

Neo Cobalancing Numbers

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Abstract

In this work, we defined neo cobalancing numbers and then determined the general terms of them in terms of balancing and Lucas-balancing or cobalancing and Lucas-cobalancing numbers. We also deduced some algebraic relations on Binet formulas, relationship with balancing, Pell, Pell-Lucas, triangular, square triangular numbers, sums of first n -terms of them, Cassini and Catalan identities.

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1. INTRODUCTION.

A positive integer n is called a balancing number ([2]) if the Diophantine equation

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r) \quad (1.1)$$

holds for some positive integer r which is called balancer. From (1.1)

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 1}}{2}. \quad (1.2)$$

Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (1.2), Behera and Panda noted that $8(0)^2 + 1 = 1$ and $8(1)^2 + 1 = 3^2$ are perfect squares. So they accepted 0 and 1 to be balancing numbers.

A positive integer n is called a cobalancing number ([16]) if the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r) \quad (1.3)$$

holds for some positive integer r which is called cobalancer. Form (1.3)

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8n + 1}}{2}. \quad (1.4)$$

From definition cobalancing numbers should be positive. But form (1.4), Panda and Ray noted that $8(0)^2 + 8(0) + 1 = 1$ is a perfect square. So they accepted 0 to be a cobalancing number just like Behera and Panda accepted 0 and 1 to be balancing numbers.

Let B_n denote the n^{th} balancing number and let b_n denote the n^{th} cobalancing number. Then $B_0 = 0, B_1 = 1, B_2 = 6, B_{n+1} = 6B_n - B_{n-1}$ and $b_0 = b_1 = 0, b_2 = 2, b_{n+1} = 6b_n - b_{n-1} + 2$ for $n \geq 2$. Also from (1.2), B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and from (1.4), b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. Thus

$$C_n = \sqrt{8B_n^2 + 1} \text{ and } c_n = \sqrt{8b_n^2 + 8b_n + 1}$$

are integers which are called the n^{th} Lucas-balancing and the n^{th} Lucas-cobalancing number, respectively.

Ray proved ([20]) that Binet formulas are $B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}, b_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, C_n = \frac{\alpha^{2n} + \beta^{2n}}{2}$ and $c_n = \frac{\alpha^{2n-1} + \beta^{2n-1}}{2}$ for $n \geq 1$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ which are the roots of the characteristic equation of Pell P_n and Pell-Lucas numbers Q_n (Note that $B_{-n} = -B_n, b_{-n} = b_{n+1}, C_{-n} = C_n, c_{-n} = -c_{n+1}, P_{-n} = (-1)^{n+1}P_n$ and $Q_{-n} = (-1)^n Q_n$) (see also [13, 15]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [11], Liptai proved that there is no Fibonacci balancing number except 1 and in [12] he proved that there is no Lucas-balancing number. In [23], Szalay considered the same problem and obtained some nice results by a different method. In [9], Kovács, Liptai, Olajos extended the concept of balancing numbers to the (a, b) -balancing numbers. Let $a > 0$ and $b \geq 0$ be coprime integers. If $(a + b) + \cdots + (a(n - 1) + b) = (a(n + 1) + b) + \cdots + (a(n + r) + b)$ for some positive integers n and r , then $an + b$ is an (a, b) -balancing number. The sequence of (a, b) -balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [10], the authors generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ with $y \geq 4$. A positive integer x with $x \leq y - 2$ is called a (k, l) -power numerical center for y if $1^k + \cdots + (x - 1)^k = (x + 1)^l + \cdots + (y - 1)^l$. They studied the number of solutions of the equation above and proved several effective and

ineffective finiteness results for (k, l) -power numerical centers. For positive integers k, x , let $\Pi_k(x) = x(x + 1) \dots (x + k - 1)$. Then it was proved in [9] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. In [35] Tengely, considered the case $k = 5$ and proved that this Diophantine equation has no integer solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [18], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers. In [14], Özdemir introduced a new non-commutative number system called hybrid numbers. In [3], Bród, Szynal-Liana and Włoch defined balancing and Lucas-balancing hybrid numbers and in [21], Rubajczyk and Szynal-Liana defined cobalancing and Lucas-cobalancing hybrid numbers. In [5], Dash, Ota, Dash defined t -balancing numbers. In [27], Tekcan and Aydın determined the general terms of t -balancing numbers. In [26], Tekcan and Erdem defined t -cobalancing numbers and determined the general terms them. In [17], Panda and Panda defined almost balancing numbers. In [19], Panda defined almost cobalancing numbers. In [31], Tekcan and Erdem determined the general terms of all almost balancing numbers and almost cobalancing numbers. In [24], Tekcan considered the sums and spectral norms of all almost balancing numbers and in [25], Tekcan derived some results on almost balancing numbers, triangular numbers and square triangular numbers. In [28, 29], Tekcan and Yıldız defined balcobalancing numbers and in [30, 32], they defined almost balcobalancing numbers. In [33], Tekcan and Akgüç defined almost neo cobalancing numbers and in [34], Tekcan defined almost neo balancing numbers.

2. NEO COBALANCING NUMBERS.

In [4], Chailangka and Pakapongpun defined neo balancing numbers. They said that an integer $n \geq 1$ is called a neo balancing number if the Diophantine equation

$$1 + 2 + \dots + (n - 1) = (n - 1) + (n - 0) + (n + 1) + (n + 2) + \dots + (n + r) \quad (2.1)$$

holds for some integer $r \geq 1$ which is called neo balancer. By considering (2.1), we say that a positive integer n is called a neo cobalancing number if the Diophantine equation

$$1 + 2 + \dots + n = (n - 1) + (n - 0) + (n + 1) + (n + 2) + \dots + (n + r) \quad (2.2)$$

holds for some positive integer r which is called neo cobalancer.

From (2.2), we get

$$r = \frac{-2n - 1 + \sqrt{8n^2 - 8n + 9}}{2} \quad (2.3)$$

and

$$n = \frac{2r + 3 + \sqrt{8r^2 + 16r + 1}}{2}. \quad (2.4)$$

From (2.3), we note that $8(1)^2 - 8(1) + 9 = 3^2$ and $8(2)^2 - 8(2) + 9 = 5^2$ are perfect squares. But in this case $r = 0$ and $r = 0$, respectively. Nevertheless, we accept 1 and 2 to be a neo cobalancing numbers just like Behera and Panda accepted 0 and 1 to be balancing numbers.

Let b_n^{neo} denote the n^{th} neo cobalancing number and let r_n^{neo} denote the n^{th} neo cobalancer. Then from (2.3), b_n^{neo} is a neo cobalancing number if and only if $8(b_n^{\text{neo}})^2 - 8b_n^{\text{neo}} + 9$ is a perfect square. Thus

$$c_n^{\text{neo}} = \sqrt{8(b_n^{\text{neo}})^2 - 8b_n^{\text{neo}} + 9} \quad (2.5)$$

is an integer which is called the n^{th} neo Lucas-cobalancing number and from (2.4), r_n^{neo} is a neo cobalancer if and only if $8(r_n^{\text{neo}})^2 + 16r_n^{\text{neo}} + 1$ is a perfect square. Thus

$$cr_n^{\text{neo}} = \sqrt{8(r_n^{\text{neo}})^2 + 16r_n^{\text{neo}} + 1} \quad (2.6)$$

is an integer which is called the n^{th} neo Lucas-cobalancer.

In order to determine the general terms of all neo cobalancing numbers, we have to determine the set of all integer solutions of some specific Pell equation (see [1, 8]). Indeed, since b_n^{neo} is a neo cobalancing number if and only if $8(b_n^{\text{neo}})^2 - 8b_n^{\text{neo}} + 9$ is a perfect square, we set

$$8(b_n^{\text{neo}})^2 - 8b_n^{\text{neo}} + 9 = y^2$$

for some integer $y \geq 1$. Then $2[(2b_n^{\text{neo}} - 1)^2 - 1] + 9 = y^2$. Taking $x = 2b_n^{\text{neo}} - 1$, we get the Pell equation

$$2x^2 - y^2 = -7. \quad (2.7)$$

Let Ω denotes the set of all integer solutions of (2.7). Then we can give the following theorem.

