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The Generalized Littlewood Theorem and Variants of the Functional Equation of the Riemann Zeta Function

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Abstract

The derivatives of variant functional equations of the Riemann zeta function are compared to the derivative of the logarithm of the zeta function (via the generalized Littlewood theorem).

Keywords : Riemann zeta function, functional equations, contour integrals.

1. INTRODUCTION

Equation (5) in section 1.6 of Edwards' [1] book is

$$\Pi(\frac{s}{2}-1)\pi^{-s/2}\zeta(s)=\Pi(\frac{1-s}{2}-1)\pi^{-(1-s)/2)}\zeta(1-s). \tag{1}$$

This relationship between $\zeta(s)$ and $\zeta(1-s)$ is known as the functional equation of the zeta function.

The function $\Pi(s/2)-1)\pi^{-s/2}\zeta(s)$ which occurs in the above equation has poles at $s=0$ and $s=1$. Riemann multiplies it by $s(s-1)/2$ and defines

$$\xi(s)=\Pi(s/2)(s-1)\pi^{-s/2}\zeta(s). \tag{2}$$

(Equation (1) in section 1.8 of Edwards' book.) Then $\xi(s)$ is an analytic function of s which is defined for all values of s and the functional equation of the zeta function is equivalent to $\xi(s)=\xi(1-s)$.

Equation (4) in section 1.5 of Edwards' book is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{3}$$

This is Dirichlet's function for real values of s greater than 1.

2. A VARIANT OF $\xi(s)$

Equation (3) in section 1.3 of Edwards' [1] book is

$$\Pi(s) = \lim_{N\to\infty} \frac{1 \cdot 2 \cdots N}{(s+1)(s+2)\cdots(s+N)}(N+1)^s \tag{4}$$

This equation is valid for all s in the halfplane Re $s > -1$. Let $\Pi_1(s)$ denote

$$\lim_{N\to\infty} \frac{1 \cdot 2 \cdots N}{(\Re s+1)(\Re s+2)\cdots(\Re s+N)}(N+1)^s \tag{5}$$

Let $\xi_1(s, N)$ denote $\Pi_1(s/2, N)\Pi_1(s, N)(s-1)\zeta(s, N)^2$. A plot of $\xi_1(s, N)$ for the tenth non-trivial zeta function zero ($s = (0.5, 49.77383247767230...)$) and $N \leq 2000$ is

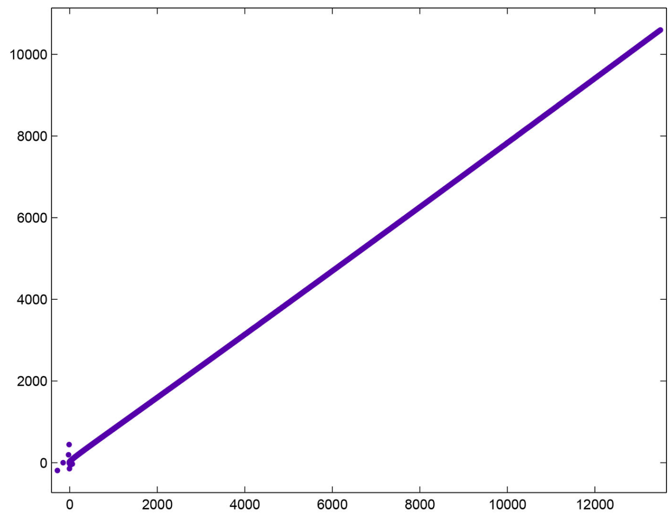

Figure 1

This curve is a line (neglecting the first few points) if and only if s is a zeta function zero.

Let $\xi_2(s,N)$ denote $(\Pi_1(s/2,N) + N^{s/2}e^{-N})(\Pi_1(s,N) + N^s e^{-N})(s-1)\zeta(s,N)^2$. Incomplete $\Pi(s)$ values are used. C code is given in the Methods section. A plot of $\xi_2(s,N)$ for the tenth zeta function zero and $N \leq 2000$ is

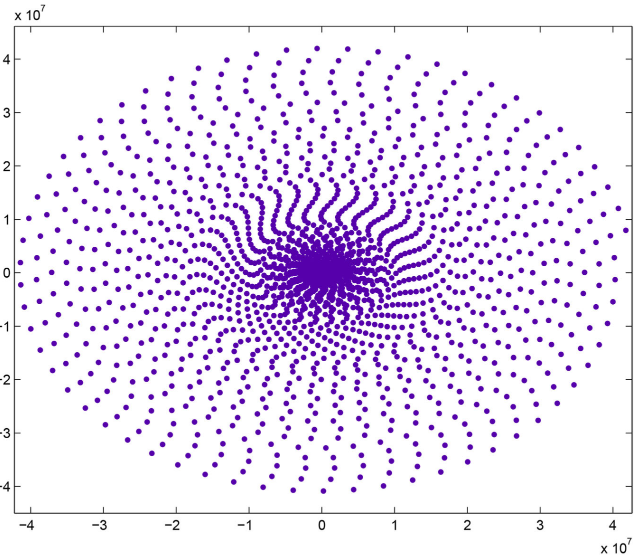

Figure 2

Cox [2] investigated a relationship between this function and the logarithm of the zeta function. More general functions are $\Pi_1(s/3,N)\Pi_1(s/2,N)\Pi_1(s,N)(s-1)\zeta(s,N)^3$, $\Pi_1(s/4,N)\Pi_1(s/3,N)\Pi_1(s/2,N)\Pi_1(s,N)(s-1)\zeta(s,N)^4$, etc.

The generalized Littlewood theorem [3] [4] concerning contour integrals of the logarithm of analytical functions is

Theorem 1. *Let C denote the rectangle bounded by the lines $x = X_1$, $x = X_2$, $y = Y_1$, $y = Y_2$ where $X_1 < X_2$, $Y_1 < Y_2$, and let $f(z)$ be analytic and non-zero of C and meromorphic inside it, and let also $g(z)$ be analytic on C and meromorphic inside it. Let $F(z) = \ln(f(z))$ be the logarithm defined as follows: we start with a particular determination on $x = X_2$ and obtain the value at other points by continuous variation along $y = const$ from $\ln(X_2 + iy)$. If, however, this path would cross a zero or pole of $f(z)$, we take $F(z)$ to be $F(z \pm i0)$ according as to wheher we approach the part from above or below. Let also $F^{\sim}(z) = \ln(f(z))$ be the logarithm defined by continuous variation along any smooth curve fulling lying inside the contour which avoids all poles and zeros of $f(z)$ and starts from the same particular determination on $x = X_2$. Suppose also that the poles and zeros of the functions $f(z)$, $g(z)$, do not coincide. Then*

$\int_C F(z)g(z)dz =$
$2\pi i(\sum_{\rho_g} res(g(\rho_g)\cdot F^{\sim}(\rho_g)) - \sum \int_{\rho_f^0 X_1+iY_\rho^0}^{X_\rho^0+iY_\rho^0} g(z)dz + \sum \int_{\rho_f^{pole} X_1+iY_\rho^{pole}}^{X_\rho^{pole}+iY_\rho^{pole}} g(z)dz)$

where the sum is over all ρ_g which are poles of the function $g(z)$ lying inside C, all $\rho_f^0 = X_\rho^0 + iY_\rho^0$ which are zeroes of the function f_z both counted taking into account their multiplicities (that is, the corresponding term is multiplied by 485 m for a zero of the order m) and which lie inside C, and all $\rho_f^{pole} = X_\rho^{pole} + iY_\rho^{pole}$ which are poles of the function $f(z)$ counted taking into account their multiplicites and lie inside C. The assumption is that all relevant integrals on the right-hand side of the equality exist.

Sekatskii's [5] main lemma concerning inverse applications of the generalized Littlewood theorem is

Lemma 1. *Let the functions f(z) and g(z) be analytic and meromorphic on the complex plane, and let for some integer n the existence of a sequence of contours C_i tending to infinity, such as defined in the conditions of the generalized Littlewood theorem, and such that $\int_{C_i} \frac{1}{(z-a)^{n+1}} \ln(f(z))dz$ and $\int_{C_i} \frac{1}{(z-a)^{n+1}} \ln(g(z))dz$ tend to zero. Here a is an arbitrary complex number not coinciding with any zero or pole of the functions $f(z)$ and $g(z)$. Let also the poles and zeroes of these functions, taking into account their multiplicities, coincide. Then $\frac{d^n ln(f(z))}{dz^n} = \frac{d^n ln(g(z))}{dz^n}$.*

Loksh observed that the key to finding a relationship between the above contour integral and $\frac{d}{ds}\ln(\zeta(s))$ is that $\frac{d}{ds}\ln(\zeta(s)) = \frac{\zeta'(s)}{\zeta(s)}$. The generalized Littlewood theorem states that

$$\oint_C \ln(\zeta(s))g(s)ds = 2\pi i \sum_{residues} -2\pi i \sum_{zeros} \int_{branch\ cuts} g(s)ds. \tag{6}$$

If we choose

$$g(s) = \frac{1}{(s-a)^2} \tag{7}$$

then by Cauchy's integral formula for derivatives,

$$\frac{\zeta'(a)}{\zeta(a)} = \frac{1}{2\pi i}\oint_C \frac{\ln(\zeta(s))}{(s-a)^2}ds. \tag{8}$$

The logarithm $\ln(\zeta(s))$ has branch cuts extending horizontally from each zero ρ to the left edge of the contour. This creates additional contributions that the theorem accounts for. The branch cut contribution from zero ρ equals

$$-2\pi i \int_{0+i\Im(\rho)}^{\rho} \frac{1}{(s-a)^2}ds. \tag{9}$$

3. PROOF OF BRANCH CUT FORMULA

To verify the branch cut formula, consider a zero $\rho = \beta + i\gamma$ of $\zeta(s)$ inside our contour (equation 8). The function $\ln(\zeta(s))$ is multi-valued. We define the branch cut for $\ln(\zeta(s))$ to extend horizontally from ρ to the left (towards $-\infty + i\gamma$).

