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Iterative Metthods for General Bifunction Equilibrium Problems

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Abstract

In this paper, some new classes of general bifunction equilibrium problems are introdcued and investigated. The auxiliary principle technique is used to consider some new predictor-corrector and inertial proximal algorithms for solving general equilibrium problems.The convergence of the proposed methods either requires partially relaxed strongly monotonicity or pseudomononicity, which are weaker conditions. As special cases, we obtain a number of known and new results for solving various classes of equilibrium and variational inequality problems. Our results represent can be viewed as significant improvement and refinement of the previously known results.

Keywords : Equilibrium problems, variational inequalities, auxiliary principle, iterative methods, convergence.

Mathematics Subject Classification (MSC2010): 49J40, 26D15, 26D10, 90C33

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1. INTRODUCTION

Equilibrium problems theory [5, 31] provides us with a unified, natural, innovative and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation. Equilibrium problems theory has witnessed an explosive growth in theoretic advances, algorithmic aspects and applications across all discipline of pure and applied sciences. This theory provides a novel and unified treatment of problems arising in economics, finance, transportation, network and structural analysis, elasticity and optimization. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative. Equilibrium problems have been extended and generalized in many directions using novel and innovative techniques, see [5, 6, 8, 9, 11, 19, 20, 21, 23, 24, 25, 26, 28, 29, 31, 32, 36] and the references therein.

Inspired and motivated by the recent research and activities going on in this fascinating and interesting field, we introduce and consider a class of equilibrium problems, which is called *general equilibrium problem.* This class includes general variational inequalities, complementarity problems and Nash equilibria problems as special cases. There are several numerical methods [1, 6, 7, 9, 10, 13, 14, 15, 16, 18, 23, 27, 28, 29, 30, 32, 35, 38, 41] including projection, its variant forms, descent and auxiliary principle for solving variational inequalities. On the other hand, there are only few iterative methods for solving equilibrium problems. Due to nature of the equilibrium problems involving the bifunction, it is known that projection methods and variant forms cannot be extended for equilibrium. This fact has motivated to use the auxiliary principle technique, which is mainly due to Lions et al [10]. The main and basic idea in this technique is to consider an auxiliary invex equilibrium problem related to the original problem. This way one defines a mapping connecting the solutions of both these problems. In this case, one has to show that the mapping connecting the solution is a contraction mapping and consequently it has a fixed point, which is the solution of the original problem. Glowinski et al. [7] used this technique to study the existence of a solution of the mixed variational inequalities. This technique has been developed in [8, 17, 19, 20, 21, 22, 23, 24, 25, 26, 28, 29, 30, 32, 41] to suggest and analyze a number of iterative methods for solving various classes of, equilibrium problems, variational-like inequalities and variational inequalities.

In this paper, we use the auxiliary principle technique to consider a class of predictor-corrector methods for solving general bifunction equilibrium problems. The convergence of these methods requires only that the operator is partially relaxed strongly monotone, which is weaker than monotonicity. Consequently, we improve the convergence results of previously known methods, which can be obtained as special cases from our results. We also use the auxiliary principle technique to suggest and analyze an inertial proximal method for solving equilibrium problem. It is worth mentioning that inertial proximal methods were introduced by Alvarez et al. [1] and Noor [23] for solving variational inequalities. We prove that the convergence of inertial proximal method requires only pseudomonotonicity, which is a weaker condition. This clearly improve the known results. Since general equilibrium problems includes equilibrium, general variational inequalities and complementarity problems as special cases, results obtained in this paper continue to hold for these problems. Our results can be considered an important and significant extension of the known results for solving equilibrium, variational inequalities and complementarity problems.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H.
For given nonlinear operators $g, T : H \to H$ and $F(.,.) : H \times H \longrightarrow H$, consider the problem of finding $u \in K$ such that

$$F(T(u), g(v)) \quad \geq 0, \qquad \forall v \in K, \tag{2.1}$$

which is called is called the *general bifunction equilibrium problem.*

Special cases

We now consider some special caases of the problem (2.1)

1. For $g = I, T = I$, where I is the identity operator, we obtain the original equilibrium problem, that is, finding $u \in K$ such that

$$F(u, v) \quad \geq \quad 0, \qquad \forall v \in K, \tag{2.2}$$

   which was introduced and studied by Blum et al.[5] and Noor et al.[31]. For the applications, mathematical formulations, generalizations and existence theory, see [5, 6, 8, 11, 19, 20, 21, 23, 24, 25, 26, 28, 29, 31, 32] and the references therein.

