \|\breve{C}(\varpi_*, \varpi_*, p_{m+1})\| + k_1\|\breve{C}(p_{m+1}, p_{m+1}, \varpi_*)\| \\ +k_2\|\breve{C}(\varpi_*, \varpi_*, p_{m+1})\| + k_2\|\breve{C}(p_m, p_m, \varpi_*)\| \\ +k_3\|\breve{C}(p_m, p_m, p_{m+1})\| + k_4\|\breve{C}(p_m, p_m, p_{m+1})\| \end{matrix}\right\}$$

(4.8)

On considering limit as $m \to \infty$ in equation (4.8) and lemma 2.7, one can have

$$\|\breve{C}(\varpi_*, \varpi_*, \check{f}\varpi_*)\| = 0 = \|\breve{C}(\check{f}\varpi_*, \check{f}\varpi_*, \varpi_*)\|.$$

Thus,

$$\check{f}\varpi_* = \varpi_*.$$

Consequently, we can get

$$\breve{C}(\tilde{\wp}\varpi_*, \tilde{\wp}\varpi_*, \varpi_*) = \breve{C}(\tilde{\wp}\varpi_*, \tilde{\wp}\varpi_*, \check{f}\varpi_*)$$
$$\leq k_1\{\breve{C}(\varpi_*, \varpi_*, \varpi_*)\} + k_2\{\breve{C}(\tilde{\wp}\varpi_*, \tilde{\wp}\varpi_*, \varpi_*) + \breve{C}(\check{f}\varpi_*, \check{f}\varpi_*, \varpi_*)\}$$

$$+k_3\left\{\frac{\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\check{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)+\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\check{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}{\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)+\check{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}\right\}$$
$$+k_4\left\{\frac{\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\check{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}{\check{C}(\varpi_*,\varpi_*,\varpi_*)+\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)+\check{C}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}\right\}.$$

Simplifying further, we arrive at
$\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\leq k_2\{\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\}$.
By using the property of cone O and partial ordering on $\mathbb{B}$, we conclude that
$\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)=0$, i.e., $\tilde{\wp}\varpi_*=\varpi_*$.
Uniqueness: Assume that u_σ be another CFP of $\tilde{\wp}$ and $\check{f}$. Then, from equation (4.2), we get

$$\check{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma)=\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\check{f}\mathrm{u}_\sigma)\leq k_1\{\check{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma)\}+k_2\left\{\begin{matrix}\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\mathrm{u}_\sigma)\\+\check{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\varpi_*)\end{matrix}\right\}$$
$$+k_3\left\{\frac{\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\check{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\varpi_*)+\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\mathrm{u}_\sigma)\check{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\mathrm{u}_\sigma)}{\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\mathrm{u}_\sigma)+\check{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\varpi_*)}\right\}$$
$$+k_4\left\{\frac{\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\check{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\mathrm{u}_\sigma)}{\check{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma)+\check{C}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\mathrm{u}_\sigma)+\check{C}(\check{f}\mathrm{u}_\sigma,\check{f}\mathrm{u}_\sigma,\varpi_*)}\right\}.$$

i.e., $\check{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma)\leq\{k_1+2k_2\}\{\check{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma)\}$
Therefore, $\check{C}(\varpi_*,\varpi_*,\mathrm{u}_\sigma)=0$, i.e., $\varpi_*=\mathrm{u}_\sigma$.
Hence, uniqueness is established.
□