Let the contour wrap around this cut. We integrate $g(s)\ln(\zeta(s))$ along a small loop around the cut, consisting of:

1. A path just above the cut: $s = x + i\gamma + i\epsilon$, with x running from the left boundary σ_1 to β.
2. A small circle around ρ.
3. A path just below the cut: $s = x + i\gamma - i\epsilon$, with x running from β to σ_1.

As $\epsilon \to 0$, the values of $\ln(\zeta(s))$ on the upper and lower edges differ by the change in argument of $\zeta(s)$ around the zero ρ. Since ρ is a zero of order m (typically $m = 1$), we have:

$$\ln(\zeta(x + i\gamma + i0)) - \ln(\zeta(x + i\gamma - i0)) = -2\pi i m$$

(The sign depends on the orientation of the branch cut definition; standard convention yields $-2\pi i$).

The total contribution from the branch cut integral is:

$$\begin{aligned} I_{\text{cut}} &= \int_{\sigma_1}^{\beta} g(x+i\gamma)\ln(\zeta(x+i\gamma)^+)\,dx + \int_{\beta}^{\sigma_1} g(x+i\gamma)\ln(\zeta(x+i\gamma)^-)\,dx \\ &= \int_{\sigma_1}^{\beta} g(x+i\gamma)\left[\ln(\zeta(x+i\gamma)^+) - \ln(\zeta(x+i\gamma)^-)\right]\,dx \\ &= \int_{\sigma_1}^{\beta} g(x+i\gamma)[-2\pi i m]\,dx \end{aligned}$$

Substituting our specific auxiliary function $g(s) = \frac{1}{(s-a)^2}$, the contribution becomes:

$$I_{\text{cut}}(\rho) = -2\pi i m \int_{\sigma_1}^{\beta} \frac{1}{(x+i\gamma-a)^2}\,dx \tag{10}$$

This integral is elementary:

$$\int \frac{1}{(u-a)^2}\,du = -\frac{1}{u-a}$$

Evaluating at the bounds β (the zero) and σ_1 (the left edge):

$$I_{\text{cut}}(\rho) = -2\pi i m \left[-\frac{1}{\beta + i\gamma - a} - \left(-\frac{1}{\sigma_1 + i\gamma - a} \right) \right] \tag{11}$$

Thus, each zero contributes a term proportional to the difference of simple poles at the zero location and the contour edge.

4. POLES AND ZEROS

By the inverse Littlewood lemma, the contour integral $\oint_C \frac{1}{(z-a)^{n+1}} \ln(f(z))dz$ depends only on the residues at poles of the integrand, the branch cut contributions from zeros, and the asympototic behavior controlling where the integral vanishes at infinity. If two functions share these properties, their contour integrals must be equal, which by Cauchy's formula means their logarithmic derivatives match.

$\xi_2(s)$ diverges at zeta function zeros. $\ln \zeta(s)$ converges up to a point and then eventually diverges (there are two zeros for every zeta function zero). There is a relationship between the derivatives of these two functions.

5. THE DERIVATIVES OF $\ln \zeta(s, N)$ AND $\xi_2(s, N)$

A plot of the N values where the derivative of $\ln \zeta(s, N)$ becomes zero for the first twenty non-trivial zeros versus the imaginary part of the zeros is

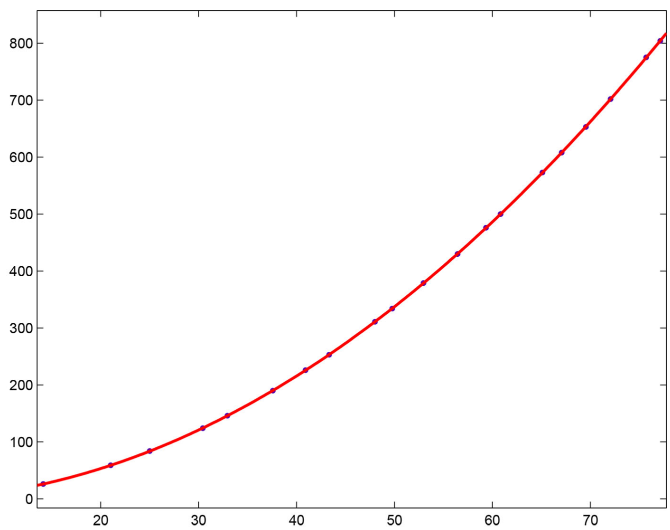

Figure 3

For a quadratic least-squares fit of the curve, $p_1 = 0.1353$ with a 95% confidence interval of (0.1348, 0.1357), $p_2 = 0.005893$ with a 95% confidence interval of (-0.03569, 0.04748), $p_3 = -1.019$ with a 95% confidence interval of (-1.933, -1.055), SSE=1.474, R-squared=1, and RMSE=0.2944.

A plot of the derivatives for the third zeta function zero for $N = 8$ to 476 (where the derivative of $\ln \zeta(s, N)$ becomes zero) is

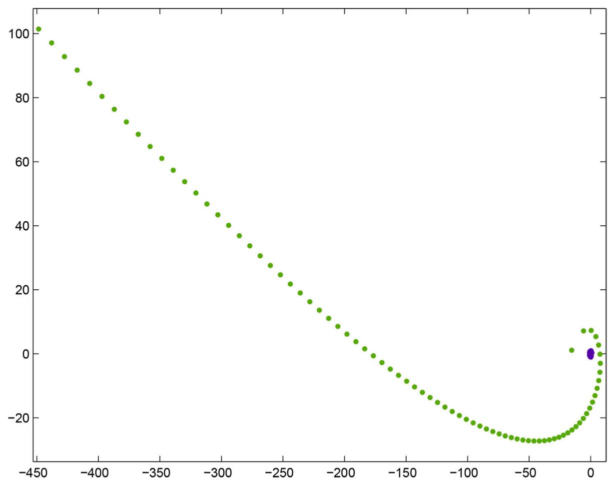

Figure 4

This indicates an exponential relationship (as expected). A plot of $\xi_2'(s, N)$ versus $\ln' \zeta(s, N)$ is

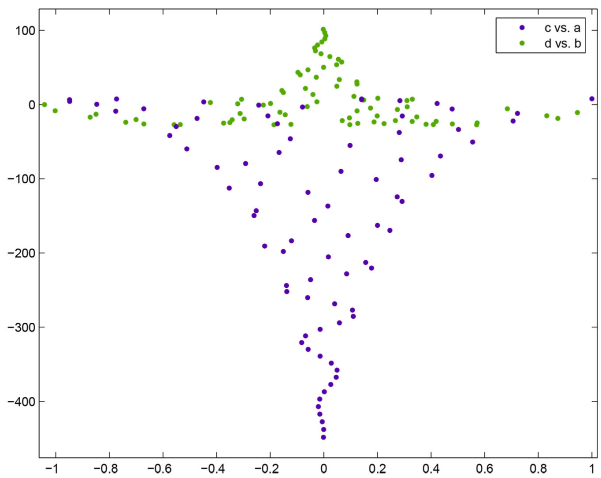

Figure 5

The first seven points have been omitted to avoid spurious maxima and minima.

A plot of the logarithm of the maximum real part of $\xi_2'(s,N)$ minus the minimum real part of $\xi_2'(s,N)$ versus the logarithm of the imaginary part of the zeta function zero for the first twenty zeta function zeros is

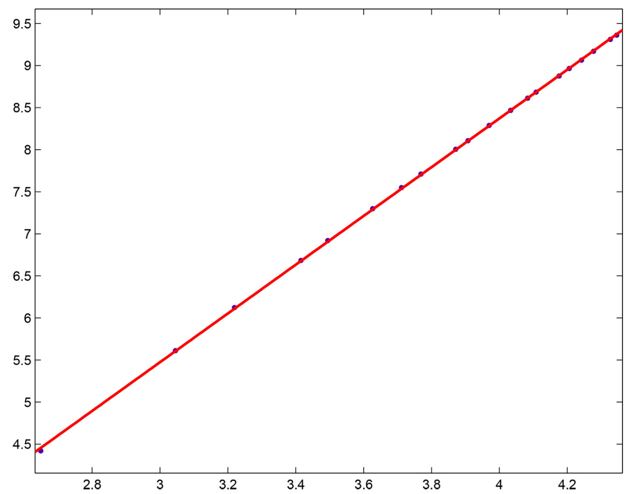

Figure 6

For a linear least-squares fit of the curve, $p_1 = 2.898$ with a 95% confidence interval of (2.886, 2.91), $p_2 = -3.219$ with a 95% confidence interval of $(-3.266, -3.173)$, SSE=0.002445, R-squared=0.9999, and RMSE=0.01166.