2. If $F(T(u), g(v)) = \langle Tu, g(v) - g(v) \rangle$, then the problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle T(u), g(v) - g(u) \rangle \quad \geq 0, \quad \forall v \in K. \tag{2.3}$$

Inequality (2.3) is known as the *general variational inequality,* which was introduced and studied by Noor [14] in 1988. It turned out that a wide class of odd order and nonsymmetric, obstacle, moving, equilibrium and optimization problems arising in pure and applied sciences can be studied via the general variational inequalities (2.2), see [14, 16, 18, 23, 28, 29] and the refrences therein.

3. If $K^* = \{u \in H : \langle u, v\rangle \geq 0, \quad \forall v \in K\}$ is a polar cone of a convex cone K in H, then problem (2.3) is equivalent to finding $u \in H$ such that

$$g(u) \in K, Tu \in K^*, \text{and} \quad \langle Tu, g(u)\rangle = 0, \tag{2.4}$$

which is known as the general complementarity problem [22] . We note that if $g(u) = u - m(u)$, where m is a point -to-point mapping, then problem (2.4) is called the quasi(implicit) complementarity problem.

4. For $F(T(u), g(v)) = \langle T(u) + A(u), v - u, \rangle$, then the problem (2.1) reduces to finding $u \in K$ such that

$$\langle T(u) + A(u), v - u, \rangle \geq 0, \quad \forall v \in K, \tag{2.5}$$

is called the mildly nonlinear variational inequality, introduced and studied by Noor [13] contains the absolute value equations and difference of the two convex functions.

5. For $g = I$, the problem (2.3) reduces to finding $u \in K$ such that

$$\langle T(u), v - u\rangle \quad \geq 0, \quad \forall v \in K \tag{2.6}$$

is known as the variational inequalities, introduced and studied by Lions and Stampacchia [10], can be viewed as an important and novel generalization of the variational principles. For the applications, generalizations, numerical methods and numerical aspects of variational inequalities, see [1, 4, 5, 6, 7, 9, 10, 13, 14, 15, 16, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 35, 38, 41] and the references threrein.

It is clear that problems (2.2)-(2.6) are special cases of the general bifunction equilibrium problems (2.1). In brief, for a suitable and appropriate choice of the operators $T, A,$ g, and the space H, one can obtain a wide class of equilibrium, variational inequalities and complementarity problems. This clearly shows that problem (2.1) is quite general unifying one and has important applications in various branches of pure and applied sciences, see [9, 12, 19, 20, 21, 22, 23, 24, 25, 26, 31, 36] and reference therein.
We also need the following concepts.

Lemma 2.1. *$\forall u, v \in H$, we have*

$$2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2. \tag{2.7}$$

Definition 2.1. *[31] The bifunction $F(.,.) : H \times H \to H$ and the operator $T : H \to H$ is said to be:*

(i) *g-partially relaxed jointly strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$F(T(u), g(v)) + F(T(v), g(z)) \leq \alpha\|g(z) - g(u)\|^2, \quad \forall u, v, z \in H.$$

(ii) *g- jointly monotone, if*

$$F(T(u), g(v)) + F(T(v), g(v) - g(u)) \leq 0, \quad \forall u, v \in H.$$

(iii) *g-jointly pseudomonotone, if*

$$F(T(u), g(v)) \leq 0 \quad \textit{implies} \quad -F(T(v), g(u)) \geq 0, \quad \forall u, v \in H.$$

We remark that if $z = u$, then g-partially relaxed strongly monotonicity is exactly g-monotonicity of the operator $F(.,.)$. For $g \equiv I$, the indentity operator, then Definition 2.1 reduces to the standard definition of partially relaxed strongly monotonicity, monotonicity, and pseudomonotonicity. It is known that monotonicity implies pseudomonotonicity, but not conversely. This implies that the concepts of partially relaxed strongly monotonicity and pseudomonotonicity are weaker than monotonicity.

3. MAIN RESULTS

In this section, we suggest and analyze a new iterative method for solving the problem (2.1) by using the auxiliary principle technique, which is orignially due to Glowinki et al. [7] as developed in [8, 11, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32, 41].

For a given $u \in K$ satisfying (2.1), consider the problem of finding a unique $w \in K$ satisfying the auxiliary variational inequality

$$\rho F(T(u), g(v)) + \langle M(w) - M(u), M(v) - M(w)\rangle \geq 0, \quad \forall v \in K, \tag{3.1}$$

where $\rho > 0$ is a constant.