Theorem 2.1. *The set of all integer solutions of (2.7) is*

$$\Omega = \{(6B_{n-1} + C_{n-1}, 4B_{n-1} + 3C_{n-1}) : n \geq 1\} \cup \{(6B_n - C_n, -4B_n + 3C_n) : n \geq 1\}.$$

Proof. For the Pell equation in (2.7), the indefinite form is $F = (2, 0, -1)$ of discriminant $\Delta = 8$. So $\tau_8 = 3 + 2\sqrt{2}$. Thus $\text{Rep} = \{[\pm 1 \ 3]\}$ and $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ (see [6, p. 121]). Here we see that $[1 \ 3]M^{n-1}$ generates all integer solutions (x_{2n-1}, y_{2n-1}) for $n \geq 1$ and $[-1 \ 3]M^n$ generates all integer solutions (x_{2n}, y_{2n}) for $n \geq 1$. It can be easily seen that the n^{th} power of M is

$$M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}$$

for $n \geq 1$. Thus

$$\begin{aligned} [x_{2n-1} \quad y_{2n-1}] &= [1 \quad 3]M^{n-1} = [6B_{n-1} + C_{n-1} \quad 4B_{n-1} + 3C_{n-1}] \\ [x_{2n} \quad y_{2n}] &= [-1 \quad 3]M^n = [6B_n - C_n \quad -4B_n + 3C_n]. \end{aligned}$$

So $\Omega = \{(6B_{n-1} + C_{n-1}, 4B_{n-1} + 3C_{n-1}) : n \geq 1\} \cup \{(6B_n - C_n, -4B_n + 3C_n) : n \geq 1\}$. □

From Theorem 2.1, we can give the following result.

Theorem 2.2. *The general terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned} b_{2n-1}^{neo} &= \frac{6B_{n-1} + C_{n-1} + 1}{2} \\ b_{2n}^{neo} &= \frac{6B_n - C_n + 1}{2} \\ c_{2n-1}^{neo} &= 4B_{n-1} + 3C_{n-1} \\ c_{2n}^{neo} &= -4B_n + 3C_n \\ r_{2n-1}^{neo} &= -B_{n-1} + C_{n-1} - 1 \\ r_{2n}^{neo} &= -5B_n + 2C_n - 1 \\ cr_{2n}^{neo} &= 16B_n - 5C_n \end{aligned}$$

for $n \geq 1$ and

$$cr_{2n-1}^{neo} = 8B_{n-1} - C_{n-1}$$

for $n \geq 2$.

Proof. Applying Theorem 2.1, we deduce that

$$b_{2n-1}^{neo} = \frac{6B_{n-1} + C_{n-1} + 1}{2}$$

since $x = 2b_n^{neo} - 1$. Thus from (2.5), we obtain

$$\begin{aligned} c_{2n-1}^{neo} &= \sqrt{8(b_{2n-1}^{neo})^2 - 8b_{2n-1}^{neo} + 9} \\ &= \sqrt{8\left(\frac{6B_{n-1} + C_{n-1} + 1}{2}\right)^2 - 8\left(\frac{6B_{n-1} + C_{n-1} + 1}{2}\right) + 9} \\ &= \sqrt{72B_{n-1}^2 + 24B_{n-1}C_{n-1} + 2C_{n-1}^2 + 7} \\ &= \sqrt{72B_{n-1}^2 + 24B_{n-1}C_{n-1} + 2(8B_{n-1}^2 + 1) + 7} \\ &= \sqrt{16B_{n-1}^2 + 24B_{n-1}C_{n-1} + 9(8B_{n-1}^2 + 1)} \end{aligned}$$

$$\begin{aligned}
&= \sqrt{16B_{n-1}^2 + 24B_{n-1}C_{n-1} + 9C_{n-1}^2} \\
&= \sqrt{(4B_{n-1} + 3C_{n-1})^2} \\
&= 4B_{n-1} + 3C_{n-1}.
\end{aligned}$$

From (2.3), we deduce that

$$\begin{aligned}
r_{2n-1}^{\text{neo}} &= \frac{-2b_{2n-1}^{\text{neo}} - 1 + c_{2n-1}^{\text{neo}}}{2} \\
&= \frac{-2\left(\frac{6B_{n-1} + C_{n-1} + 1}{2}\right) - 1 + (4B_{n-1} + 3C_{n-1})}{2} \\
&= -B_{n-1} + C_{n-1} - 1.
\end{aligned}$$

Finally from (2.6), we get

$$\begin{aligned}
cr_{2n-1}^{\text{neo}} &= \sqrt{8(r_{2n-1}^{\text{neo}})^2 + 16r_{2n-1}^{\text{neo}} + 1} \\
&= \sqrt{8(-B_{n-1} + C_{n-1} - 1)^2 + 16(-B_{n-1} + C_{n-1} - 1) + 1} \\
&= \sqrt{8B_{n-1}^2 - 16B_{n-1}C_{n-1} + 8C_{n-1}^2 - 7} \\
&= \sqrt{8B_{n-1}^2 - 16B_{n-1}C_{n-1} + 8(8B_{n-1}^2 + 1) - 7} \\
&= \sqrt{64B_{n-1}^2 - 16B_{n-1}C_{n-1} + (8B_{n-1}^2 + 1)} \\
&= \sqrt{64B_{n-1}^2 - 16B_{n-1}C_{n-1} + C_{n-1}^2} \\
&= \sqrt{(8B_{n-1} - C_{n-1})^2} \\
&= 8B_{n-1} - C_{n-1}.
\end{aligned}$$

The others can be proved similarly. □

In Table 2.1, the first ten terms of all neo cobalancing numbers is given.

Apart from Theorem 2.2, we can give the general terms of all neo cobalancing numbers in terms of cobalancing and Lucas-cobalancing numbers instead of balancing and Lucas-balancing numbers as follows.

Theorem 2.3. *The general terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned}
b_{2n-1}^{\text{neo}} &= -b_n + c_n \\
b_{2n}^{\text{neo}} &= b_n + c_n + 1 \\
c_{2n-1}^{\text{neo}} &= 8b_n - c_n + 4
\end{aligned}$$

Table 2.1 Neo cobalancing numbers

i	b_n^{neo}	r_n^{neo}	c_n^{neo}	cr_n^{neo}
1	1	0	3	1
2	2	0	5	1
3	5	1	13	5
4	10	3	27	11
5	27	10	75	31
6	56	22	157	65
7	155	63	437	181
8	324	133	915	379
9	901	372	2547	1055
10	1886	780	5333	2209

$$\begin{aligned}
 c_{2n}^{neo} &= 8b_n + c_n + 4 \\
 r_{2n-1}^{neo} &= \frac{10b_n - 3c_n + 3}{2} \\
 r_{2n}^{neo} &= \frac{6b_n - c_n + 1}{2} \\
 cr_{2n}^{neo} &= -4b_n + 3c_n - 2
 \end{aligned}$$

for $n \geq 1$ and

$$cr_{2n-1}^{neo} = -12b_n + 5c_n - 6$$

for $n \geq 2$.

Proof. Applying Theorem 2.2, we deduce that

$$\begin{aligned}
 b_{2n-1}^{neo} &= \frac{6B_{n-1} + C_{n-1} + 1}{2} \\
 &= \frac{6\left(\frac{\alpha^{2n-2} - \beta^{2n-2}}{4\sqrt{2}}\right) + \left(\frac{\alpha^{2n-2} + \beta^{2n-2}}{2}\right) + 1}{2} \\
 &= \frac{\alpha^{2n-2}(3 + \sqrt{2}) + \beta^{2n-2}(-3 + \sqrt{2}) + 2\sqrt{2}}{4\sqrt{2}} \\
 &= \alpha^{2n-2}\left(\frac{-\alpha + 2\sqrt{2}\alpha}{4\sqrt{2}}\right) + \beta^{2n-2}\left(\frac{\beta + 2\sqrt{2}\beta}{4\sqrt{2}}\right) + \frac{1}{2} \\
 &= \alpha^{2n-2}\left(\frac{-\alpha}{4\sqrt{2}} + \frac{\alpha}{2}\right) + \beta^{2n-2}\left(\frac{\beta}{4\sqrt{2}} + \frac{\beta}{2}\right) + \frac{1}{2} \\
 &= -\left(\frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}\right) + \left(\frac{\alpha^{2n-1} + \beta^{2n-1}}{2}\right)
 \end{aligned}$$

$$= -b_n + c_n.$$

The others can be proved similarly. \square

3. BINET FORMULAS AND RECURRENCE RELATIONS.

Theorem 3.1. *Binet formulas for neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned} b_{2n-1}^{neo} &= \frac{(3 + \sqrt{2})\alpha^{2n-2} - (3 - \sqrt{2})\beta^{2n-2} + 2\sqrt{2}}{4\sqrt{2}} \\ b_{2n}^{neo} &= \frac{(3 - \sqrt{2})\alpha^{2n} - (3 + \sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} \\ c_{2n-1}^{neo} &= \frac{(3 + \sqrt{2})\alpha^{2n-2} + (3 - \sqrt{2})\beta^{2n-2}}{2} \\ c_{2n}^{neo} &= \frac{(3 - \sqrt{2})\alpha^{2n} + (3 + \sqrt{2})\beta^{2n}}{2} \\ r_{2n-1}^{neo} &= \frac{(-1 + 2\sqrt{2})\alpha^{2n-2} + (1 + 2\sqrt{2})\beta^{2n-2} - 4\sqrt{2}}{4\sqrt{2}} \\ r_{2n}^{neo} &= \frac{(-5 + 4\sqrt{2})\alpha^{2n} + (5 + 4\sqrt{2})\beta^{2n} - 4\sqrt{2}}{4\sqrt{2}} \\ cr_{2n}^{neo} &= \frac{(-5 + 4\sqrt{2})\alpha^{2n} - (5 + 4\sqrt{2})\beta^{2n}}{2} \end{aligned}$$

for $n \geq 1$ and

$$cr_{2n-1}^{neo} = \frac{(-1 + 2\sqrt{2})\alpha^{2n-2} - (1 + 2\sqrt{2})\beta^{2n-2}}{2}$$

for $n \geq 2$.