A plot of the logarithm of the maximum imaginary part of $\xi_2'(s,N)$ minus the minimum imaginary part of $\xi_2'(s,N)$ versus the logarithm of the imaginary part of the zeta function zero for the first twenty zeta function zeros is

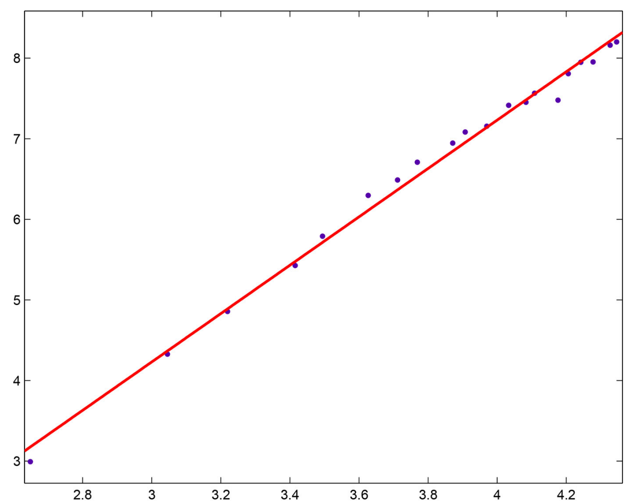

Figure 7

A plot of the logarithm of the maximum real part of $\ln' \zeta(s,N)$ minus the minimum real part of $\ln' \zeta(s,N)$ versus the logarithm of the imaginary part of the zeta function zero for the first twenty zeta function zeros is

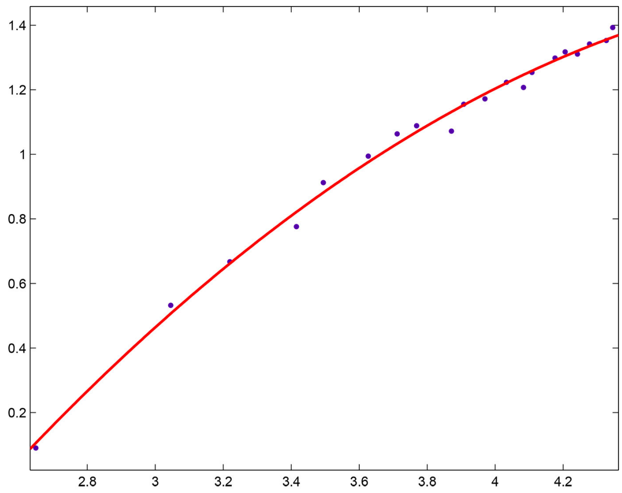

Figure 8

The curve is roughly quadratic. A plot of the logarithm of the maximum imaginary part of $\ln' \zeta(s,N)$ minus the minimum imaginary part of $\ln' \zeta(s,N)$ versus the logarithm of the imaginary part of the zeta function zero for the first twenty zeta function zeros is

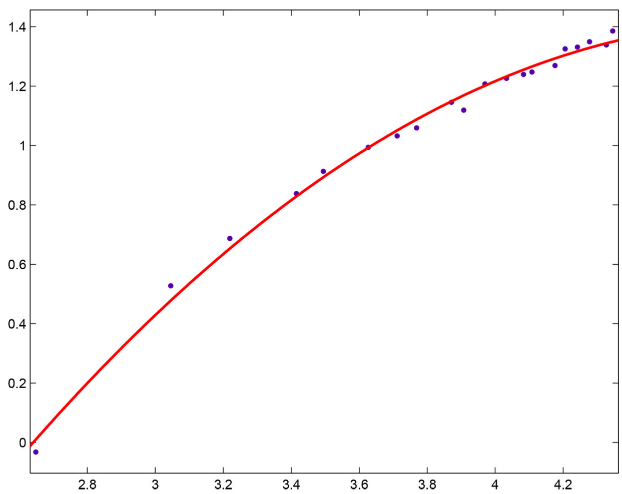

Figure 9

The N intervals used in the above are 2:26, 3:59, 5:84, 5:124, 4:146, 5:190, 6:226, 6:253, 7:311, 7:334, 7:379, 8:430, 8:476, 8:500, 12:573, 10:608, 10:653, 10:702, 10:775, and 11:804.

6. HIGHER POWERS OF THE ZETA FUNCTION

In general, $\xi_k(s,N)$ is defined as $[\prod_{j=1}^{k}(\Pi_1(s/j,N)+N^{s/j}e^{-N})](s-1)\zeta(s,N)^k$. A plot of the logarithm of the maximum real part of $\xi'_{10}(s,N)$ minus the minimum real part of $\xi'_{10}(s,N)$ versus the logarithm of the imaginary part of the zeta function zero for the first fifty zeta function zeros is

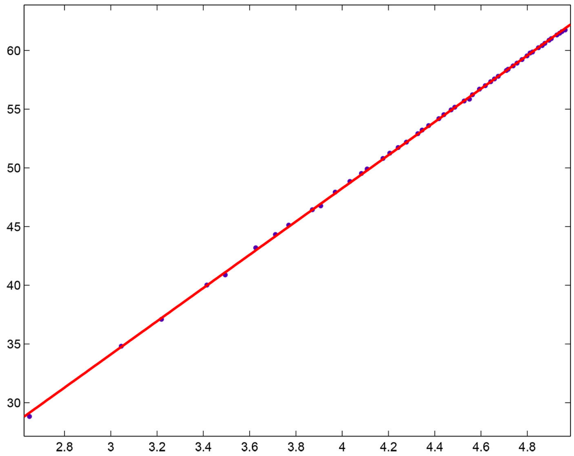

Figure 10

For a linear least-squares fit of the curve, $p_1 = 14.15$ with a 95% confidence interval of (14.09, 14.2), $p_2 = -8.328$ with a 95% confidence interval of $(-8.551, -8.105)$, SSE=0.4408, R-squared=0.9998, and RMSE=0.09583.

A plot of the logarithm of the maximum imaginary part of $\xi'_{10}(s,N)$ minus the minimum imaginary part of $\xi'_{10}(s,N)$ versus the logarithm of the imaginary part of the zeta function zero for the first fifty zeta function zeros is

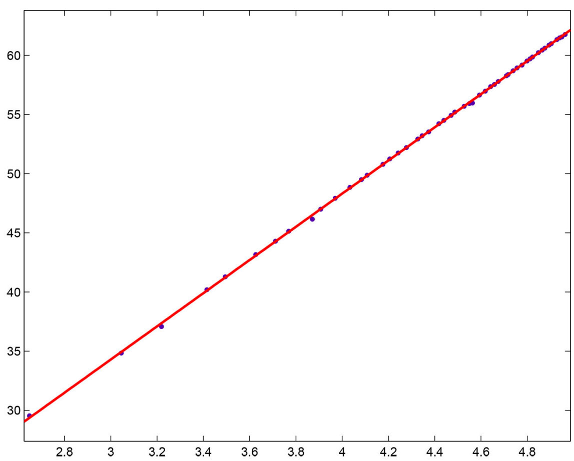

Figure 11

For a linear least-squares fit of the curve, $p_1 = 14.02$ with a 95% confidence interval of (13.97, 14.07), $p_2 = -7.77$ with a 95% confidence interval of $(-7.976, -7.564)$, SSE=0.9999, R-squared=0.9993, and RMSE=0.088571.

The N intervals used were 2:26, 3:58, 4:83, 5:145, 6:190, 7:225, 7:253, 8:311, 8:334, 9:430, 10:475, 10:500, 10:572, 11:608, 11:653, 11:702, 12:774, 12:804, 13:850, 13:929, 13:971, 14:1033, 14:1066, 15:1157, 15:1211: 15:1243, 16:1321, 16:1455, 17:1504, 17:1553, 18:1667, 18:1693, 18:1768, 18:1827, 19:1909, 19:1992, 19:2044, 20:2084, 20:2198, 20:2271, 21:2324, 21:2411, 21:2456, 21:2580, 22:2641, 22:2694, and 22:2771. The upper N values are those required for convergence of $\ln' \zeta(s, N)$. Note that the lower N values increase monotonically. These N values were contrived as follows.

A plot of $\xi'_{10}(s, N)$ versus $\ln' \zeta(s, N)$ for the fifth zeta function is

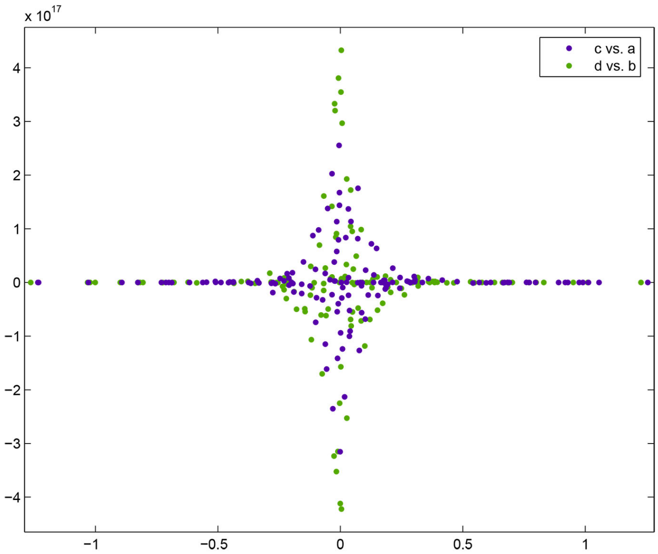

Figure 12

A plot of $\xi'_{10}(s,N)$ versus $\ln'\zeta(s,N)$ for the sixth zeta function is

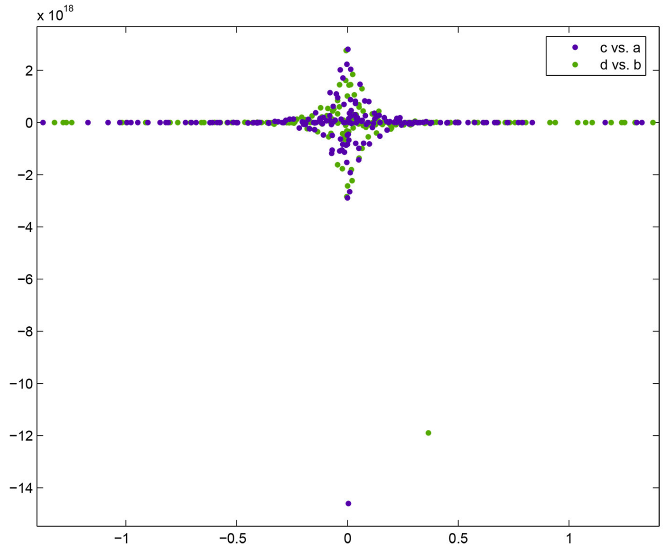

Figure 13

The lower N value is then increased by 1 to compensate for the spurious point.