We note that, if $w = u$, then clearly w is a solution of the general bifunction equilibrium problems (2.1). This observation enables us to suggest the following predictor-corrector method for solving the general equilibrium problems (2.1).

Algorithm 3.1. *For a given u_0, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho F(T(w_n), g(v)) + \langle M(u_{n+1}) - M(w_n), M(v) - M(u_{n+1})\rangle \geq 0, \forall v \in K \tag{3.2}$$

$$\beta F(T(y_n), g(v)) + \langle M(w_n) - M(y_n), M(v) - M(w_n)\rangle \geq 0, \quad \forall v \in K, \tag{3.3}$$

$$\mu F(T(u_n), g(v)) + \langle M(y_n) - M(u_n), M(v) - M(y_n)\rangle \geq 0, \quad \forall v \in K, \tag{3.4}$$

where $\rho > 0$, $\mu > 0$ nd $\beta > 0$ are constants.

Algorithm 3.1 is a three-step predictor-corrector iterative method for solving general bifunction equilibrium problems. Note that if $g = I$, $M = I$, the identity operator, then Algorithm 3.1 reduces to a new predictor-corrector method for solving the equilibrium problems (2.2).

Algorithm 3.2. *For a given u_0,compute u_{n+1} by the iterative schemes*

$$\begin{aligned}
\rho F(T(w_n), v) &+ \langle u_{n+1} - w_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K,\\
\beta F(T(y_n), v) &+ \langle w_n - y_n, v - w_n\rangle \geq 0, \quad \forall v \in K,\\
\mu F(T(u_n), v) &+ \langle y_n - u_n, v - y_n\rangle \geq 0, \quad \forall v \in K.
\end{aligned}$$

For $F(u, g(v)) = \langle T(u), M(v) - M(u)\rangle$,

Algorithm 3.1 reduces to:

Algorithm 3.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
\langle \rho T(w_n) &+ M(u_{n+1}) - M(w_n), M(v) - M(u_{n+1})\rangle \geq 0, \quad \forall v \in K\\
\langle \beta T(y_n) &+ M(w_n) - M(y_n), M(v) - M(y_n)\rangle \geq 0, \quad \forall v \in K\\
\langle \mu T(u_n) &+ M(y_n) - M(u_n), M(v) - M(y_n)\rangle \geq 0, \quad \forall v \in K.
\end{aligned}$$

Algorithm 3.3 is called the predictor-corrector method for solving general variational inequalities (2.3), see Noor [18, 23].

Using the technique of the projection, Algorithm 3.3 can be written as

Algorithm 3.4. *For a given u_0, compute u_{n+1} by the iterative schemes*

$$\begin{aligned}
M(u_{n+1}) &= P_K[M(w_n) - \rho T(w_n)],\\
M(w_n) &= P_K[M(y_n) - \beta T(y_n)],\\
M(y_n) &= P_K[M(u_n) - \mu T(u_n)],
\end{aligned}$$

which can be written as

$$M(u_{n+1}) = P_K[I - \rho TM^{-1}]P_K[I - \beta TM^{-1}]P_K[I - \mu TM^{-1}]M(u_n),$$

where P_K is the projection of H onto the convex and closed convex set K.

Algorithm 3.4 is three-step forward-backward splitting method for solving general variational inequalities (2.3), which is due to Noor [18]. These three step iterative methods are known as Noor iterations. For the applcaions and generalizations of Noor iterations, see [2, 3, 12, 27, 33, 34, 38, 39, 40] and the references therein. For suitable and appropriate choice of the operators T, g, M convex set and the space H, one can obtain various new and known methods for solving equilibrium problems, variational inequalities and complementarity problems.

For the convergence analysis of Algorithm 3.1, we need the following result.

Lemma 3.1. *Let $\bar{u} \in K$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.1. If $F(.,.) : H \times H \longrightarrow H$ is partially relaxed jointly monotone, then*

$$\|M(u_{n+1}) - M(\bar{u})\|^2 \leq \|M(w_n) - M(\bar{u})\|^2 - \|M(u_{n+1}) - M(w_n)\|^2 \quad (3.5)$$

$$\|M(w_n) - M(\bar{u})\|^2 \leq \|M(y_n) - M(\bar{u})\|^2 - \|M(w_n) - M(y_n)\|^2 \quad (3.6)$$

$$\|M(y_n) - M(\bar{u})\|^2 \leq \|M(u_n) - M(\bar{u})\|^2 - \|M(y_n) - M(u_n)\|^2. \quad (3.7)$$

Proof. Let $\bar{u} \in H$ be solution of (2.1). Then

$$\rho F(T(\bar{u}), g(v)) \geq 0, \quad \forall v \in K, \quad (3.8)$$

$$\beta F(T(\bar{u}), g(v)) \geq 0, \quad \forall v \in K, \quad (3.9)$$

$$\mu F(T(\bar{u}), g(v)) \geq, \quad \forall v \in K, \quad (3.10)$$

where $\rho > 0$, $\beta > 0$ and $\mu > 0$ are constants.