Followings are the two corollaries that arise from the theorems stated above in this section: **Corollary 4.5.** Consider a normal cone O with Y be normal constant and $(\mathfrak{A},\check{C})$ as a complete C_S-MS with maps $\tilde{\wp},\check{f}:\mathfrak{A}\to\mathfrak{A}$ having type-I contraction for some $m\in\mathbb{N}$. If for any $\varpi,w\in\mathfrak{A}$ with $\varpi\neq w$ and for $k_1,k_2,k_3\geq 0$, $2k_1+k_2+k_3<1$ satisfies the following condition:

$$\check{C}(\tilde{\wp}^m\varpi,\tilde{\wp}^m\varpi,\check{f}^m w)\leq k_1\{\check{C}(\varpi,\varpi,w)\}+k_2\{\check{C}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,\varpi)+\check{C}(\check{f}w,\check{f}w,w)\}$$
$$+k_3\left\{\frac{\check{C}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,\varpi)\check{C}(\check{f}w,\check{f}w,\varpi)+\check{C}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,w)\check{C}(\check{f}w,\check{f}w,w)}{\check{C}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,w)+\check{C}(\check{f}w,\check{f}w,\varpi)}\right\}$$
$$+k_4\left\{\frac{\check{C}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,\varpi)\check{C}(\check{f}w,\check{f}w,w)}{\check{C}(\varpi,\varpi,w)+\check{C}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,w)+\check{C}(\check{f}w,\check{f}w,\varpi)}\right\},\quad(4.9)$$

then self-maps $\tilde{\wp}$ and $\check{f}$ have a unique CFP in $\mathfrak{A}$.
Proof: Letting $\tilde{\wp}^m=\tilde{\wp}$, $\check{f}^m=\check{f}$ in equation (4.9), one can have proof using Theorem 4.3. □

Corollary 4.6. Consider a normal cone O with Y be normal constant and $(\mathfrak{A},\check{C})$ as a complete C_S-MS with maps $\tilde{\wp},\check{f}:\mathfrak{A}\to\mathfrak{A}$ having type-II contraction for some $m\in\mathbb{N}$.

If for any $\varpi, \omega \in \breve{\mathfrak{U}}$ with $\varpi \neq \omega$ and for $k_1, k_2, k_3 \geq 0$, $2k_1 + k_2 + k_3 < 1$ satisfies the following condition:

$$\breve{c}(\tilde{\wp}^m\varpi, \tilde{\wp}^m\varpi, \check{f}^m\omega) \leq k_1\{\breve{c}(\varpi,\varpi,\omega)\} + k_2\{\breve{c}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,\omega) + \breve{c}(\check{f}\omega,\check{f}\omega,\varpi)\}$$

$$+k_3\left\{\frac{\breve{c}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,\varpi)\breve{c}(\check{f}\omega,\check{f}\omega,\varpi) + \breve{c}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,\omega)\breve{c}(\check{f}\omega,\check{f}\omega,\omega)}{\breve{c}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,\omega) + \breve{c}(\check{f}\omega,\check{f}\omega,\varpi)}\right\}$$

$$+k_4\left\{\frac{\breve{c}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,\varpi)\breve{c}(\check{f}\omega,\check{f}\omega,\omega)}{\breve{c}(\varpi,\varpi,\omega) + \breve{c}(\tilde{\wp}\varpi,\tilde{\wp}\varpi,\omega) + \breve{c}(\check{f}\omega,\check{f}\omega,\varpi)}\right\}, \quad (4.10)$$

then self-maps $\tilde{\wp}$ and $\check{f}$ have a unique CFP in $\breve{\mathfrak{U}}$.

Proof: Letting $\tilde{\wp}^m = \tilde{\wp}$, $\check{f}^m = \check{f}$ in equation (4.10), one can have proof using Theorem 4.4. □

5. APPLICATIONS

In this section, we established some allied applications of our results for some periodic point on complete C_S-MS. Initially, we stated some linked definitions i.e., notion of property (P) stated as (refer: [2]):

(i) Assume a self-map $\tilde{\wp} : \breve{\mathfrak{U}} \to \breve{\mathfrak{U}}$, then map $\tilde{\wp}$ have property (P) if $Fix(\tilde{\wp}^m) = Fix(\tilde{\wp})$ for every $m \in \mathbb{N}$, where $Fix(\tilde{\wp}) : \{\varpi \in \breve{\mathfrak{U}}: \tilde{\wp}\,\varpi = \varpi\}$. (5.1)