Proof. Recall that $B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}$ and $C_n = \frac{\alpha^{2n} + \beta^{2n}}{2}$. Thus from Theorem 2.2, we easily get

$$\begin{aligned} b_{2n-1}^{neo} &= \frac{6B_{n-1} + C_{n-1} + 1}{2} \\ &= \frac{6\left(\frac{\alpha^{2n-2} - \beta^{2n-2}}{4\sqrt{2}}\right) + \left(\frac{\alpha^{2n-2} + \beta^{2n-2}}{2}\right) + 1}{2} \\ &= \frac{\alpha^{2n-2}\left(\frac{3}{2\sqrt{2}} + \frac{1}{2}\right) + \beta^{2n-2}\left(\frac{-3}{2\sqrt{2}} + \frac{1}{2}\right) + 1}{2} \\ &= \frac{(3 + \sqrt{2})\alpha^{2n-2} - (3 - \sqrt{2})\beta^{2n-2} + 2\sqrt{2}}{4\sqrt{2}}. \end{aligned}$$

The others can be proved similarly. \square

Theorem 3.2. *Recurrence relations for neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned} b_n^{neo} &= b_{n-1}^{neo} + 6b_{n-2}^{neo} - 6b_{n-3}^{neo} - b_{n-4}^{neo} + b_{n-5}^{neo} \\ c_n^{neo} &= c_{n-1}^{neo} + 6c_{n-2}^{neo} - 6c_{n-3}^{neo} - c_{n-4}^{neo} + c_{n-5}^{neo} \\ r_n^{neo} &= r_{n-1}^{neo} + 6r_{n-2}^{neo} - 6r_{n-3}^{neo} - r_{n-4}^{neo} + r_{n-5}^{neo} \end{aligned}$$

for $n \geq 6$ and

$$cr_n^{neo} = cr_{n-1}^{neo} + 6cr_{n-2}^{neo} - 6cr_{n-3}^{neo} - cr_{n-4}^{neo} + cr_{n-5}^{neo}$$

for $n \geq 7$.

Proof. Let n be even, say $n = 2k$ for some positive $k \geq 1$. Then from Theorem 3.1, we get

$$\begin{aligned} b_{n-1}^{neo} + 6b_{n-2}^{neo} - 6b_{n-3}^{neo} - b_{n-4}^{neo} + b_{n-5}^{neo} &= \frac{(3 + \sqrt{2})\alpha^{2k-2} - (3 - \sqrt{2})\beta^{2k-2} + 2\sqrt{2}}{4\sqrt{2}} \\ &+ 6 \left(\frac{(3 - \sqrt{2})\alpha^{2k-2} - (3 + \sqrt{2})\beta^{2k-2} + 2\sqrt{2}}{4\sqrt{2}} \right) \\ &- 6 \left(\frac{(3 + \sqrt{2})\alpha^{2k-4} - (3 - \sqrt{2})\beta^{2k-4} + 2\sqrt{2}}{4\sqrt{2}} \right) \\ &- \left(\frac{(3 - \sqrt{2})\alpha^{2k-4} - (3 + \sqrt{2})\beta^{2k-4} + 2\sqrt{2}}{4\sqrt{2}} \right) \\ &+ \left(\frac{(3 + \sqrt{2})\alpha^{2k-6} - (3 - \sqrt{2})\beta^{2k-6} + 2\sqrt{2}}{4\sqrt{2}} \right) \\ &= \left(\frac{(21 - 5\sqrt{2})\alpha^{2k-2} - (21 + 5\sqrt{2})\beta^{2k-2} + 14\sqrt{2}}{4\sqrt{2}} \right) \\ &- \left(\frac{(21 + 5\sqrt{2})\alpha^{2k-4} - (21 - 5\sqrt{2})\beta^{2k-4} + 14\sqrt{2}}{4\sqrt{2}} \right) \\ &+ \left(\frac{(3 + \sqrt{2})\alpha^{2k-6} - (3 - \sqrt{2})\beta^{2k-6} + 2\sqrt{2}}{4\sqrt{2}} \right) \\ &= \frac{(3 - \sqrt{2})\alpha^{2k} - (3 + \sqrt{2})\beta^{2k} + 2\sqrt{2}}{4\sqrt{2}} \\ &= b_{2k}^{neo} \\ &= b_n^{neo}. \end{aligned}$$

The others can be proved similarly. □

Apart from Theorem 3.2, we can give the following result which can be proved similarly.

Theorem 3.3. *For neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers, we have*

$$\begin{aligned} b_{2n+1}^{neo} &= \frac{7b_{2n}^{neo} - 3b_{2n-1}^{neo} - 1}{2} \\ c_{2n+1}^{neo} &= \frac{7c_{2n}^{neo} - 3c_{2n-1}^{neo}}{2} \\ r_{2n+1}^{neo} &= \frac{7r_{2n}^{neo} - 3r_{2n-1}^{neo} + 2}{2} \end{aligned}$$

for $n \geq 1$ and

$$\begin{aligned} b_{2n}^{neo} &= \frac{7b_{2n-1}^{neo} - 2b_{2n-2}^{neo} - 1}{3} \\ c_{2n}^{neo} &= \frac{7c_{2n-1}^{neo} - 2c_{2n-2}^{neo}}{3} \\ r_{2n}^{neo} &= \frac{7r_{2n-1}^{neo} - 2r_{2n-2}^{neo} + 2}{3} \\ cr_{2n}^{neo} &= \frac{7cr_{2n-1}^{neo} - 2cr_{2n-2}^{neo}}{3} \\ cr_{2n+1}^{neo} &= \frac{7cr_{2n}^{neo} - 3cr_{2n-1}^{neo}}{2} \end{aligned}$$

for $n \geq 2$.

4. SUMS OF NEO COBALANCING NUMBERS.

Recall that the sum of first n -terms of all balancing numbers can be given in terms of same balancing numbers, that is,

$$\begin{aligned} \sum_{i=1}^n B_i &= \frac{5B_n - B_{n-1} - 1}{4}, \quad n \geq 1 & \sum_{i=1}^n b_i &= \frac{5b_n - b_{n-1} + 2 - 2n}{4}, \quad n \geq 1 \\ \sum_{i=1}^n C_i &= \frac{5C_n - C_{n-1} - 2}{4}, \quad n \geq 1 & \sum_{i=1}^n c_i &= \frac{5c_n - c_{n-1} - 2}{4}, \quad n \geq 2. \end{aligned}$$

Similarly we can give the following theorem.

Theorem 4.1. *The sums of first n -terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\sum_{i=1}^n b_i^{neo} = \frac{5b_n^{neo} + 5b_{n-1}^{neo} - b_{n-2}^{neo} - b_{n-3}^{neo} + 2n - 8}{4}$$

$$\sum_{i=1}^n c_i^{neo} = \frac{5c_n^{neo} + 5c_{n-1}^{neo} - c_{n-2}^{neo} - c_{n-3}^{neo}}{4}$$

$$\sum_{i=1}^n r_i^{neo} = \frac{5r_n^{neo} + 5r_{n-1}^{neo} - r_{n-2}^{neo} - r_{n-3}^{neo} - 4n + 12}{4}$$

for $n \geq 4$ and

$$\sum_{i=1}^n cr_i^{neo} = \frac{5cr_n^{neo} + 5cr_{n-1}^{neo} - cr_{n-2}^{neo} - cr_{n-3}^{neo} - 8}{4}$$

for $n \geq 5$.