Now taking $v = u_{n+1}$ in (3.8) and $v = \bar{u}$ in (3.2), we have

$$\rho F(T(\bar{u}), g(u_{n+1})) \geq 0 \quad (3.11)$$

$$-\rho F(T(w_n), g(\bar{u})) + \langle M(u_{n+1}) - M(w_n), M(\bar{u}) - M(u_{n+1})\rangle \geq 0. \quad (3.12)$$

Adding (3.11) and (3.12), we have

$$\langle M(u_{n+1}) - M(w_n), M(\bar{u}) - M(u_{n+1})\rangle \geq -\rho\{F(T(w_n), g(\bar{u})) + F(T(\bar{u}), g(u_{n+1})) \geq 0, \quad (3.13)$$

where we have used the fact that $F(.,.)$ is partially relaxed jointly monotone .

Setting $u = M(\bar{u}) - M(u_{n+1})$ and $v = M(u_{n+1}) - M(w_n)$ in (2.7), we obtain

$$2\langle M(u_{n+1}) - M(w_n), M(\bar{u}) - M(u_{n+1})\rangle = \|M(\bar{u}) - M(w_n)\|^2$$
$$-\|M(\bar{u}) - M(u_{n+1})\|^2 - \|M(u_{n+1}) - M(w_n)\|. \qquad (3.14)$$

Combining (3.13) and (3.14), we have

$$\|M(u_{n+1}) - M(\bar{u})\|^2 \leq \|M(w_n) - M(\bar{u})\|^2 - \|M(u_{n+1}) - M(w_n)\|^2,$$

the required (3.5).
Taking $v = \bar{u}$ in (3.3) and $v = w_n$ in (3.9), we have

$$\beta F(T(\bar{u}), g(w_n)) \geq 0 \qquad (3.15)$$

and

$$\beta F(y_n, g(\bar{u})) + \langle M(w_n) - M(y_n), M(\bar{u}) - M(w_n)\rangle \geq 0. \qquad (3.16)$$

Adding (3.15) and (3.16) and rearranging the terms, we have

$$\langle M(w_n) - M(y_n), M(\bar{u}) - M(w_n)\rangle \geq 0, \qquad (3.17)$$

since $F(.,.)$ is g-partially relaxed jontly monotone operator.

Now taking $v = M(w_n) - M(y_n)$ and $u = M(\bar{u}) - M(w_n)$ in (2.7), (3.17) can be written as

$$\|M(\bar{u}) - M(w_n)\|^2 \leq \|M(\bar{u}) - M(y_n)\|^2 - ||M(y_n) - M(w_n)||^2,$$

the required (3.6).

Similarly, by taking $v = \bar{u}$ in (3.4) and $v = u_{n+1}$ in (3.10) and using g-partially relaxed jointly monotonicity of $F(.,.)$, we have

$$\langle M(y_n) - M(u_n), M(\bar{u}) - M(y_n)\rangle \geq 0. \qquad (3.18)$$

Letting $v = y_n - u_n$, and $u = \bar{u} - y_n$ in (2.7), and combining the resultant with (3.18), we have

$$\|M(y_n) - M(\bar{u})\|^2 \leq \|M(\bar{u}) - M(u_n)\|^2 - \|M(y_n) - M(u_n)\|^2,$$

the required (3.7). □

Theorem 3.1. *Let $M : H \longrightarrow H$ be invertible and u_{n+1} be the approximate solution obtained from Algorithm 3.1 . If $\bar{u} \in K$ be the exact solution of (2.1), then $\lim_{n\to\infty} u_n = \bar{u}$.*