(ii) Assume two self-maps $\tilde{\wp}, \check{f}$ on $\breve{\mathfrak{U}}$, then self-maps $\tilde{\wp}$ and $\check{f}$ have property (P) if $Fix(\tilde{\wp}^m) \cap Fix(\check{f}^m) = Fix(\tilde{\wp}) \cap Fix(\check{f})$ for every $m \in \mathbb{N}$, where $Fix(\tilde{\wp}): \{\varpi \in \breve{\mathfrak{U}}: \tilde{\wp}\,\varpi = \check{f}\varpi\}$
(5.2)

Theorem 5.1. Consider a normal cone O with Y be a normal constant and $(\breve{\mathfrak{U}}, \breve{c})$ as a complete C_S-MS with two self-maps $\tilde{\wp}, \check{f}: \breve{\mathfrak{U}} \to \breve{\mathfrak{U}}$ satisfies the hypothesis of Theorem 4.3. Then, self-maps $\tilde{\wp}$ and $\check{f}$ own the property (P).

Proof: From the given hypothesis of Theorem 4.3, self-maps $\tilde{\wp}$ and $\check{f}$ have CFP in $\breve{\mathfrak{U}}$. Assume that $\varpi_* \in Fix(\tilde{\wp}^m) \cap Fix(\check{f}^m)$. Then, by using equation (4.1) and lemma 2.7, we imply that

$$\breve{c}(\check{f}\varpi_*, \check{f}\varpi_*, \varpi_*) = \breve{c}(\varpi_*, \varpi_*, \check{f}\varpi_*) = \breve{c}\{\tilde{\wp}(\tilde{\wp}^{m-1}\varpi_*), \tilde{\wp}(\tilde{\wp}^{m-1}\varpi_*), \check{f}(\check{f}^m\varpi_*)\}$$

$$\leq k_1\{\breve{c}(\tilde{\wp}^{m-1}\varpi_*, \tilde{\wp}^{m-1}\varpi_*, \tilde{\wp}^m\varpi_*)\}$$

$$+k_2\{\breve{c}(\tilde{\wp}^m\varpi_*, \tilde{\wp}^m\varpi_*, \tilde{\wp}^{m-1}\varpi_*) + \breve{c}(\check{f}^{m+1}\varpi_*, \check{f}^{m+1}\varpi_*, \tilde{\wp}^m\varpi_*)\}$$

$$+k_3\left\{\frac{\breve{c}(\tilde{\wp}^m\varpi_*, \tilde{\wp}^m\varpi_*, \tilde{\wp}^{m-1}\varpi_*)\breve{c}(\check{f}^{m+1}\varpi_*, \check{f}^{m+1}\varpi_*, \tilde{\wp}^{m-1}\varpi_*) + \breve{c}(\tilde{\wp}^m\varpi_*, \tilde{\wp}^m\varpi_*, \check{f}^m\varpi_*)\breve{c}(\check{f}^{m+1}\varpi_*, \check{f}^{m+1}\varpi_*, \tilde{\wp}^m\varpi_*)}{\breve{c}(\tilde{\wp}^m\varpi_*, \tilde{\wp}^m\varpi_*, \check{f}^m\varpi_*) + \breve{c}(\check{f}^{m+1}\varpi_*, \check{f}^{m+1}\varpi_*, \tilde{\wp}^{m-1}\varpi_*)}\right\}$$

$$+k_4\left\{\frac{\breve{c}(\tilde{\wp}^m\varpi_*,\tilde{\wp}^m\varpi_*,\tilde{\wp}^{m-1}\varpi_*)\breve{c}(\check{f}^{m+1}\varpi_*,\check{f}^{m+1}\varpi_*,\tilde{\wp}^m\varpi_*)}{\breve{c}(\tilde{\wp}^{m-1}\varpi_*,\tilde{\wp}^{m-1}\varpi_*,\tilde{\wp}^m\varpi_*)+\breve{c}(\check{f}^{m+1}\varpi_*,\check{f}^{m+1}\varpi_*,\tilde{\wp}^{m-1}\varpi_*)+\breve{c}(\tilde{\wp}^m\varpi_*,\tilde{\wp}^m\varpi_*,\check{f}^m\varpi_*)}\right\}.$$