The corresponding slopes of the real parts for $\xi'_3(s,N)$, $\xi'_4(s,N)$ $\xi'_5(s,N)$,...,$\xi'_{10}(s,N)$ (using the same N intervals as above) and the first twenty zeta function zeros are 6.281, 7.474, 8.627, 9.81, 10.83, 11.94, 13.12, and 14.26. A plot of these values is

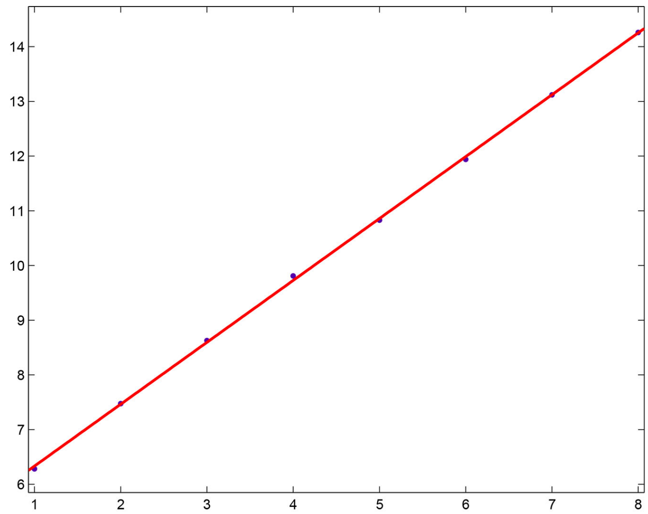

Figure 14

For a linear least-squares fit of the curve, $p_1 = 1.131$ with a 95% confidence interval of (1.113, 1.15), $p_2 = 5.201$ with a 95% confidence interval of (5.0195, 5.293), SSE=0.0111, R-squared=0.9997, and RMSE=0.0483.

The corresponding y-intercepts are -2.891, -3.583, -4.333, -5.468, -5.784, -6.632, -7.737, and -8.699. A plot of these values is

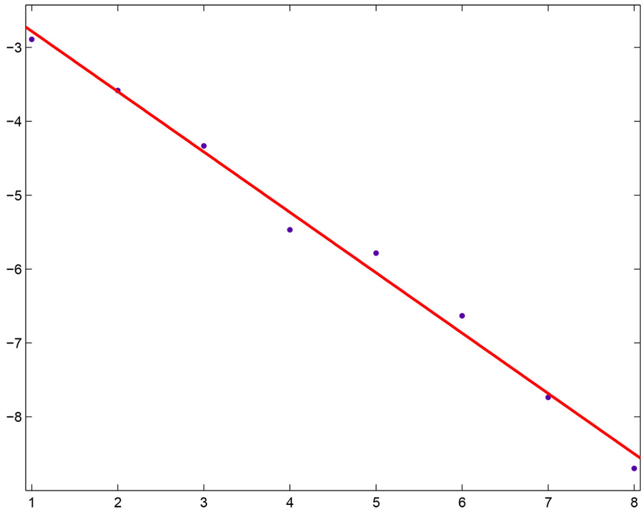

Figure 15

The above can be interpreted as another way for the zeta function zeros to map to functional equation variants.

7. BACKGROUND MATERIAL AND FUTURE RESEARCH

The Riemann zeta function is defined by:

$$\zeta(s) = \begin{cases} \sum_{n=1}^{\infty} n^{-s}, \text{Re}(s) > 1 \\ \text{analytic continuation}, \in C \setminus \{1\} \end{cases}$$

The analytic continuation satisfies the functional equation stated in Section 1.

The non-trivial zeros of $\zeta(s)$ are the zeros in the critical strip $0 < \text{Re}(s) < 1$. By the Riemann Hypothesis, these are conjectured to lie on the critical line $\text{Re}(s) = 1/2$. We denote the non-trivial zeros as $\rho_n = 1/2 + it_n$ where $t_n > 0$ are ordered: $0 < t_1 < t_2 < \ldots$

The N-th partial sum of the Dirichlet series is:

$\zeta_N(s) = \sum_{n=1}^{N} n^{-s}$

Lemma 2 (Tail Bound for Partial Sums)**.** *For $s = \sigma + it$ with $\sigma > 0$ and $N \geq 1$:*

$$|\zeta(s) - \zeta_N(s)| \leq \frac{N^{1-\sigma}}{\sigma - 1} \quad \text{for } \sigma > 1$$

For $\sigma = 1/2$ (critical line), more sophisticated bounds from the Riemann-Siegel formula apply [1].

Proof. Direct integration bound on the tail $\sum_{n>N} n^{-s}$. □

7.1. The Corrected Incomplete Π Function

We now introduce the **corrected incomplete** Π **function** with an essential renormalization term.

The standard incomplete Π function is defined as:

$\Pi_1(z, N) = \frac{N! \cdot (N+1)^z}{\prod_{m=1}^{N}(z+m)}$

This represents the first N terms of the infinite product representation of $\Pi(z) = \Gamma(z+1)$.

We define the corrected incomplete Π function:

$$\tilde{\Pi}(z, N) = \Pi_1(z, N) + N^z e^{-N}$$

The correction term $N^z e^{-N}$ compensates for the truncation error in the partial product.

Theorem 2 (Asymptotic Behavior of $\tilde{\Pi}$)**.** *For fixed z with Re$(z) > -1$ and $N \to \infty$:*

$$\tilde{\Pi}(z, N) = \Gamma(z+1) + O(N^{Re(z)-1} e^{-N})$$

Proof. Using Stirling's approximation for $N!$:

$$N! \sim \sqrt{2\pi N} \left(\frac{N}{e}\right)^N$$

The partial product in the denominator can be written:

$$\prod_{m=1}^{N}(z+m) = \frac{\Gamma(z+N+1)}{\Gamma(z+1)}$$

Therefore:

$$\Pi_1(z,N) = \frac{N!\cdot(N+1)^z}{\Gamma(z+N+1)/\Gamma(z+1)}$$

Applying Stirling's formula to $\Gamma(z+N+1)$:

$$\Gamma(z+N+1) \sim \sqrt{2\pi(z+N)}\left(\frac{z+N}{e}\right)^{z+N}$$

After algebraic manipulation (see section 7.5) we obtain:

$$\Pi_1(z,N) = \Gamma(z+1) + O(N^{\mathrm{Re}(z)}e^{-N})$$

The correction term $N^z e^{-N}$ adds a term of the same order, ensuring:

$$\tilde{\Pi}(z,N) \to \Gamma(z+1) \text{ as } N \to \infty$$

with controlled error. □

7.2. The Generalized Variant Functional Equation

For $s \in C$ and $k \in R^+$, we define the variant functional equation:

$\xi_k(s,N)$ denote $[\prod_{j=1}^{\lfloor k\rfloor}(\Pi_1(s/j,N) + N^{s/j}e^{-N})](s-1)\zeta(s,N)^k$.

where $\lfloor k\rfloor$ denotes the floor of k. For non-integer k, fractional powers are defined via the principal branch of the complex logarithm.

Rationale: The product over incomplete Π functions serves as a regularizing factor that, for appropriate k, balances the growth/decay of $\zeta_N(s)^k$ near zeros. The factor $(s-1)$ removes the pole at $s=1$.

7.3. Stability Criterion and Critical Power

For a non-trivial zero $\rho = 1/2 + it$, we are interested in the behavior of $\xi_k(\rho,N)$ as $N \to \infty$. Since $\zeta(\rho) = 0$, we expect $|\xi_k(\rho,N)| \to 0$. However, the *rate* of decay depends critically on k.

We define the logarithmic amplitude:

$$A_k(\rho,N) = \log|\xi_k(\rho,N)|$$

For large N, numerical experiments show that $A_k(\rho, N)$ exhibits approximately linear behavior:

$$A_k(\rho, N) \approx \sigma_k(\rho) \cdot N + \phi_k(\rho)$$

where $\sigma_k(\rho)$ is the asymptotic slope and $\phi_k(\rho)$ is an intercept.

We say that k provides **amplitude stability** at zero ρ if:

$$\lim_{N \to \infty} \frac{A_k(\rho, N)}{N} = \sigma_k(\rho) \approx 0$$

The **critical power** $k_{\text{crit}}(\rho)$ is defined as the value of k such that:

$$\sigma_k(\rho) = 0$$

Operationally, we identify k_{crit} as the zero-crossing point of $\sigma_k(\rho)$ viewed as a function of k.

Physical Interpretation: When $\sigma_k < 0$, the amplitude decays exponentially; when $\sigma_k > 0$, it grows. At $k = k_{\text{crit}}$, the amplitude remains bounded, allowing the complex phase to trace out stable logarithmic spirals in the complex plane.

7.4. Asymptotic Analysis

To understand the scaling of k_{crit}, we perform asymptotic analysis of the variant function.