Proof. Let $\bar{u} \in K$ be a solution of (2.1). Then from (3.5), (3.6) and (3.7), it follows that the sequence $\{\|M(w_n) - M(\bar{u})\|\}, \quad \{\|M(y_n) - M(\bar{u}\|\}$ and $\{\|M(\bar{u}) - M(u_n)\|\}$ are nonincreasing and consequently $\{u_n\}, \{w_n\}$ and $\{y_n\}$ are bounded. Furthermore, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \|M(u_{n+1}) - M(w_n)\|^2 &\leq \|M(w_0) - M(\bar{u})\|^2, \\
\sum_{n=0}^{\infty} \|M(w_n) - M(y_n)\|^2 &\leq \|M(y_0) - M(\bar{u})\|^2 \\
\sum_{n=0}^{\infty} \|M(y_n) - M(u_n)\|^2 &\leq \|M(u_0) - M(\bar{u}\|^2
\end{aligned}$$

which imply that

$$\begin{aligned}
\lim_{n\to\infty} \|M(u_{n+1}) - M(w_n)\| &= 0 \\
\lim_{n\to\infty} \|M(w_n) - M(y_n)\| &= 0 \\
\lim_{n\to\infty} \|M(y_{n+1}) - M(u_n)\| &= 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} \|M(u_{n+1}) - M(u_n)\| \leq \{&\lim_{n\to\infty} \|M(u_{n+1}) - M(w_n)\| \\
&+ \lim_{n\to\infty} \|M(w_n) - M(y_n)\| + \lim_{n\to\infty} \|M(y_n) - M(u_n)\|\} = 0. \quad (3.19)
\end{aligned}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing w_n and y_n by u_{n_j} in (3.2),(3.3) and (3.4), taking the limit $j \longrightarrow \infty$ and using (3.19), we have

$$F(\hat{u}, g(v)) \geq 0, \quad \forall v \in K,$$

which implies that $\hat{u}$ solves the general equilibrium problems (2.1) and

$$\|M(u_{n+1}) - M(\bar{u})\|^2 \leq \|M(u_n) - M(\bar{u})\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} M(u_n) = M(\hat{u}).$$

Since M is invertible, thus

$$\lim_{n\to\infty}(u_n) = \hat{u},$$

the required result. □

We now show that the auxiliary principle technique can be used to suggest and analyze an inertial proximal method for solving general bifunction equilibrium problems (2.1). Inertial proximal methods have been introduced by Noor [23], for solving variational inequalities. Alvarez and Attouch [1] studied these proximal methods for maximal monotone operators associated with the discretization of the second-order differential equations in time. It is known that inertial proximal method includes the classical proximal method as a special case.

For a given $u \in K$ satisfying (2.1), consider the auxiliary problem of finding $w \in K$ such that

$$\rho F(T(w), g(v)) + \langle M(w) - M(u) - \alpha(M(u) - M(u)), M(v) - M(w)\rangle \geq 0, \forall v \in K, \quad (3.20)$$

where $\rho > 0$ and $\alpha > 0$ are constants.
Note that if $w = u$, then w is a solution of (2.1). This fact enables us to suggest the following iterative method for solving general equilibrium problems (2.1).

Remark 3.1. *We would like to mention that the auxiliarly problems such as (3.1) and (3.20) are all distinctly different from each others. This shows shows that one a suggest auxililrly problems associated with one problem according to his own choice. This is the fascinating and novel aspects of the auxililary principle technique. We has just mention this idea for the interested research to explore these concepts. We have also proved that the resolvent method and projection methods can be obtained as special cases of the marvolous approach. For excellent discussion this technique, see [7, 9, 11, 21, 23, 28, 29, 30, 32, 36, 41] and the references therein.*

Algorithm 3.5. *For a given u_0, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(T(u_{n+1}), g(v)) + \big\langle M(u_{n+1}) - M(u_n) - \alpha_n(M(u_n) - M(u_{n-1})), M(v) - M(u_{n+1})\big\rangle \geq 0, \forall v \in K, \quad (3.21)$$

where $\rho > 0$ and $\alpha_n > 0$ are constants.

Algorithm 3.5 is known as the inertial proximal method.

For $\alpha_n = 0$ Algorithm 3.5 reduces to:

Algorithm 3.6. *For a given u_0, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(T(u_{n+1}), g(v)) + \langle M(u_{n+1}) - M(u_n), M(v) - M(u_{n+1})\rangle \geq 0, \quad \forall v \in K,$$

which is known as the proximal method for solving general equilibrium problem (2.1). For $g = I$, where I is the identity operator, we obtain an inertial proximal method for equilibrium problems (2.2).

If $F(u, g(v)) = \langle T(u), g(v) - g(u)\rangle, T : H \longrightarrow H$ a nonlinear operator, Algorithm 3.5 is exactly the same as considered for solving general variational inequalities (2.3). In a similar way, one can obtain a variant form of inertial proximal methods for solving variational inequalities and equilibrium problems as special cases.