$$\leq k_1\{\breve{c}(\tilde{\wp}^{m-1}\varpi_*,\tilde{\wp}^{m-1}\varpi_*,\varpi_*)\}+k_2\{\breve{c}(\varpi_*,\varpi_*,\tilde{\wp}^{m-1}\varpi_*)+\breve{c}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)\}$$

$$+k_3\left\{\frac{\breve{c}(\varpi_*,\varpi_*,\tilde{\wp}^{m-1}\varpi_*)\breve{c}(\check{f}\varpi_*,\check{f}\varpi_*,\tilde{\wp}^{m-1}\varpi_*)+\breve{c}(\varpi_*,\varpi_*,\varpi_*)\breve{c}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}{\breve{c}(\varpi_*,\varpi_*,\varpi_*)+\breve{c}(\check{f}\varpi_*,\check{f}\varpi_*,\tilde{\wp}^{m-1}\varpi_*)}\right\}$$

$$+k_4\left\{\frac{\breve{c}(\varpi_*,\varpi_*,\tilde{\wp}^{m-1}\varpi_*)\breve{c}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)}{\breve{c}(\tilde{\wp}^{m-1}\varpi_*,\tilde{\wp}^{m-1}\varpi_*,\varpi_*)+\breve{c}(\check{f}\varpi_*,\check{f}\varpi_*,\tilde{\wp}^{m-1}\varpi_*)+\breve{c}(\varpi_*,\varpi_*,\varpi_*)}\right\},$$

i.e., $\breve{c}(\varpi_*,\varpi_*,\tilde{\wp}^m\varpi_*)\leq\breve{\beta}\{\breve{c}(\varpi_*,\varpi_*,\tilde{\wp}^{m-1}\varpi_*)\}$,

where $\breve{\beta}=\frac{k_1+k_2+k_3+k_4}{1-k_2}<1$.

Moreover, this means that

$$\breve{c}(\varpi_*,\varpi_*,\tilde{\wp}\varpi_*)=\breve{c}(\tilde{\wp}^m\varpi_*,\tilde{\wp}^m\varpi_*,\tilde{\wp}^{m+1}\varpi_*)\leq\breve{\beta}\{\breve{c}(\tilde{\wp}^m\varpi_*,\tilde{\wp}^m\varpi_*,\tilde{\wp}^{m-1}\varpi_*)\}$$
$$\leq\cdots\leq\breve{\beta}^m\{\breve{c}(\varpi_*,\varpi_*,\tilde{\wp}\varpi_*)\}. \quad (5.3)$$

Then, from equation (5.3) and obey normality, on can have

$$\|\breve{c}(\varpi_*,\varpi_*,\tilde{\wp}\varpi_*)\|\leq Y\breve{\beta}^m\|\breve{c}(\varpi_*,\varpi_*,\tilde{\wp}\varpi_*)\|. \quad (5.4)$$

On considering as limit $m\to\infty$ in equation (5.4) and lemma 2.7, we imply that

$$\|\breve{c}(\varpi_*,\varpi_*,\tilde{\wp}\varpi_*)\|=0=\|\breve{c}(\tilde{\wp}\varpi_*,\tilde{\wp}\varpi_*,\varpi_*)\|.$$

Hence, we conclude that

$$\tilde{\wp}\varpi_*=\varpi_* \text{ and } Fix(\tilde{\wp})\cap Fix(\check{f})=Fix(\tilde{\wp}^m)\cap Fix(\check{f}^m). \qquad \square$$

Theorem 5.2. Consider a normal cone O with Y be a normal constant and $(\tilde{\mathfrak{A}},\breve{c})$ as a complete C_S-MS with two self-maps $\tilde{\wp},\check{f}:\tilde{\mathfrak{A}}\to\tilde{\mathfrak{A}}$ satisfies the hypothesis of Theorem 4.4. Then, self-maps $\tilde{\wp}$ and $\check{f}$ own the property (P).