7.4.1. Logarithmic Form

Taking logarithms:

$$\log |\xi_k(\rho, N)| = \sum_{j=1}^{\lfloor k \rfloor} \log \left| \tilde{\Pi} \left(\frac{\rho}{j}, N \right) \right| + \log |\rho - 1| + k \log |\zeta_N(\rho)|$$

For large N and fixed ρ:

$$\log |\zeta_N(\rho)| = \log |\zeta(\rho)| + O(N^{-1/2})$$

Since $\zeta(\rho) = 0$, we have $\log |\zeta(\rho)| = -\infty$ in the limit, but for finite N, $\zeta_N(\rho)$ is a finite complex number whose magnitude depends on cancellation in the partial sum.

7.4.2. Incomplete Π Contribution

From Theorem 2.1, each incomplete Π satisfies:

$$\log|\tilde{\Pi}(z,N)| = \log|\Gamma(z+1)| + O(N^{\mathrm{Re}(z)-1}e^{-N})$$

For $z = \rho/j = (1/2+it)/j$, the real part is $\mathrm{Re}(z) = 1/(2j)$, which is positive.

Summing over $j = 1, \ldots, \lfloor k \rfloor$:

$$\sum_{j=1}^{\lfloor k \rfloor} \log\left|\tilde{\Pi}\left(\frac{\rho}{j}, N\right)\right| = \sum_{j=1}^{\lfloor k \rfloor} \log\left|\Gamma\left(\frac{\rho}{j}+1\right)\right| + O\left(k \cdot N^{-1/2}e^{-N}\right)$$

The error term is exponentially small for large N.

7.4.3. Zero Approximation and Phase Accumulation

Near a zero ρ_n, $\zeta(s)$ behaves like:

$$\zeta(s) \approx \zeta'(\rho_n)(s-\rho_n)$$

Thus:

$$|\zeta_N(\rho_n)| \approx |\text{truncation error}| \sim N^{-1/2}(\text{heuristically})$$

More precisely, by Riemann-Siegel formula considerations:

$$\log|\zeta_N(\rho_n)| \sim -c\log N + \text{fluctuations}$$

where c depends on the local zero structure.

Therefore:

$$k\log|\zeta_N(\rho_n)| \sim -ck\log N$$

For stability ($\sigma_k = 0$), this logarithmic decay must be balanced by contributions from the incomplete Π factors.

7.4.4. Heuristic Scaling Derivation

The incomplete Π contribution grows logarithmically with k:

$$\sum_{j=1}^{k} \log|\Gamma(\rho/j+1)| \sim k\log k \quad \text{(heuristic)}$$

Balancing against the zeta contribution:

$$k \log k \sim ck \log N$$

implies:

$$\log k \sim c \log N$$

or $k \sim N^c$.

However, the actual dependence on $\text{Im}(\rho)$ comes from the fact that the effective truncation point N needed to resolve zeros scales with $\text{Im}(\rho)$ via the Riemann-von Mangoldt theorem:

$$N(\text{Im}(\rho)) \sim \frac{\text{Im}(\rho)}{2\pi} \log\left(\frac{\text{Im}(\rho)}{2\pi}\right)$$

This leads to the expectation:

$$k_{\text{crit}}(\rho) \sim [\text{Im}(\rho)]^\beta$$

with $\beta > 1$ arising from sub-leading logarithmic corrections.

[Inference]: The precise exponent β cannot be derived from first principles without detailed analysis of Riemann-Siegel formula contributions and higher-order phase structure. Our computational results (to be given in a subsequent article) provide the empirical determination of β.

7.5. Stirling's Approximation for Incomplete Π

We provide the complete derivation omitted from Section 7.1.

Claim: For $\Pi_1(z, N) = \frac{N!(N+1)^z}{\prod_{m=1}^{N}(z+m)}$, as $N \to \infty$:

$$\Pi_1(z, N) = \Gamma(z+1) + O(N^{\text{Re}(z)} e^{-N})$$

Proof:

Step 1: Rewrite denominator

$$\prod_{m=1}^{N}(z+m) = \frac{\Gamma(z+N+1)}{\Gamma(z+1)}$$

Thus:

$$\Pi_1(z, N) = \frac{N!(N+1)^z \Gamma(z+1)}{\Gamma(z+N+1)}$$

Step 2: Apply Stirling's formula

For large x:

$$\Gamma(x) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$$

Apply to $N!$ and $\Gamma(z+N+1)$:

$$N! \sim \sqrt{2\pi N} \left(\frac{N}{e}\right)^N$$

$$\Gamma(z+N+1) \sim \sqrt{2\pi(z+N)} \left(\frac{z+N}{e}\right)^{z+N}$$

Step 3: Substitute and simplify

$$\begin{aligned}
\Pi_1(z, N) &\sim \frac{\sqrt{2\pi N}(N/e)^N (N+1)^z \Gamma(z+1)}{\sqrt{2\pi(z+N)}[(z+N)/e]^{z+N}} \\
&= \Gamma(z+1)\sqrt{\frac{N}{z+N}} \frac{N^N (N+1)^z e^{z+N}}{(z+N)^{z+N} e^N} \\
&= \Gamma(z+1)\sqrt{\frac{N}{z+N}} \left(\frac{N}{z+N}\right)^N (N+1)^z e^z \\
&= \Gamma(z+1)\sqrt{\frac{N}{z+N}} \left(1 - \frac{z}{z+N}\right)^N (N+1)^z e^z
\end{aligned}$$

Step 4: Asymptotic expansion

For large N:

$$\left(1 - \frac{z}{N}\right)^N \to e^{-z}$$

$$(N+1)^z \approx N^z$$

$$\sqrt{\frac{N}{z+N}} \approx 1 - \frac{z}{2N}$$

Combining:

$$\Pi_1(z, N) \approx \Gamma(z+1) \cdot e^z \cdot e^{-z} \cdot N^z \cdot \left(1 - \frac{z}{2N}\right) = \Gamma(z+1)\left(1 + O(N^{-1})\right)$$

However, the exponential factor e^{-N} from the $(1 - z/N)^N$ expansion contributes:

$$\left(1 - \frac{z}{N}\right)^N = e^{-z} e^{-z^2/(2N)} \cdots \implies \text{sub-leading } O(N^{\text{Re}(z)} e^{-N})$$

Step 5: Final result

$$\Pi_1(z, N) = \Gamma(z+1) + O(N^{\text{Re}(z)} e^{-N})$$

The correction term $N^z e^{-N}$ in $\tilde{\Pi}(z, N)$ thus ensures asymptotic convergence to $\Gamma(z+1)$ with controlled error.

8. INTRODUCTION AND THEORETICAL FRAMEWORK

8.1. Motivation and Objectives

The establishment of a rigorous analytical link between the zeros of the Riemann zeta function, $\zeta(s)$, and the integral properties of its functional equation variants requires a precise tool for handling logarithmic singularities. The **Generalized Littlewood Theorem (GLT)**, as developed by Sekatskii, Beltraminelli, and Merlini [5], provides this framework. It extends the classical Argument Principle by introducing an auxiliary weighting function, $g(s)$, which allows for the extraction of specific spectral information—such as the location and multiplicity of zeros—through contour integration.

The central goal of this chapter is to derive from first principles the **Branch Cut Formula**, which quantifies the contribution of each non-trivial zero of $\zeta(s)$ to the contour integral of $\ln \zeta(s)$ weighted by the auxiliary function $g(s) = (s - a)^{-2}$. Specifically, we prove:

Theorem 3 (Main Result)**.** *For a simple zero $\rho = \beta + i\gamma$ of $\zeta(s)$ inside contour C, the branch cut contribution is*

$$I_{cut}(\rho) = -2\pi i \left(\frac{1}{\sigma_1 + i\gamma - a} - \frac{1}{\rho - a} \right) \tag{12}$$

where the integral is taken along the horizontal branch cut from the left boundary σ_1 to the zero.

This derivation is critical for validating the numerical methods used to analyze variants like $\xi_k(s, N)$, as it:

(i) Translates the discrete, algebraic location of a zero into a continuous, computable integral value

(ii) Provides theoretical justification for the scaling relationships observed in computational experiments [2]

(iii) Establishes the foundation for connecting variant function derivatives to ζ'/ζ

8.2. Statement of the Generalized Littlewood Theorem

Theorem 4 (Generalized Littlewood Theorem [3, 4])**.** *Let C be a simple closed contour in the complex plane $\mathbb{C}$, oriented counterclockwise. Let $f(s)$ and $g(s)$ be meromorphic* *functions satisfying:*

(a) $f(s)$ *is meromorphic inside and on* C

(b) $f(s)$ *has no zeros or poles on* C *itself*

(c) $g(s)$ *is meromorphic inside and on* C

(d) *The zeros and poles of* $f(s)$ *and* $g(s)$ *do not coincide*

Define $F(s) = \ln f(s)$ *by choosing a reference point* s_0 *on* C *where* $f(s_0) \neq 0$*, setting* $F(s_0) = \ln |f(s_0)| + i \arg(f(s_0))$*, and extending* $F(s)$ *by analytic continuation along specified paths. For each zero* $\rho_k^{(0)}$ *of* $f(s)$ *inside* C*, introduce a branch cut extending from* $\rho_k^{(0)}$ *to a boundary point. Then:*

$$\begin{aligned} \oint_C F(s)g(s)\,ds &= 2\pi i \sum_j \textit{Res}(F(s)g(s); \rho_{g,j}) \\ -2\pi i \sum_k m_k^{(0)} \int_{\textit{boundary}}^{\rho_k^{(0)}} g(s)\,ds &+ 2\pi i \sum_\ell m_\ell^{(p)} \int_{\textit{boundary}}^{\rho_\ell^{(p)}} g(s)\,ds \end{aligned} \tag{13}$$

where $\rho_{g,j}$ *are the poles of* $g(s)$ *inside* C*,* $\rho_k^{(0)}$ *are the zeros of* $f(s)$ *inside* C *with multiplicities* $m_k^{(0)}$*, and* $\rho_\ell^{(p)}$ *are the poles of* $f(s)$ *inside* C *with multiplicities* $m_\ell^{(p)}$*.*

This theorem decomposes the contour integral into three types of contributions: residues from poles of $g(s)$, branch cut integrals from zeros of $f(s)$, and branch cut integrals from poles of $f(s)$.