Proof. We note that $b_n^{neo} = 6b_{n-2}^{neo} - b_{n-4}^{neo} - 2$ for $n \geq 5$. Let n be odd, say $n = 2k - 1$ for some integer $k \geq 1$. Thus

$$\begin{aligned} b_5^{neo} &= 6b_3^{neo} - b_1^{neo} - 2 \\ b_6^{neo} &= 6b_4^{neo} - b_2^{neo} - 2 \\ b_7^{neo} &= 6b_5^{neo} - b_3^{neo} - 2 \\ &\dots \\ b_{2k-1}^{neo} &= 6b_{2k-3}^{neo} - b_{2k-5}^{neo} - 2 \end{aligned}$$

and hence

$$(b_5^{neo} + b_6^{neo} + \dots + b_{2k-1}^{neo}) = 6(b_3^{neo} + b_4^{neo} + \dots + b_{2k-3}^{neo}) - (b_1^{neo} + b_2^{neo} + \dots + b_{2k-5}^{neo}) - 2(2k-5).$$

Thus

$$\begin{aligned} &(b_1^{neo} + b_2^{neo} + \dots + b_{2k-1}^{neo}) - (b_1^{neo} + b_2^{neo} + b_3^{neo} + b_4^{neo}) \\ &= 6(b_1^{neo} + b_2^{neo} + \dots + b_{2k-3}^{neo} + b_{2k-2}^{neo} + b_{2k-1}^{neo}) \\ &\quad - 6(b_1^{neo} + b_2^{neo} + b_{2k-2}^{neo} + b_{2k-1}^{neo}) \\ &\quad - (b_1^{neo} + b_2^{neo} + \dots + 4b_{2k-5}^{neo} + b_{2k-4}^{neo} + b_{2k-3}^{neo} + b_{2k-2}^{neo} + b_{2k-1}^{neo}) \\ &\quad + b_{2k-4}^{neo} + b_{2k-3}^{neo} + b_{2k-2}^{neo} + b_{2k-1}^{neo} - 2(2k-5). \end{aligned}$$

So

$$\begin{aligned} \sum_{i=1}^{2k-1} b_i^{neo} - (b_1^{neo} + b_2^{neo} + b_3^{neo} + b_4^{neo}) &= 6 \sum_{i=1}^{2k-1} b_i^{neo} - 6(b_1^{neo} + b_2^{neo} + b_{2k-2}^{neo} + b_{2k-1}^{neo}) \\ &\quad - \sum_{i=1}^{2k-1} b_i^{neo} + (b_{2k-4}^{neo} + b_{2k-3}^{neo} + b_{2k-2}^{neo} + b_{2k-1}^{neo}) \\ &\quad - 2(2k-5) \end{aligned}$$

and hence

$$\sum_{i=1}^{2k-1} b_i^{neo} = \frac{5b_{2k-1}^{neo} + 5b_{2k-2}^{neo} - b_{2k-3}^{neo} - b_{2k-4}^{neo} + 2(2k - 5)}{4}.$$

Since $n = 2k - 1$, we conclude that

$$\sum_{i=1}^n b_i^{neo} = \frac{5b_n^{neo} + 5b_{n-1}^{neo} - b_{n-2}^{neo} - b_{n-3}^{neo} + 2n - 8}{4}.$$

The others can be proved similarly. □

We also note that

$$\begin{aligned} \sum_{i=1}^n (-1)^i B_i &= \begin{cases} 2B_{\frac{n}{2}}^2 + B_{\frac{n}{2}} C_{\frac{n}{2}} & n \geq 2 \text{ even} \\ -2B_{\frac{n+1}{2}}(b_{\frac{n+1}{2}} + \frac{1}{2}) & n \geq 1 \text{ odd} \end{cases} \\ \sum_{i=1}^n (-1)^i b_i &= \begin{cases} 2B_{\frac{n}{2}}^2 & n \geq 2 \text{ even} \\ -2b_{\frac{n+1}{2}}^2 - 2b_{\frac{n+1}{2}} & n \geq 1 \text{ odd} \end{cases} \\ \sum_{i=1}^n (-1)^i C_i &= \begin{cases} B_n + 8B_{\frac{n}{2}}^2 & n \geq 2 \text{ even} \\ -B_n - 8(b_{\frac{n+1}{2}} + \frac{1}{2})^2 & n \geq 1 \text{ odd} \end{cases} \\ \sum_{i=1}^n (-1)^i c_i &= \begin{cases} B_n & n \geq 2 \text{ even} \\ -B_n & n \geq 1 \text{ odd.} \end{cases} \end{aligned}$$

Similarly we can give the following theorem.

Theorem 4.2. *For the sums of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers, we have*

$$\begin{aligned} \sum_{i=1}^n (-1)^i b_i^{neo} &= \frac{1}{4} \begin{cases} 5b_n^{neo} - 5b_{n-1}^{neo} - b_{n-2}^{neo} + b_{n-3}^{neo} & n \geq 4 \text{ even} \\ -5b_n^{neo} + 5b_{n-1}^{neo} + b_{n-2}^{neo} - b_{n-3}^{neo} - 2 & n \geq 5 \text{ odd} \end{cases} \\ \sum_{i=1}^n (-1)^i c_i^{neo} &= \frac{1}{4} \begin{cases} 5c_n^{neo} - 5c_{n-1}^{neo} - c_{n-2}^{neo} + c_{n-3}^{neo} - 4 & n \geq 4 \text{ even} \\ -5c_n^{neo} + 5c_{n-1}^{neo} + c_{n-2}^{neo} - c_{n-3}^{neo} - 4 & n \geq 5 \text{ odd} \end{cases} \\ \sum_{i=1}^n (-1)^i r_i^{neo} &= \frac{1}{4} \begin{cases} 5r_n^{neo} - 5r_{n-1}^{neo} - r_{n-2}^{neo} + r_{n-3}^{neo} - 2 & n \geq 4 \text{ even} \\ -5r_n^{neo} + 5r_{n-1}^{neo} + r_{n-2}^{neo} - r_{n-3}^{neo} + 2 & n \geq 5 \text{ odd} \end{cases} \\ \sum_{i=1}^n (-1)^i cr_i^{neo} &= \frac{1}{4} \begin{cases} 5cr_n^{neo} - 5cr_{n-1}^{neo} - cr_{n-2}^{neo} + cr_{n-3}^{neo} - 4 & n \geq 6 \text{ even} \\ -5cr_n^{neo} + 5cr_{n-1}^{neo} + cr_{n-2}^{neo} - cr_{n-3}^{neo} - 4 & n \geq 5 \text{ odd.} \end{cases} \end{aligned}$$

Proof. It can be proved as in the same way that Theorem 4.1 was proved. □

In [22, Lemma 1], Santana and Diaz-Barrero proved that the sum of first nonzero $4n + 1$ terms of Pell numbers is a perfect square and is

$$\sum_{i=1}^{4n+1} P_i = \left(\sum_{i=0}^n \binom{2n+1}{2i} 2^i \right)^2.$$

Similarly we can give the following result.

Theorem 4.3. *The sum of first nonzero $4n + 1$ terms of Pell numbers is*

$$\sum_{i=1}^{4n+1} P_i = \left(\frac{-b_{2n+3}^{neo} + 5b_{2n+2}^{neo} - 2}{3} \right)^2.$$

Proof. From Theorem 3.1, we notice that

$$-b_{2n+3}^{neo} + 5b_{2n+2}^{neo} = \frac{(-3 + 3\sqrt{2})\alpha^{2n+2} - (3 + 3\sqrt{2})\beta^{2n+2} + 4}{2}.$$

Since $\sum_{i=1}^n P_i = \frac{P_{n+1} + P_n - 1}{2}$ and $P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$, we get

$$\begin{aligned} \sum_{i=1}^{4n+1} P_i &= \frac{P_{4n+2} + P_{4n+1} - 1}{2} \\ &= \frac{\frac{\alpha^{4n+2} - \beta^{4n+2}}{2\sqrt{2}} + \frac{\alpha^{4n+1} - \beta^{4n+1}}{2\sqrt{2}} - 1}{2} \\ &= \frac{\alpha^{4n+2}(1 + \alpha^{-1}) + \beta^{4n+2}(-1 - \beta^{-1})}{2\sqrt{2}} - \frac{1}{2} \\ &= \frac{\alpha^{4n+2} + \beta^{4n+2}}{4} - \frac{1}{2} \\ &= \frac{(27 - 18\sqrt{2})\alpha^{4n+4} + (27 + 18\sqrt{2})\beta^{4n+4} - 18}{36} \\ &= \frac{1}{36} \left\{ \begin{aligned} &(-3 + 3\sqrt{2})^2 \alpha^{4n+4} - 2(-3 + 3\sqrt{2})(3 + 3\sqrt{2})(\alpha\beta)^{2n+2} \\ &+ (3 + 3\sqrt{2})^2 \beta^{4n+4} \end{aligned} \right\} \\ &= \left(\frac{(-3 + 3\sqrt{2})\alpha^{2n+2} - (3 + 3\sqrt{2})\beta^{2n+2}}{6} \right)^2 \\ &= \left(\frac{(-3 + 3\sqrt{2})\alpha^{2n+2} - (3 + 3\sqrt{2})\beta^{2n+2} + 4}{3} \right)^2 \\ &= \left(\frac{-b_{2n+3}^{neo} + 5b_{2n+2}^{neo} - 2}{3} \right)^2 \end{aligned}$$

as we wanted. □

As in Theorem 4.3, we can give the following theorem which can be proved similarly.