Proof: By using the hypothesis of Theorem 4.4, self-mappings $\tilde{\wp}$ and $\check{f}$ have CFP in $\tilde{\mathfrak{A}}$. Consider $\varpi_*\in Fix(\tilde{\wp}^m)\cap Fix(\check{f}^m)$ an arbitrary point.

Then, from equation (4.2) and lemma 2.7, one can have

$$\breve{c}(\check{f}\varpi_*,\check{f}\varpi_*,\varpi_*)=\breve{c}(\varpi_*,\varpi_*,\check{f}\varpi_*)=\breve{c}\{\tilde{\wp}(\tilde{\wp}^{m-1}\varpi_*),\tilde{\wp}(\tilde{\wp}^{m-1}\varpi_*),\check{f}(\check{f}^m\varpi_*)\}$$

$$\leq k_1\{\breve{c}(\tilde{\wp}^{m-1}\varpi_*,\tilde{\wp}^{m-1}\varpi_*,\tilde{\wp}^m\varpi_*)\}$$

$$+k_2\{\breve{c}(\tilde{\wp}^m\varpi_*,\tilde{\wp}^m\varpi_*,\check{f}^m\varpi_*)+\breve{c}(\check{f}^{m+1}\varpi_*,\check{f}^{m+1}\varpi_*,\tilde{\wp}^{m-1}\varpi_*)\}$$

$$+k_3\left\{\frac{\breve{c}(\tilde{\wp}^m\varpi_*,\tilde{\wp}^m\varpi_*,\tilde{\wp}^{m-1}\varpi_*)\breve{c}(\check{f}^{m+1}\varpi_*,\check{f}^{m+1}\varpi_*,\tilde{\wp}^{m-1}\varpi_*)+\breve{c}(\tilde{\wp}^m\varpi_*,\tilde{\wp}^m\varpi_*,\check{f}^m\varpi_*)\breve{c}(\check{f}^{m+1}\varpi_*,\check{f}^{m+1}\varpi_*,\tilde{\wp}^m\varpi_*)}{\breve{c}(\tilde{\wp}^m\varpi_*,\tilde{\wp}^m\varpi_*,\check{f}^m\varpi_*)+\breve{c}(\check{f}^{m+1}\varpi_*,\check{f}^{m+1}\varpi_*,\tilde{\wp}^{m-1}\varpi_*)}\right\}$$

$$+k_4\left\{\frac{\breve{C}(\breve{\wp}^m\varpi_*,\breve{\wp}^m\varpi_*,\breve{\wp}^{m-1}\varpi_*)\breve{C}(\breve{f}^{m+1}\varpi_*,\breve{f}^{m+1}\varpi_*,\breve{\wp}^m\varpi_*)}{\begin{matrix}\breve{C}(\breve{\wp}^{m-1}\varpi_*,\breve{\wp}^{m-1}\varpi_*,\breve{\wp}^m\varpi_*)+\breve{C}(\breve{f}^{m+1}\varpi_*,\breve{f}^{m+1}\varpi_*,\breve{\wp}^{m-1}\varpi_*)\\+\breve{C}(\breve{\wp}^m\varpi_*,\breve{\wp}^m\varpi_*,\breve{f}^m\varpi_*)\end{matrix}}\right\}.$$

$$\leq k_1\{\breve{C}(\breve{\wp}^{m-1}\varpi_*,\breve{\wp}^{m-1}\varpi_*,\varpi_*)\}+k_2\{\breve{C}(\varpi_*,\varpi_*,\varpi_*)+\breve{C}(\breve{f}\varpi_*,\breve{f}\varpi_*,\breve{\wp}^{m-1}\varpi_*)\}$$