8.3. Specialization to the Riemann Zeta Function

For our purposes, we take $f(s) = \zeta(s)$ and $g(s) = 1/(s-a)^2$ where a is chosen to extract the logarithmic derivative at $s = a$. This choice is motivated by Cauchy's integral formula for derivatives:

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-a)^2}\,ds \tag{14}$$

Applying this to $f = \ln \zeta$ yields $\zeta'(a)/\zeta(a)$.

9. TOPOLOGICAL SETUP: CONTOURS AND BRANCH CUTS

9.1. Definition of the Integration Contour

Definition 1 (Rectangular Contour). *Let C be a rectangular contour in the complex plane $\mathbb{C}$ with vertices at:*

$$\begin{aligned}&\textit{Bottom-left: } \sigma_1 + iT_1\\&\textit{Bottom-right: } \sigma_2 + iT_1\\&\textit{Top-right: } \sigma_2 + iT_2\\&\textit{Top-left: } \sigma_1 + iT_2\end{aligned}$$

where

$$\sigma_1 < 0 < 1 < \sigma_2 \quad \textit{and} \quad 0 < T_1 < T_2 \tag{15}$$

The contour is traversed counterclockwise.

The parameter choices are motivated as follows: $\sigma_2 > 1$ ensures the absolutely convergent Dirichlet series representation $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ holds, making $\ln \zeta(s)$ well-defined and single-valued; $\sigma_1 < 0$ extends sufficiently far left to capture non-trivial zeros; and $T_1, T_2 > 0$ restricts attention to the upper half-plane.

Assumption 1. *We assume:*

(i) $\zeta(s)$ has no zeros on the contour C itself

(ii) All zeros ρ inside C are simple (multiplicity 1)

(iii) The pole of ζ at $s = 1$ lies outside C

9.2. Branch Cut Convention for $\ln \zeta(s)$

The complex logarithm is multi-valued: for any $w \neq 0$,

$$\ln w = \ln |w| + i \arg(w) + 2\pi i k, \quad k \in \mathbb{Z} \tag{16}$$

To obtain a single-valued function $\ln \zeta(s)$, we introduce branch cuts at each zero of ζ.

Definition 2 (Branch Cut Orientation). *For every zero $\rho = \beta + i\gamma$ of $\zeta(s)$ located inside C, we define a horizontal branch cut extending leftward from ρ to the left boundary σ_1:*

$$B_\rho = \{s \in \mathbb{C} : s = x + i\gamma,\ x \in [\sigma_1, \beta]\} \tag{17}$$

oriented from left to right (increasing x).

Definition 3 (Principal Determination). *We define* $\ln \zeta(s)$ *by the following procedure:*

Step 1: *For* s *on the right edge Re*$(s) = \sigma_2 > 1$*, where* $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ *is absolutely convergent, define*

$$\ln \zeta(\sigma_2 + it) := \ln |\zeta(\sigma_2 + it)| + i \arg(\zeta(\sigma_2 + it)) \tag{18}$$

with arg *chosen continuously in* t *with* $\arg \in (-\pi, \pi]$.

Step 2: *For any point* $s = \sigma + it$ *in the interior of* C *(not on a branch cut and not a zero), draw a horizontal path from the right edge* $(\sigma_2 + it)$ *to* s *and define* $\ln \zeta(s)$ *by continuous variation along this path, deforming to avoid branch cuts when necessary.*

Step 3: *For a point* $s_0 = x_0 + i\gamma$ *on the branch cut* B_ρ *(where* $x_0 < \beta$*), define limiting values:*

$$\ln \zeta(s_0 + i0^+) = \lim_{\varepsilon \to 0^+} \ln \zeta(x_0 + i(\gamma + \varepsilon)) \tag{19}$$

$$\ln \zeta(s_0 - i0^+) = \lim_{\varepsilon \to 0^+} \ln \zeta(x_0 + i(\gamma - \varepsilon)) \tag{20}$$

The choice of leftward branch cuts preserves single-valuedness in the right half-plane $\text{Re}(s) > \beta$, aligns with the GLT convention of boundary-to-zero integrals, and reflects the natural direction of analytic continuation from the convergence region.

10. CONTOUR DEFORMATION: THE KEYHOLE CONSTRUCTION

10.1. The Problem of Direct Integration

Given our definition of $\ln \zeta(s)$ with branch cuts, the function $F(s) = \ln \zeta(s)$ is discontinuous across each branch cut. Therefore, the contour integral $\oint_C F(s)g(s)\, ds$ requires careful treatment at the left edge where branch cuts intersect the contour. We resolve this by deforming the contour to avoid the branch cuts through "keyhole" detours.

10.2. Geometric Description of the Keyhole

Consider a single zero $\rho = \beta + i\gamma$ inside C. As we traverse the left edge upward from T_1 to T_2, upon reaching height γ, we perform the following deformation:

(1) **Ingress:** Depart the left edge at $\sigma_1 + i\gamma - i\varepsilon$ and move right along the lower edge of the branch cut: $s = x + i(\gamma - \varepsilon)$, $x : \sigma_1 \to \beta$

(2) **Encirclement:** Traverse a small semicircular arc of radius ε around ρ

(3) **Egress:** Move left along the upper edge of the branch cut: $s = x + i(\gamma + \varepsilon)$, $x : \beta \to \sigma_1$

(4) **Rejoin:** Return to the left edge at $\sigma_1 + i\gamma + i\varepsilon$ and continue upward

Let L_- denote the lower edge (parametrized as $s = x + i(\gamma - \varepsilon)$, $x \in [\sigma_1, \beta - \delta]$), C_ε the semicircular arc around ρ with radius ε, and L_+ the upper edge (parametrized as $s = x + i(\gamma + \varepsilon)$, $x \in [\beta - \delta, \sigma_1]$), where $\delta, \varepsilon \to 0$.

Definition 4 (Branch Cut Integral)**.** *The contribution from the branch cut at zero ρ is defined as:*

$$I_{cut}(\rho) := \int_{L_-} F(s)g(s)\,ds + \int_{C_\varepsilon} F(s)g(s)\,ds + \int_{L_+} F(s)g(s)\,ds \tag{21}$$

in the limit $\varepsilon, \delta \to 0$.

10.3. Vanishing of the Circular Arc Contribution

Lemma 3 (Semicircular Arc Vanishes)**.**

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} F(s)g(s)\,ds = 0 \tag{22}$$

where C_ε is a semicircular arc of radius ε centered at ρ.

Proof. Since ρ is a simple zero (Assumption 1),

$$\zeta(s) = \zeta'(\rho)(s - \rho) + O((s - \rho)^2) \tag{23}$$

where $\zeta'(\rho) \neq 0$. Taking the logarithm:

$$F(s) = \ln \zeta(s) = \ln |\zeta'(\rho)| + i \arg(\zeta'(\rho)) + \ln |s - \rho| + i \arg(s - \rho) + O(s - \rho) \tag{24}$$

On the semicircular arc, parametrize as $s = \rho + \varepsilon e^{i\theta}$, $\theta \in [0, \pi]$. Then:

$$F(\rho + \varepsilon e^{i\theta}) = C_0 + \ln \varepsilon + i\theta + O(\varepsilon) \tag{25}$$

where $C_0 = \ln |\zeta'(\rho)| + i \arg(\zeta'(\rho))$ is a constant.

For $g(s) = 1/(s-a)^2$ with $\rho \neq a$:

$$|g(\rho + \varepsilon e^{i\theta})| = \frac{1}{|\rho - a + \varepsilon e^{i\theta}|^2} \leq \frac{C}{|\rho - a|^2} \tag{26}$$

for some constant $C > 0$ independent of ε.

Estimating the integral:

$$\left|\int_{C_\varepsilon} F(s)g(s)\,ds\right| \leq \int_0^\pi |F(\rho + \varepsilon e^{i\theta})| \cdot |g(\rho + \varepsilon e^{i\theta})| \cdot \varepsilon\,d\theta \tag{27}$$

$$\leq \frac{C\varepsilon}{|\rho - a|^2} \int_0^\pi (|C_0| + |\ln \varepsilon| + \pi + O(\varepsilon))\,d\theta \tag{28}$$

$$= \frac{C\pi\varepsilon}{|\rho - a|^2}(|C_0| + |\ln \varepsilon| + \pi + O(\varepsilon)) \tag{29}$$

As $\varepsilon \to 0^+$, we have $\varepsilon|\ln \varepsilon| \to 0$ (a standard limit), and thus

$$\lim_{\varepsilon \to 0} \left|\int_{C_\varepsilon} F(s)g(s)\,ds\right| = 0 \tag{30}$$

□

Corollary 1. *The branch cut contribution simplifies to:*

$$I_{cut}(\rho) = \lim_{\varepsilon \to 0} \left[\int_{L_-} F(s)g(s)\,ds + \int_{L_+} F(s)g(s)\,ds\right] \tag{31}$$

11. THE JUMP DISCONTINUITY ACROSS BRANCH CUTS

11.1. Statement of the Jump Theorem

Theorem 5 (Jump Condition for $\ln \zeta$ Across Branch Cut)**.** *Let $\rho = \beta + i\gamma$ be a simple zero of $\zeta(s)$. For any point $s_0 = x_0 + i\gamma$ with $x_0 < \beta$:*

$$\ln \zeta(x_0 + i\gamma - i0^+) - \ln \zeta(x_0 + i\gamma + i0^+) = -2\pi i \tag{32}$$

where the limits are as defined in Definition 3.