Theorem 4.4. *For the sums of Pell, Pell-Lucas, balancing and Lucas-cobalancing numbers, we have*

1. *the sum of first nonzero $4n - 1$ terms of Pell numbers $+1$ is a perfect square and is*

$$1 + \sum_{i=1}^{4n-1} P_i = (b_{2n+1}^{neo} - b_{2n}^{neo})^2.$$

2. *the sum of odd ordered Pell-Lucas numbers from 1 to $2n$ is a perfect square and is*

$$\sum_{i=1}^{2n} Q_{2i-1} = \left(\frac{2b_{2n+1}^{neo} + 2b_{2n}^{neo} - 2}{3} \right)^2.$$

3. *the half of the sum of odd ordered Pell-Lucas numbers from 0 to $2n$ is a perfect square and is*

$$\frac{\sum_{i=0}^{2n} Q_{2i+1}}{2} = \left(\frac{-b_{2n+3}^{neo} + 5b_{2n+2}^{neo} - 2}{3} \right)^2.$$

4. *the sum of odd ordered balancing numbers from 1 to $2n$ is a perfect square and is*

$$\sum_{i=1}^{2n} B_{2i-1} = \left(\frac{b_{4n+1}^{neo} + b_{4n}^{neo} - 1}{6} \right)^2.$$

5. *the sum of first $4n+2$ terms of Lucas-cobalancing numbers $+1$ is a perfect square and is*

$$1 + \sum_{i=1}^{4n+2} c_i = (b_{4n+3}^{neo} - b_{4n+2}^{neo})^2.$$

Panda and Ray proved in [15, Theorem 3.4] that the sum of first $2n - 1$ Pell numbers is

$$\sum_{i=1}^{2n-1} P_i = B_n + b_n. \quad (4.1)$$

Later Gözeri, Özkoç and Tekcan proved in [7, Theorem 2.5] that the sum of Pell-Lucas numbers from 0 to $2n - 1$ is

$$\sum_{i=0}^{2n-1} Q_i = C_n + c_n.$$

Since $R_n = b_n$, (4.1) becomes

$$\sum_{i=1}^{2n-1} P_i = B_n + R_n. \tag{4.2}$$

As in (4.2), we can give the following result.

Theorem 4.5. *The sum of first $2n - 1$ Pell numbers is*

$$\sum_{i=1}^{2n-1} P_i = b_{2n-1}^{neo} + r_{2n}^{neo}.$$

Proof. We easily deduce that

$$\begin{aligned} \sum_{i=1}^{2n-1} P_i &= \frac{P_{2n} + P_{2n-1} - 1}{2} \\ &= \frac{\frac{\alpha^{2n} - \beta^{2n}}{2\sqrt{2}} + \frac{\alpha^{2n-1} - \beta^{2n-1}}{2\sqrt{2}} - 1}{2} \\ &= \frac{\alpha^{2n}(1 + \alpha^{-1}) + \beta^{2n}(-1 - \beta^{-1})}{4\sqrt{2}} - \frac{1}{2} \\ &= \frac{\alpha^{2n} + \beta^{2n}}{4} - \frac{1}{2} \\ &= \frac{1}{4\sqrt{2}} \left\{ \begin{array}{l} \alpha^{2n}[(3 + \sqrt{2})(-1 + \sqrt{2})^2 - 5 + 4\sqrt{2}] \\ + \beta^{2n}[-(3 - \sqrt{2})(-1 - \sqrt{2})^2 + 5 + 4\sqrt{2}] \end{array} \right\} - \frac{1}{2} \\ &= \frac{1}{4\sqrt{2}} \left\{ \begin{array}{l} \alpha^{2n}[(3 + \sqrt{2})\alpha^{-2} - 5 + 4\sqrt{2}] \\ + \beta^{2n}[-(3 - \sqrt{2})\beta^{-2} + 5 + 4\sqrt{2}] \end{array} \right\} - \frac{1}{2} \\ &= \frac{1}{4\sqrt{2}} \left\{ \begin{array}{l} (3 + \sqrt{2})\alpha^{2n-2} - (3 - \sqrt{2})\beta^{2n-2} + 2\sqrt{2} \\ + (-5 + 4\sqrt{2})\alpha^{2n} + (5 + 4\sqrt{2})\beta^{2n} - 4\sqrt{2} \end{array} \right\} \\ &= b_{2n-1}^{neo} + r_{2n}^{neo} \end{aligned}$$

by Theorem 3.1. □

5. NEO COBALANCING NUMBERS AND BALANCING NUMBERS.

In Theorems 2.2, we deduced the general terms of all neo cobalancing numbers in terms of balancing and Lucas-balancing numbers and in Theorem 2.3, we deduced the general terms of them in terms of cobalancing and Lucas-cobalancing numbers. Conversely, we can give the general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers in terms of neo cobalancing numbers as follows:

Theorem 5.1. *The general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers are*

$$\begin{aligned} B_n &= \frac{b_{2n+1}^{neo} + b_{2n}^{neo} - 1}{6} \\ b_n &= \frac{b_{2n+1}^{neo} - 2b_{2n}^{neo} - 1}{3} \\ C_n &= b_{2n+1}^{neo} - b_{2n}^{neo} \\ c_n &= \frac{-b_{2n+1}^{neo} + 5b_{2n}^{neo} - 2}{3} \end{aligned}$$

for $n \geq 1$.

Proof. Recall that $B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}$. So we get

$$\begin{aligned} B_n &= \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} \\ &= \frac{6\left(\frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}\right)}{6} \\ &= \frac{\alpha^{2n}(3+\sqrt{2}+3-\sqrt{2}) + \beta^{2n}(-3+\sqrt{2}-3-\sqrt{2})\beta^{2n} + 4\sqrt{2}}{4\sqrt{2}} - 1 \\ &= \frac{6}{6} \\ &= \frac{(3+\sqrt{2})\alpha^{2n} - (3-\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} + \frac{(3-\sqrt{2})\alpha^{2n} - (3+\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} - 1 \\ &= \frac{b_{2n+1}^{neo} + b_{2n}^{neo} - 1}{6} \end{aligned}$$

by Theorem 3.1. The others can be proved similarly. \square

6. NEO COBALANCING NUMBERS AND PELL AND PELL-LUCAS NUMBERS.

Theorem 6.1. *The general terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned} b_{2n-1}^{neo} &= \frac{4P_{2n-2} + P_{2n-3} + 1}{2} \\ b_{2n}^{neo} &= \frac{2P_{2n} - P_{2n-1} + 1}{2} \\ c_{2n-1}^{neo} &= 5P_{2n-2} + 3P_{2n-3} \\ c_{2n}^{neo} &= P_{2n} + 3P_{2n-1} \\ r_{2n-1}^{neo} &= \frac{P_{2n-2} + 2P_{2n-3} - 2}{2} \end{aligned}$$

$$r_{2n}^{neo} = \frac{-P_{2n} + 4P_{2n-1} - 2}{2}$$

$$cr_{2n}^{neo} = 3P_{2n} - 5P_{2n-1}$$

for $n \geq 1$ and

$$cr_{2n-1}^{neo} = 3P_{2n-2} - P_{2n-3}$$

for $n \geq 2$.