$$+k_3\left\{\frac{\breve{C}(\varpi_*,\varpi_*,\breve{\wp}^{m-1}\varpi_*)\breve{C}(\breve{f}\varpi_*,\breve{f}\varpi_*,\breve{\wp}^{m-1}\varpi_*)+\breve{C}(\varpi_*,\varpi_*,\varpi_*)\breve{C}(\breve{f}\varpi_*,\breve{f}\varpi_*,\varpi_*)}{\breve{C}(\varpi_*,\varpi_*,\varpi_*)+\breve{C}(\breve{f}\varpi_*,\breve{f}\varpi_*,\breve{\wp}^{m-1}\varpi_*)}\right\}$$

$$+k_4\left\{\frac{\breve{C}(\varpi_*,\varpi_*,\breve{\wp}^{m-1}\varpi_*)\breve{C}(\breve{f}\varpi_*,\breve{f}\varpi_*,\varpi_*)}{\breve{C}(\breve{\wp}^{m-1}\varpi_*,\breve{\wp}^{m-1}\varpi_*,\varpi_*)+\breve{C}(\breve{f}\varpi_*,\breve{f}\varpi_*,\breve{\wp}^{m-1}\varpi_*)+\breve{C}(\varpi_*,\varpi_*,\varpi_*)}\right\},$$

i.e., $\breve{C}(\varpi_*,\varpi_*,\breve{\wp}^m\varpi_*)\leq\breve{\gamma}\{\breve{C}(\varpi_*,\varpi_*,\breve{\wp}^{m-1}\varpi_*)\}$,

where $\breve{\gamma}=\frac{k_1+k_2+k_3+k_4}{1-k_2}<1$.

Moreover, this means that

$$\breve{C}(\varpi_*,\varpi_*,\breve{\wp}\varpi_*)=\breve{C}(\breve{\wp}^m\varpi_*,\breve{\wp}^m\varpi_*,\breve{\wp}^{m+1}\varpi_*)\leq\breve{\gamma}\{\breve{C}(\breve{\wp}^m\varpi_*,\breve{\wp}^m\varpi_*,\breve{\wp}^{m-1}\varpi_*)\}$$
$$\leq\cdots\leq\breve{\gamma}^m\{\breve{C}(\varpi_*,\varpi_*,\breve{\wp}\varpi_*)\}. \quad (5.5)$$

Then, from equation (5.5) and using normality, one can have

$$\|\breve{C}(\varpi_*,\varpi_*,\breve{\wp}\varpi_*)\|\leq\Upsilon\breve{\gamma}^m\|\breve{C}(\varpi_*,\varpi_*,\breve{\wp}\varpi_*)\|. \quad (5.6)$$

On considering as limit $m\to\infty$ in equation (5.6) and lemma 2.7, we imply that

$\|\breve{C}(\varpi_*,\varpi_*,\breve{\wp}\varpi_*)\|=0=\|\breve{C}(\breve{\wp}\varpi_*,\breve{\wp}\varpi_*,\varpi_*)\|$.

Hence, we conclude that

$$\breve{\wp}\varpi_*=\varpi_* \text{ and } Fix(\breve{\wp}^m)\cap Fix(\breve{f}^m)=Fix(\breve{\wp})\cap Fix(\breve{f}). \qquad \square$$

6. CONCLUSION

In this manuscript, we have established CFP theorems for generalized contraction self-maps in complete C_S-MS. We demonstrate fixed and periodic-point results for new contractive type self-maps including some applications in complete C_S-MS. This study overgeneralizes, **consolidates** and **prolong** respective results from the existing literature in C_S-MS.

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Abbreviations: C_S-MS: cone S-metric space; s.t.; such that; *CFP*: common fixed-point; *FP*: fixed-point; *MS*: metric space.

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