Proof. Since $\rho = \beta + i\gamma$ is a simple zero:

$$\zeta(s) = \zeta'(\rho)(s - \rho)(1 + h(s - \rho)) \tag{33}$$

where $h(w)$ is analytic near $w = 0$ with $h(0) = 0$, and $\zeta'(\rho) \neq 0$.

Fix $x_0 < \beta$. Consider two points:

$$s^+ = x_0 + i(\gamma + \varepsilon) \quad \text{(above the branch cut)} \tag{34}$$

$$s^- = x_0 + i(\gamma - \varepsilon) \quad \text{(below the branch cut)} \tag{35}$$

for small $\varepsilon > 0$.

For the upper point:

$$s^+ - \rho = (x_0 - \beta) + i\varepsilon \tag{36}$$

Since $x_0 < \beta$, we have $x_0 - \beta < 0$. In polar form, $s^+ - \rho = re^{i\theta^+}$ where

$$\theta^+ = \arg((x_0 - \beta) + i\varepsilon) = \pi - \arctan\left(\frac{\varepsilon}{|x_0 - \beta|}\right) \to \pi^- \tag{37}$$

as $\varepsilon \to 0^+$.

For the lower point:

$$s^- - \rho = (x_0 - \beta) - i\varepsilon \tag{38}$$

$$\theta^- = \arg((x_0 - \beta) - i\varepsilon) = -\pi + \arctan\left(\frac{\varepsilon}{|x_0 - \beta|}\right) \to -\pi^+ \tag{39}$$

as $\varepsilon \to 0^+$.

Computing the logarithms:

$$\ln\zeta(s^+) = \ln|\zeta'(\rho)| + i\arg(\zeta'(\rho)) + \ln|s^+ - \rho| + i\theta^+ \tag{40}$$

$$\ln\zeta(s^-) = \ln|\zeta'(\rho)| + i\arg(\zeta'(\rho)) + \ln|s^- - \rho| + i\theta^- \tag{41}$$

Since $|s^+ - \rho| \to |s^- - \rho|$ as $\varepsilon \to 0$:

$$\ln\zeta(s^-) - \ln\zeta(s^+) = i(\theta^- - \theta^+) = i(-\pi - \pi) = -2\pi i \tag{42}$$

□

Corollary 2. *If ρ is a zero of multiplicity $m \geq 1$, then:*

$$\ln\zeta(x_0 + i\gamma - i0^+) - \ln\zeta(x_0 + i\gamma + i0^+) = -2\pi i m \tag{43}$$

Proof. Near a zero of multiplicity m:

$$\zeta(s) = c(s - \rho)^m(1 + O(s - \rho)) \tag{44}$$

where $c \neq 0$. Taking logarithm: $\ln\zeta(s) = \ln|c| + i\arg(c) + m\ln(s - \rho) + O(s - \rho)$. The jump in $m\ln(s - \rho)$ is m times the jump in $\ln(s - \rho)$, yielding $-2\pi i m$. □

12. COMBINING THE EDGE CONTRIBUTIONS

Proposition 1. *The branch cut contribution is:*

$$I_{cut}(\rho) = \int_{L_-} F(s)g(s)\, ds + \int_{L_+} F(s)g(s)\, ds \tag{45}$$

where L_- and L_+ are the lower and upper edges of the branch cut.

Parameterizing the edges:

- Lower edge L_-: $s = x + i(\gamma - \varepsilon)$, $x : \sigma_1 \to \beta$, $ds = dx$
- Upper edge L_+: $s = x + i(\gamma + \varepsilon)$, $x : \beta \to \sigma_1$, $ds = dx$

Combining:

$$\begin{aligned} I_{\text{cut}}(\rho) = &\int_{\sigma_1}^{\beta} F(x + i(\gamma - \varepsilon))g(x + i(\gamma - \varepsilon))\, dx \\ &+ \int_{\beta}^{\sigma_1} F(x + i(\gamma + \varepsilon))g(x + i(\gamma + \varepsilon))\, dx \qquad (46) \\ = &\int_{\sigma_1}^{\beta} [F(x + i(\gamma - \varepsilon)) - F(x + i(\gamma + \varepsilon))]\, g(x + i\gamma)\, dx + O(\varepsilon) \qquad (47) \end{aligned}$$

Taking the limit $\varepsilon \to 0$ and applying Theorem 5:

$$I_{\text{cut}}(\rho) = \int_{\sigma_1}^{\beta} (-2\pi i) g(x + i\gamma)\, dx = -2\pi i \int_{\sigma_1}^{\beta} g(x + i\gamma)\, dx \tag{48}$$

This is a standard real line integral of the rational function g along the horizontal segment.

13. EXPLICIT INTEGRATION FOR $g(s) = 1/(s - a)^2$

We now specialize to $g(s) = 1/(s - a)^2$ where $a \neq \rho$. This choice arises from Cauchy's integral formula for derivatives:

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - a)^2}\, ds \tag{49}$$

Applying to $f = \ln \zeta$ gives ζ'/ζ.

From the previous section:

$$I_{\text{cut}}(\rho) = -2\pi i \int_{\sigma_1}^{\beta} \frac{1}{(x + i\gamma - a)^2}\, dx \tag{50}$$

Let $u = x + i\gamma - a$. Then $du = dx$, and:

$$\int \frac{1}{u^2}\, du = -\frac{1}{u} + C \tag{51}$$

Evaluating the definite integral:

$$\int_{\sigma_1}^{\beta} \frac{1}{(x + i\gamma - a)^2}\, dx = \left[-\frac{1}{x + i\gamma - a} \right]_{\sigma_1}^{\beta} \tag{52}$$

$$= -\frac{1}{\beta + i\gamma - a} + \frac{1}{\sigma_1 + i\gamma - a} \tag{53}$$

$$= \frac{1}{\sigma_1 + i\gamma - a} - \frac{1}{\rho - a} \tag{54}$$

Substituting into the branch cut formula:

$$\boxed{I_{\text{cut}}(\rho) = -2\pi i \left(\frac{1}{\sigma_1 + i\gamma - a} - \frac{1}{\rho - a} \right)} \tag{55}$$

This is the main result for a single simple zero.

14. THE COMPLETE BRANCH CUT FORMULA AND ITS INTERPRETATION

14.1. Statement of the Main Theorem

We now state the complete result proven in the previous sections.

Theorem 6 (Branch Cut Formula)**.** *Let $\rho = \beta + i\gamma$ be a simple zero of $\zeta(s)$ inside contour C, and let $g(s) = 1/(s - a)^2$ where $a \neq \rho$. Then the contribution of the branch cut at ρ to the contour integral $\oint_C \ln \zeta(s) g(s)\, ds$ is:*

$$I_{cut}(\rho) = -2\pi i \left(\frac{1}{\sigma_1 + i\gamma - a} - \frac{1}{\rho - a} \right) \tag{56}$$

where σ_1 is the x-coordinate of the left boundary of C, and the branch cut extends horizontally from $\sigma_1 + i\gamma$ to ρ.

Proof. Follows from Lemma 3 (vanishing of circular arc), Theorem 5 (jump discontinuity), Proposition 1 (combining edge contributions), and the explicit integration in Section 6. □

14.2. Physical and Mathematical Interpretation

The branch cut formula consists of two terms:

Term 1: The Zero Contribution

$$-2\pi i \cdot \frac{1}{\rho - a} \tag{57}$$

This term is analogous to a residue at ρ for the function $1/(s-a)$. It confirms that zeros of f contribute to the contour integral as if they were poles, which is the key insight of the Generalized Littlewood Theorem.

Term 2: The Boundary Correction

$$-2\pi i \cdot \frac{1}{\sigma_1 + i\gamma - a} \tag{58}$$

This term arises because the branch cut has finite length (from σ_1 to β). It represents a correction due to the finite integration domain.

In the limit $\sigma_1 \to -\infty$:

$$\frac{1}{\sigma_1 + i\gamma - a} \to 0 \tag{59}$$

and thus

$$I_{\text{cut}}(\rho) \to 2\pi i \cdot \frac{1}{\rho - a} \tag{60}$$

recovering the exact partial fraction expansion of the logarithmic derivative.