Proof. From Theorem 3.1, we get

$$b_{2n}^{neo} = \frac{(3 - \sqrt{2})\alpha^{2n} - (3 + \sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}}$$

$$= \frac{\alpha^{2n}(\frac{2-\alpha^{-1}}{2\sqrt{2}}) - \beta^{2n}(\frac{2-\beta^{-1}}{2\sqrt{2}}) + 2\sqrt{2}}{2}$$

$$= \frac{2(\frac{\alpha^{2n}-\beta^{2n}}{2\sqrt{2}}) - (\frac{\alpha^{2n-1}-\beta^{2n-1}}{2\sqrt{2}}) + 1}{2}$$

$$= \frac{2P_{2n} - P_{2n-1} + 1}{2}.$$

The others can be proved similarly. □

Conversely, we can give the general terms of even and odd ordered Pell numbers in terms of neo cobalancing numbers and neo Lucas-cobalancing as follows:

Theorem 6.2. *The general terms of even and odd ordered Pell numbers are*

$$P_{2n} = \frac{6b_{2n}^{neo} + c_{2n}^{neo} - 3}{7} \quad \text{and} \quad P_{2n-1} = \frac{2b_{2n-1}^{neo} + 2c_{2n-1}^{neo} - 1}{7}$$

for $n \geq 1$.

Proof. Applying Binet formulas, we deduce that

$$P_{2n} = \frac{\alpha^{2n} - \beta^{2n}}{2\sqrt{2}}$$

$$= \frac{\alpha^{2n}[\frac{6(3-\sqrt{2})}{4\sqrt{2}} + \frac{3-\sqrt{2}}{2}] + \beta^{2n}[\frac{-6(3+\sqrt{2})}{4\sqrt{2}} + \frac{3+\sqrt{2}}{2}]}{7}$$

$$= \frac{6[\frac{(3-\sqrt{2})\alpha^{2n} - (3+\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}}] + [(\frac{3-\sqrt{2}}{2})\alpha^{2n} + (\frac{3+\sqrt{2}}{2})\beta^{2n}] - 3}{7}$$

$$= \frac{6b_{2n}^{neo} + c_{2n}^{neo} - 3}{7}.$$

The others can be proved similarly. □

Thus we construct one-to-one correspondence between all neo cobalancing numbers and Pell numbers.

Similarly, we can give the general terms of all neo cobalancing numbers in terms of Pell-Lucas numbers as follows:

Theorem 6.3. *The general terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned} b_{2n-1}^{neo} &= \frac{5Q_{2n-2} + 3Q_{2n-3} + 4}{8} \\ b_{2n}^{neo} &= \frac{Q_{2n} + 3Q_{2n-1} + 4}{8} \\ c_{2n-1}^{neo} &= \frac{4Q_{2n-2} + Q_{2n-3}}{2} \\ c_{2n}^{neo} &= \frac{2Q_{2n} - Q_{2n-1}}{2} \\ r_{2n-1}^{neo} &= \frac{3Q_{2n-2} - Q_{2n-3} - 8}{8} \\ r_{2n}^{neo} &= \frac{3Q_{2n} - 5Q_{2n-1} - 8}{8} \\ cr_{2n}^{neo} &= \frac{-Q_{2n} + 4Q_{2n-1}}{2} \end{aligned}$$

for $n \geq 1$ and

$$cr_{2n-1}^{neo} = \frac{Q_{2n-2} + 2Q_{2n-3}}{2}$$

for $n \geq 2$.

Proof. It can be proved as in the same way that Theorem 6.1 was proved. □

Conversely, we can give the general terms of even and odd ordered Pell-Lucas numbers in terms of all neo cobalancing numbers as follows.

Theorem 6.4. *The general terms of even and odd ordered Pell-Lucas numbers are*

$$Q_{2n} = \frac{8c_{2n}^{neo} + 2cr_{2n}^{neo}}{7} \quad \text{and} \quad Q_{2n-1} = \frac{20b_{2n-1}^{neo} + 4r_{2n-1}^{neo} - 6}{7}$$

for $n \geq 1$.

Proof. It can be proved as in the same way that Theorem 6.2 was proved. □

Thus we construct one-to-one correspondence between all neo cobalancing numbers and Pell-Lucas numbers.

7. NEO COBALANCING NUMBERS AND TRIANGULAR AND SQUARE TRIANGULAR NUMBERS.

Triangular numbers denoted by T_n are the numbers of the form

$$T_n = \frac{n(n + 1)}{2}.$$

It is known that there is a correspondence between balancing and also cobalancing numbers and triangular numbers. From (1.1), we get

$$\frac{(n + r)(n + r + 1)}{2} = n^2.$$

So

$$T_{B_n+R_n} = B_n^2.$$

Similarly from (1.3), we get

$$\frac{(n + r)(n + r + 1)}{2} = n^2 + n.$$

So

$$T_{b_n+r_n} = b_n^2 + b_n. \tag{7.1}$$

As in (7.1), we can give the following theorem.

Theorem 7.1. b_n^{neo} is a neo cobalancing number if and only if $(b_n^{neo})^2 - b_n^{neo} + 1$ is a triangular number, that is,

$$T_{b_n^{neo}+r_n^{neo}} = (b_n^{neo})^2 - b_n^{neo} + 1.$$

Proof. From (2.2), we get $n^2 + n = 4n - 2 + 2nr + r^2 + r$ and hence

$$\frac{(n + r)(n + r + 1)}{2} = n^2 - n + 1.$$

Thus

$$T_{b_n^{neo}+r_n^{neo}} = (b_n^{neo})^2 - b_n^{neo} + 1$$

as we wanted. □

There are infinitely many triangular numbers that are also square numbers which are called square triangular numbers and is denoted by S_n . So

$$S_n = s_n^2 = \frac{t_n(t_n + 1)}{2},$$

where s_n and t_n are the sides of the corresponding square and triangle. Their Binet formulas are

$$S_n = \frac{\alpha^{4n} + \beta^{4n} - 2}{32}, s_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} \text{ and } t_n = \frac{\alpha^{2n} + \beta^{2n} - 2}{4} \quad (7.2)$$

for $n \geq 1$.

We can give the general terms of all neo cobalancing numbers in terms of s_n and t_n as follows.

Theorem 7.2. *The general terms of neo cobalancing numbers, neo Lucas-cobalancing numbers, neo cobalancers and neo Lucas-cobalancers are*

$$\begin{aligned} b_{2n-1}^{neo} &= 3s_{n-1} + t_{n-1} + 1 \\ b_{2n}^{neo} &= 3s_n - t_n \\ c_{2n-1}^{neo} &= 4s_{n-1} + 6t_{n-1} + 3 \\ c_{2n}^{neo} &= -4s_n + 6t_n + 3 \\ r_{2n-1}^{neo} &= -s_{n-1} + 2t_{n-1} \\ r_{2n}^{neo} &= -5s_n + 4t_n + 1 \\ cr_{2n}^{neo} &= 16s_n - 10t_n - 5 \end{aligned}$$

for $n \geq 1$ and

$$cr_{2n-1}^{neo} = 8s_{n-1} - 2t_{n-1} - 1$$

for $n \geq 2$.

Proof. From Theorem 3.1, we get

$$\begin{aligned} b_{2n-1}^{neo} &= \frac{(3 + \sqrt{2})\alpha^{2n-2} - (3 - \sqrt{2})\beta^{2n-2} + 2\sqrt{2}}{4\sqrt{2}} \\ &= \alpha^{2n-2} \left(\frac{3}{4\sqrt{2}} + \frac{1}{4} \right) - \beta^{2n-2} \left(\frac{3}{4\sqrt{2}} - \frac{1}{4} \right) - \frac{1}{2} + 1 \\ &= 3 \left(\frac{\alpha^{2n-2} - \beta^{2n-2}}{4\sqrt{2}} \right) + \left(\frac{\alpha^{2n-2} + \beta^{2n-2} - 2}{4} \right) + 1 \\ &= 3s_{n-1} + t_{n-1} + 1. \end{aligned}$$

by (7.2). The others can be proved similarly. \square

Conversely, we can give the following theorem.

Theorem 7.3. *The general terms of S_n , s_n and t_n are*

$$S_n = \frac{(b_{2n+1}^{neo})^2 + (b_{2n}^{neo})^2 + 2b_{2n+1}^{neo}b_{2n}^{neo} - 2b_{2n+1}^{neo} - 2b_{2n}^{neo} + 1}{36}$$

$$s_n = \frac{b_{2n+1}^{neo} + b_{2n}^{neo} - 1}{6}$$

$$t_n = \frac{b_{2n+1}^{neo} - b_{2n}^{neo} - 1}{2}$$

for $n \geq 1$.