We now study the convergence analysis of Algorithm 3.5 using the technique of Noor [23]. For the sake of completeness and to convey an idea of the techniques involved, we sketch the main points only.

Theorem 3.2. *Let $\bar{u}$ be a solution of (2.1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.5. If $F(.,.) : H \times H \longrightarrow H$ is g-jointly pseudomonotone, then*

$$\begin{aligned} \|M(\bar{u}) - M(u_{n+1})\|^2 \leq \ & \|M(\bar{u}) - M(u_n)\|^2 \\ & - \|M(u_{n+1}) - M(u_n) - \alpha_n(M(u_n) - M(u_{n-1}))\|^2 \\ & + \alpha_n\{\|M(\bar{u}) - M(u_n)\|^2 - \|M(\bar{u}) - M(u_{n-1})\|^2 \\ & + 2\|M(u_n) - M(u_{n-1})\|^2\}. \end{aligned} \tag{3.22}$$

Proof. Let $\bar{u} \in K$ be a solution of (2.1). Then

$$F(T(\bar{u}), g(v)) \geq 0, \quad \forall v \in K,$$

which implies that

$$-F(T(v), g(\bar{u})) \geq 0, \quad \forall \in K, \tag{3.23}$$

since the bifunction $F(.,.)$ is g-jointly pseudomonotone. Taking $v = u_{n+1}$ in (3.23), we have

$$-F(T(u_{n+1}), g(\bar{u})) \geq 0. \tag{3.24}$$

Setting $v = \bar{u}$ in (3.21), we obtain

$$\rho F(Tu_{n+1}), g(\bar{u})) + \big\langle M(u_{n+1}) - M(u_n) \\ -\alpha(M(u_n) - M(u_{n-1})), M(\bar{u}) - M(u_{n+1})\big\rangle \geq 0. \quad (3.25)$$

Adding (3.24) and (3.25), we have

$$\langle M(u_{n+1}) - M(u_n) - \alpha_n(M(u_n) - M(u_{n-1})), M(\bar{u}) - M(u_{n+1})\rangle \geq 0,$$

which can be written as

$$\big\langle M(u_{n+1}) - M(u_n), M(\bar{u}) - M(_{n+1})\big\rangle \\ \geq \alpha_n\langle M(u_n) - M(u_{n-1}), M(\bar{u}) - M(u_n) + M(u_n) - g(u_{n+1})\rangle, \quad (3.26)$$

Using Lemma 2.1 and rearranging the terms in (3.26), one can easily obtain (3.22), the required result. □

Theorem 3.3. *Let H be a finite dimensional subspace and M be an invertible operator. Let u_{n+1} be the approximate solution obtained from Algorithm 3.5 and $\bar{u} \in M$ be a solution of (2.1). If there exists $\alpha \in (0, 1)$ such that $0 \leq \alpha_n \leq \alpha, \forall n \in N$ and*

$$\sum_{n=1}^{\infty} \alpha_n \|M(u_n) - M(u_n)\|^2 \leq \infty,$$

then $\lim_{n\longrightarrow\infty}(u_n) = \bar{u}$.

Proof.

Proof. Its proof is similar to that of Theorem 3.2. See also [1, 23]. □

4. CONCLUSION

Some new classes of general bifunction equilibrium problems are introduced and investigated. The auxiliary principle technique is used to suggest new multi step multi-step iterative methods for solving the general bifunction equilibrium problems.. Convergence analysis of the proposed method is discussed for strongly monotonicity and pseudo monontoncity, which are weaker conditions. It is an open problem to compare these proposed methods with other methods. These new methods are new novel generations of the Noor (three step) iterations, which contain Ishikwaw (two step) iterations and Mann iteration as special cases. Applying the technique and ideas of Ashish et. al.[2, 3], Natarajan et al.[12], Paimsang et al [33], Rattanaseeha et al.[34], Suantai et al [38], Tomar et al.[39] and Yadav et al. [40], can one explore the Julia set and Mandelbrot set in Noor orbit using the new Noor (three step) iterations for solving the equilibrium problems in the fixed point theory, fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. This is an open problem, which deserves further research efforts.

5. DATA AVAILABILITY:

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study

6. CONFLICT INTEREST:

Authors have no conflict of interest.

7. AUTHORS CONTRIBUTIONS:

All authors contributed equally to the conception, design of the work, analysis, interpretation of data, reviewing it critically and final approval of the version for publication.

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