Proof. From (7.2), we get

$$S_n = \frac{\alpha^{4n} + \beta^{4n} - 2}{32}$$

$$= \frac{1}{36} \left(\frac{9}{8}(\alpha^{4n} + \beta^{4n}) - \frac{9}{4}(\alpha\beta)^{2n} \right)$$

$$= \frac{1}{36} \left\{ \begin{aligned} &\left(\frac{1}{4} + \frac{3\sqrt{2}}{8}(\alpha^{2n} - \beta^{2n}) + \frac{11}{32}(\alpha^{4n} + \beta^{4n}) + \frac{3\sqrt{2}}{16}(\alpha^{4n} - \beta^{4n}) \right) \\ &\quad + \frac{1}{4}(\alpha^{2n} + \beta^{2n}) - \frac{7}{16}(\alpha\beta)^{2n} \\ &\left(\frac{1}{4} + \frac{3\sqrt{2}}{8}(\alpha^{2n} - \beta^{2n}) + \frac{11}{32}(\alpha^{4n} + \beta^{4n}) - \frac{3\sqrt{2}}{16}(\alpha^{4n} - \beta^{4n}) \right) \\ &\quad - \frac{1}{4}(\alpha^{2n} + \beta^{2n}) - \frac{7}{16}(\alpha\beta)^{2n} \\ &+ \left(\frac{1}{2} + \frac{3\sqrt{2}}{4}(\alpha^{2n} - \beta^{2n}) + \frac{7}{16}(\alpha^{4n} + \beta^{4n}) - \frac{11}{8}(\alpha\beta)^{2n} \right) \\ &\quad + \left(-\frac{3\sqrt{2}}{2}(\alpha^{2n} - \beta^{2n}) - 1 \right) \end{aligned} \right\}$$

$$= \frac{1}{36} \left\{ \begin{aligned} &\left(\frac{(3+\sqrt{2})\alpha^{2n} - (3-\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} \right)^2 + \left(\frac{(3-\sqrt{2})\alpha^{2n} - (3+\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} \right)^2 \\ &+ 2 \left(\frac{(3+\sqrt{2})\alpha^{2n} - (3-\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} \right) \left(\frac{(3-\sqrt{2})\alpha^{2n} - (3+\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} \right) \\ &- 2 \left(\frac{(3+\sqrt{2})\alpha^{2n} - (3-\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} \right) - 2 \left(\frac{(3-\sqrt{2})\alpha^{2n} - (3+\sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} \right) + 1 \end{aligned} \right\}$$

$$= \frac{(b_{2n+1}^{neo})^2 + (b_{2n}^{neo})^2 + 2b_{2n+1}^{neo}b_{2n}^{neo} - 2b_{2n+1}^{neo} - 2b_{2n}^{neo} + 1}{36}$$

by Theorem 3.1. The others can be proved similarly. □

Thus we construct one-to-one correspondence between all neo cobalancing numbers and square triangular numbers.

Finally, we want to construct a correspondence between triangular and square triangular numbers via neo cobalancing numbers, that is, we want to find out that for which neo cobalancing numbers (or neo cobalancers) m , the equation

$$T_m = S_n$$

holds. The answer is given below.

Theorem 7.4. For triangular numbers T_n and square triangular numbers S_n , we have

1. $T_{\frac{b_{2n+1}^{neo} - b_{2n}^{neo} - 1}{2}} = S_n$

$$2. T_{\frac{b_{2n+1}^{neo} + 3r_{2n+1}^{neo} - 1}{7}} = S_n$$

$$3. T_{\frac{5r_{2n+1}^{neo} - r_{2n+1}^{neo} + 1}{6}} = S_n$$

for $n \geq 1$.

Proof. (1) From Theorem 3.1, we note that $b_{2n+1}^{neo} - b_{2n}^{neo} = \frac{\alpha^{2n} + \beta^{2n}}{2}$. So we get

$$\begin{aligned} T_{\frac{b_{2n+1}^{neo} - b_{2n}^{neo} - 1}{2}} &= \frac{\left(\frac{b_{2n+1}^{neo} - b_{2n}^{neo} - 1}{2}\right)\left(\frac{b_{2n+1}^{neo} - b_{2n}^{neo} - 1}{2} + 1\right)}{2} \\ &= \frac{(b_{2n+1}^{neo} - b_{2n}^{neo} - 1)(b_{2n+1}^{neo} - b_{2n}^{neo} + 1)}{8} \\ &= \frac{\left(\frac{\alpha^{2n} + \beta^{2n}}{2} - 1\right)\left(\frac{\alpha^{2n} + \beta^{2n}}{2} + 1\right)}{8} \\ &= \frac{\left(\frac{\alpha^{2n} + \beta^{2n}}{2}\right)^2 - 1}{8} \\ &= \frac{\alpha^{4n} + \beta^{4n} - 2}{32} \\ &= S_n \end{aligned}$$

by (7.2). The others can be proved similarly. □

8. CASSINI AND CATALAN IDENTITIES.

Note that Cassini identity for Fibonacci numbers F_n is

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

for $n \geq 1$ and Catalan identity for Fibonacci numbers F_n is

$$F_n^2 - F_{n-m}F_{n+m} = (-1)^{n-m} F_m^2$$

for $n \geq m \geq 1$. Similarly for all neo cobalancing numbers, we can give the following theorems.

Theorem 8.1. *Cassini identities for all neo cobalancing numbers are*

$$\begin{aligned} b_{2n}^{neo} b_{2n+2}^{neo} - (b_{2n+1}^{neo})^2 &= -2B_n b_{n+1} - 1 \\ b_{2n-1}^{neo} b_{2n+1}^{neo} - (b_{2n}^{neo})^2 &= 2B_n (b_n + 1) - 1 \\ c_{2n}^{neo} c_{2n+2}^{neo} - (c_{2n+1}^{neo})^2 &= -c_{2n+1} + 7 \end{aligned}$$

$$\begin{aligned}
 c_{2n-1}^{neo} c_{2n+1}^{neo} - (c_{2n}^{neo})^2 &= c_{2n} + 7 \\
 r_{2n}^{neo} r_{2n+2}^{neo} - (r_{2n+1}^{neo})^2 &= \frac{(c_n + 2)(2 - C_n) - 1}{4} \\
 r_{2n-1}^{neo} r_{2n+1}^{neo} - (r_{2n}^{neo})^2 &= \frac{(c_n - 2)(C_{n-1} - 2) - 1}{4} \\
 cr_{2n}^{neo} cr_{2n+2}^{neo} - (cr_{2n+1}^{neo})^2 &= -c_{2n} - 7
 \end{aligned}$$

for $n \geq 1$ and

$$cr_{2n-1}^{neo} cr_{2n+1}^{neo} - (cr_{2n}^{neo})^2 = c_{2n-1} - 7$$

for $n \geq 2$.

Proof. Applying Theorem 3.1, we deduce that

$$\begin{aligned}
 b_{2n}^{neo} b_{2n+2}^{neo} - (b_{2n+1}^{neo})^2 &= \left(\frac{(3 - \sqrt{2})\alpha^{2n} - (3 + \sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} \right) \\
 &\quad \times \left(\frac{(3 - \sqrt{2})\alpha^{2n+2} - (3 + \sqrt{2})\beta^{2n+2} + 2\sqrt{2}}{4\sqrt{2}} \right) \\
 &\quad - \left(\frac{(3 + \sqrt{2})\alpha^{2n} - (3 - \sqrt{2})\beta^{2n} + 2\sqrt{2}}{4\sqrt{2}} \right)^2 \\
 &= \frac{1}{32} \left\{ \begin{array}{l} 4\sqrt{2}(\alpha^{2n} - \beta^{2n}) - 2\sqrt{2}(\alpha^{4n} - \beta^{4n}) \\ -2(\alpha^{4n} + \beta^{4n}) - 28 \end{array} \right\} \\
 &= \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} + \frac{\alpha^{4n}(-2 - 2\sqrt{2}) + \beta^{4n}(-2 + 2\sqrt{2})}{32} - \frac{7}{8} \\
 &= \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} - \frac{\alpha^{4n+1} + \beta^{4n+1}}{16} - \frac{7}{8} \\
 &= -2 \left(\frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} \right) \left(\frac{\alpha^{2n+1} - \beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2} \right) - 1 \\
 &= -2B_n b_{n+1} - 1.
 \end{aligned}$$

The other cases can be proved similarly. □

Apart from Theorem 8.1, we can give the Cassini identities for all neo cobalancing numbers in terms of neo cobalancing numbers as follows:

Theorem 8.2. *Cassini identities for all neo cobalancing numbers are*

$$b_{2n}^{neo} b_{2n+2}^{neo} - (b_{2n+1}^{neo})^2 = -1 - \frac{(b_{2n+1}^{neo} + b_{2n}^{neo} - 1)(b_{2n+3}^{neo} - 2b_{2n+2}^{neo} - 1)}{9}$$

$$\begin{aligned}
b_{2n-1}^{neo} b_{2n+1}^{neo} - (b_{2n}^{neo})^2 &= -1 + \frac{(b_{2n+1}^{neo} + b_{2n}^{neo} - 1)(b_{2n+1}^{neo} - 2b_{2n}^{neo} + 2)}{9} \\
c_{2n}^{neo} c_{2n+2}^{neo} - (c_{2n+1}^{neo})^2 &= 7 - \frac{c_{4n+2}^{neo} - c_{4n+1}^{neo}}{2} \\
c_{2n-1}^{neo} c_{2n+1}^{neo} - (c_{2n}^{neo})^2 &= 7 + \frac{c_{4n}^{neo} - c_{4n-1}^{neo}}{2} \\
r_{2n}^{neo} r_{2n+2}^{neo} - (r_{2n+1}^{neo})^2 &= \frac{5}{24} - \frac{r_{2n+1}^{neo} + r_{2n}^{neo}}{3} - \frac{r_{4n+1}^{neo} - r_{4n}^{neo}}{8} \\
r_{2n-1}^{neo} r_{2n+1}^{neo} - (r_{2n}^{neo})^2 &= \frac{-1}{8} + \frac{5r_{4n+1}^{neo} - 13r_{4n}^{neo}}{24} - \frac{r_{2n+1}^{neo} + r_{2n}^{neo}}{3} \\
cr_{2n}^{neo} cr_{2n+2}^{neo} - (cr_{2n+1}^{neo})^2 &= -7 - \frac{cr_{4n+1}^{neo} + cr_{4n}^{neo}}{6}
\end{aligned}$$

for $n \geq 1$ and

$$cr_{2n-1}^{neo} cr_{2n+1}^{neo} - (cr_{2n}^{neo})^2 = -7 + \frac{3cr_{4n}^{neo} - cr_{4n+1}^{neo}}{2}$$

for $n \geq 2$.

Proof. It can be proved as in the same way that Theorem 8.1 was proved. \square

Finally as in Theorem 8.2, we can give the following result concerning Catalan identities for all neo cobalancing numbers.

Theorem 8.3. *Catalan identities for neo cobalancing numbers are*

1. $(b_{2n}^{neo})^2 - b_{2n-2m}^{neo} b_{2n+2m}^{neo} = b_{2n}^{neo} - (b_{2m+1}^{neo} - b_{2m}^{neo})(b_{2n}^{neo} - \frac{1}{2}) + (7b_{4m+1}^{neo} - 7b_{4m}^{neo} - 15)/16$ for $n > m \geq 1$.
2. $(b_{2n-1}^{neo})^2 - b_{2n-2m-1}^{neo} b_{2n+2m-1}^{neo} = b_{2n-1}^{neo} - (b_{2m+1}^{neo} - b_{2m}^{neo})(b_{2n-1}^{neo} - \frac{1}{2}) + (7b_{4m+1}^{neo} - 7b_{4m}^{neo} - 15)/16$ for $n > m \geq 1$.
3. $(b_{2n}^{neo})^2 - b_{2n-2m+1}^{neo} b_{2n+2m-1}^{neo} = b_{2n}^{neo} + (b_{4n+1}^{neo} - 5b_{4n}^{neo})/24 - (b_{2n-2m+1}^{neo} + b_{2n+2m-1}^{neo})/2 + (21b_{4m-1}^{neo} - 21b_{4m-2}^{neo} - 17)/48$ for $n \geq m \geq 1$.
4. $(b_{2n-1}^{neo})^2 - b_{2n-2m}^{neo} b_{2n+2m-2}^{neo} = b_{2n-1}^{neo} - (b_{2n-2m}^{neo} + b_{2n+2m-2}^{neo})/2 + (5b_{4n-2}^{neo} - b_{4n-1}^{neo})/24 + (21b_{4m-1}^{neo} - 21b_{4m-2}^{neo} - 25)/48$ for $n > m \geq 1$.

for neo Lucas-cobalancing numbers are

1. $(c_{2n}^{neo})^2 - c_{2n-2m}^{neo} c_{2n+2m}^{neo} = (-21c_{4m}^{neo} + 7c_{4m-1}^{neo} + 28)/8$ for $n > m \geq 1$.
2. $(c_{2n-1}^{neo})^2 - c_{2n-2m-1}^{neo} c_{2n+2m-1}^{neo} = (-21c_{4m}^{neo} + 7c_{4m-1}^{neo} + 28)/8$ for $n > m \geq 1$.

3. $(c_{2n}^{neo})^2 - c_{2n-2m+1}^{neo}c_{2n+2m-1}^{neo} = (c_{4n-1}^{neo} - c_{4n}^{neo} + 7)/2 - (7c_{4m-1}^{neo} + 7c_{4m-2}^{neo} - 42)/12$
for $n \geq m \geq 1$.
4. $(c_{2n-1}^{neo})^2 - c_{2n-2m}^{neo}c_{2n+2m-2}^{neo} = c_{4n}^{neo} - 2c_{4n-1}^{neo} - (7c_{4m-1}^{neo} + 7c_{4m-2}^{neo} - 42)/12$ for
 $n > m \geq 1$.

for neo cobalancers are

1. $(r_{2n}^{neo})^2 - r_{2n-2m}^{neo}r_{2n+2m}^{neo} = -2r_{2n}^{neo} + (r_{2n}^{neo} + 1)(10r_{2m+1}^{neo} - 2r_{2m}^{neo} + 8)/3 - (35r_{4m+1}^{neo} - 7r_{4m}^{neo} + 103)/48$ for $n > m \geq 1$.
2. $(r_{2n-1}^{neo})^2 - r_{2n-2m-1}^{neo}r_{2n+2m-1}^{neo} = -2r_{2n-1}^{neo} + (r_{2n-1}^{neo} + 1)(10r_{2m+1}^{neo} - 2r_{2m}^{neo} + 8)/3 - (35r_{4m+1}^{neo} - 7r_{4m}^{neo} + 103)/48$ for $n > m \geq 1$.
3. $(r_{2n}^{neo})^2 - r_{2n-2m+1}^{neo}r_{2n+2m-1}^{neo} = (-10r_{4n+1}^{neo} + 26r_{4n}^{neo} - 35r_{4m-1}^{neo} + 7r_{4m-2}^{neo} + 9)/48 - 2r_{2n}^{neo} + r_{2n-2m+1}^{neo} + r_{2n+2m-1}^{neo}$ for $n \geq m \geq 1$.
4. $(r_{2n-1}^{neo})^2 - r_{2n-2m}^{neo}r_{2n+2m-2}^{neo} = -2r_{2n-1}^{neo} + r_{2n-2m}^{neo} + r_{2n+2m-2}^{neo} + (r_{4n-3}^{neo} - r_{4n-4}^{neo})/8 - (35r_{4m-1}^{neo} - 7r_{4m-2}^{neo} + 7)/48$ for $n > m \geq 1$.

and for neo Lucas-cobalancers are

1. $(cr_{2n}^{neo})^2 - cr_{2n-2m}^{neo}cr_{2n+2m}^{neo} = (14cr_{4m+1}^{neo} - 7cr_{4m}^{neo} - 21)/6$ for $n > m \geq 1$.
2. $(cr_{2n-1}^{neo})^2 - cr_{2n-2m-1}^{neo}cr_{2n+2m-1}^{neo} = (14cr_{4m+1}^{neo} - 7cr_{4m}^{neo} - 21)/6$ for $n > m+1 \geq 0$.
3. $(cr_{2n}^{neo})^2 - cr_{2n-2m+1}^{neo}cr_{2n+2m-1}^{neo} = (-3cr_{4n}^{neo} + cr_{4n+1}^{neo})/2 + (7cr_{4m}^{neo} - 7cr_{4m-1}^{neo} - 14)/4$ for $n > m \geq 1$.
4. $(cr_{2n-1}^{neo})^2 - cr_{2n-2m}^{neo}cr_{2n+2m-2}^{neo} = (-9cr_{4n}^{neo} + 19cr_{4n-1}^{neo})/4 + (7cr_{4m}^{neo} - 7cr_{4m-1}^{neo} - 14)/4$ for $n > m \geq 1$.

Proof. It can be proved as in the same way that Theorem 8.1 was proved. □

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