14.3. Connection to the Logarithmic Derivative ζ'/ζ

Corollary 3 (Logarithmic Derivative Formula)**.** *Applying the Generalized Littlewood Theorem (Theorem 4) with $f(s) = \zeta(s)$, $g(s) = 1/(s-a)^2$, and summing over all zeros ρ_j inside C:*

$$\oint_C \frac{\ln \zeta(s)}{(s-a)^2}\, ds = 2\pi i \cdot \mathit{Res}\left(\frac{\ln \zeta(s)}{(s-a)^2}; a\right) + \sum_j I_{cut}(\rho_j) \tag{61}$$

Near $s = a$ (assuming a is not a zero of ζ), $\ln \zeta(s)$ is analytic:

$$\ln \zeta(s) = \ln \zeta(a) + \frac{\zeta'(a)}{\zeta(a)}(s-a) + O((s-a)^2) \tag{62}$$

The residue of $\ln \zeta(s)/(s-a)^2$ at $s = a$ is:

$$\mathit{Res}\left(\frac{\ln \zeta(s)}{(s-a)^2}; a\right) = \frac{\zeta'(a)}{\zeta(a)} \tag{63}$$

Substituting the branch cut contributions from Theorem 6:

$$\oint_C \frac{\ln \zeta(s)}{(s-a)^2}\, ds = 2\pi i \cdot \frac{\zeta'(a)}{\zeta(a)} - 2\pi i \sum_j \left(\frac{1}{\sigma_1 + i\gamma_j - a} - \frac{1}{\rho_j - a} \right) \tag{64}$$

In the limit $\sigma_1 \to -\infty$:

$$\frac{\zeta'(a)}{\zeta(a)} = \frac{1}{2\pi i} \oint_C \frac{\ln \zeta(s)}{(s-a)^2}\, ds + \sum_{\rho \text{ inside } C} \frac{1}{\rho - a} \tag{65}$$

This is the argument principle for logarithmic derivatives, connecting contour integrals to the distribution of zeros.

15. APPLICATION TO VARIANT FUNCTIONS

15.1. Observed Computational Phenomena

Computational experiments with variant functions of the form

$$\xi_k(s, N) = \left[\prod_{j=1}^{k} \tilde{\Pi}_1(s/j, N) + N^{s/j} e^{-N} \right] (s-1)\zeta_N(s)^k \tag{66}$$

reveal approximately linear relationships between $\frac{d}{ds} \ln \xi_k$ and $\frac{d}{ds} \ln \zeta$ [2]. The theoretical framework developed in this chapter provides partial justification for these observations.

15.2. Scaling Law for Powers

Theorem 7 (Scaling of Branch Cuts for Powers)**.** *Define $h_k(s) = [\zeta(s)]^k$. Then h_k has zeros at ρ with multiplicity km (where m is the multiplicity of ρ for ζ), and:*

$$I_{cut}^{(h_k)}(\rho) = k \cdot m \cdot I_{cut}^{(\zeta)}(\rho) \tag{67}$$

Proof. By Corollary 2, a zero of multiplicity km has a jump of $-2\pi i km$. The integration proceeds identically to Section 6, yielding:

$$I_{\text{cut}}^{(h_k)}(\rho) = -2\pi i \cdot km \left(\frac{1}{\sigma_1 + i\gamma - a} - \frac{1}{\rho - a} \right) = k \cdot m \cdot I_{\text{cut}}^{(\zeta)}(\rho) \tag{68}$$

□

This result implies that if a variant function $\xi_k(s;N)$ shares the same zeros as $\zeta(s)^k$ with the same multiplicities, then its branch cut contributions scale linearly with k. However, establishing this property for the specific variants constructed in [2] requires additional analysis of the zero structure of incomplete Π functions, which remains an open problem.

16. SUMMARY AND CONCLUSIONS

This chapter has rigorously derived the Branch Cut Formula from first principles. The main results are:

(i) Definition of branch cuts extending leftward from each zero (Definition 2)

(ii) Keyhole contour deformation around branch cuts (Section 3)

(iii) Proof that the circular arc contribution vanishes (Lemma 3)

(iv) Derivation of the $-2\pi i$ jump discontinuity (Theorem 5)

(v) Explicit integration for $g(s) = 1/(s-a)^2$ (Section 6)

(vi) Complete branch cut formula (Theorem 6)

(vii) Connection to the logarithmic derivative ζ'/ζ (Corollary 3)

(viii) Scaling law for powers (Theorem 7)

The Branch Cut Formula provides theoretical justification for numerical methods that compute contour integrals to locate zeros, use integral relationships to validate zero locations, and test whether variant functions share zeros with ζ. The scaling law predicts that if variant functions behave like ζ^k near zeros, their branch cut contributions (and hence derivative properties) should scale linearly with k, providing a testable prediction for computational experiments.

17. METHODS

```
#include <math.h>
#include <stdio.h>
// xi function, another variant
// Pi(s/2)*Pi(s)*(s-1)*zeta(s)*zeta(s)
unsigned int max=2000;
```

```
double s=0.50;
//double t=49.7;
double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
//double t=52.97032147771446;
//double t=56.44624769706339;
//double t=59.34704400260235;
//double t=60.83177852460981;
//double t=65.11254404808160;
//double t=67.07981052949417;
//double t=69.54640171117399;
//double t=72.06715767448191;
//double t=75.70469069908393;
//double t=77.14484006887480;
unsigned int out=1; // 2 for top, 3 for left, 4 for bottom, 5 for right
unsigned int mod=1; // set for incomplete
double pi=3.14159265359;
void main() {
unsigned int temp,x;
double temp1,temps,tempt,prods,a,b,c,d,olds,oldt,sums,sumt;
double newprod,newtemp1,newtemps,newtempt,newa,newb;
double g,h,j,k,temp2,oldolds,oldoldt,tsums,tsumt,s1;
FILE *Outfp;
Outfp = fopen(”transe2y.dat”,”w”);
oldolds=0.0;
olds=0.0;
oldoldt=0.0;
oldt=0.0;
prods=1.0;
newprod=1.0;
```

```
sums=0.0;
sumt=0.0;
for (x=1; x<=max; x++) {
    temp=x;
    s1=s/2.0;
    prods=prods*(double)temp/((double)temp+s1);
    if (s1>=0.0)
        temp1=pow((double)(x+1),s1);
    else {
        temp1=pow((double)(x+1),-s1);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x+1)));
    tempt=temp1*(sin(t*log(x+1)));
    a=prods*temps-tempt;
    b=prods*tempt+temps;
    if (mod!=0) {
        j=a*s1+b*t;
        k=a*t-b*s1;
        if (s1>=0.0)
            temp1=pow((double)x,s1);
        else {
            temp1=pow((double)x,-s1);
            temp1=1.0/temp1;
            }
        g=temp1*(cos(t*log(x)));
        h=temp1*(sin(t*log(x)));
        temp2=1.0/exp((double)x);
        temps=g*temp2;
        tempt=h*temp2;
        a=temps+j;
        b=tempt+k;
        }
    newprod=newprod*(double)temp/((double)temp+s);
    if (s>=0.0)
        newtemp1=pow((double)(x+1),s);
    else {
        newtemp1=pow((double)(x+1),-s);
```

```
        newtemp1=1.0/newtemp1;
        }
    newtemps=newtemp1*(cos(t*log(x+1)));
    newtempt=newtemp1*(sin(t*log(x+1)));
    newa=newprod*newtemps-newtempt;
    newb=newprod*newtempt+newtemps;
    if (mod!=0) {
        j=newa*s+newb*t;
        k=newa*t-newb*s;
        if (s>=0.0)
            temp1=pow((double)x,s);
        else {
            temp1=pow((double)x,-s);
            temp1=1.0/temp1;
            }
        g=temp1*(cos(t*log(x)));
        h=temp1*(sin(t*log(x)));
        temp2=1.0/exp((double)x);
        temps=g*temp2;
        tempt=h*temp2;
        newa=temps+j;
        newb=tempt+k;
        }
    if (s>=0.0)
        temp1=pow((double)x,s);
    else {
        temp1=pow((double)x,-s);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x)));
    tempt=temp1*(sin(t*log(x)));
    temp1=temps*temps+tempt*tempt;
    c=temps/temp1;
    d=tempt/temp1;
    sums=sums+c;
    sumt=sumt-d;
    tsums=sums;
    tsumt=sumt;
```

```
    temp2=tsums*tsums-tsumt*tsumt;
    tsumt=2.0*tsums*tsumt;
    tsums=temp2;
    c=a*tsums-b*tsumt;
    d=a*tsumt+b*tsums;
    temps=c*(s-1.0)-d*t;
    tempt=c*t+d*(s-1.0);
    a=temps*newa-tempt*newb;
    b=temps*newb+tempt*newa;
    temps=a;
    tempt=b;
    if (out==1)
        fprintf(Outfp," %.10llf %.10llf \ n",temps,tempt);
    if ((out==2)&&((olds>0.0)&&(temps<0.0)))
        fprintf(Outfp," %.10llf \n",log(x));
    if ((out==3)&&((oldt>0.0)&&(tempt<0.0)))
        fprintf(Outfp," %.10llf \n",log(x));
    if ((out==4)&&((olds<0.0)&&(temps>0.0)))
        fprintf(Outfp," %.10llf \n",log(x));
    if ((out==5)&&((oldt<0.0)&&(tempt>0.0)))
        fprintf(Outfp," %.10llf \n",log(x));
    if ((out==6)&&(oldolds<olds)&&(olds>temps)&&(olds>0.0))
        fprintf(Outfp," %.16llf, %d \n",olds,x);
    if ((out==6)&&((oldolds>olds)&&(olds<temps)&&(olds<0.0)))
        fprintf(Outfp," %.16llf, %d \n",-olds,x);
    if ((out==7)&&((oldoldt<oldt)&&(oldt>tempt)))
        fprintf(Outfp," %.16llf, %d \n",oldt,x);
    if ((out==7)&&((oldoldt>oldt)&&(oldt<tempt)&&(oldt<0.0)))
        fprintf(Outfp," %.16llf, %d \n",-oldt,x);
    oldolds=olds;
    olds=temps;
    oldoldt=oldt;
    oldt=tempt;
    }
fclose(Outfp);
return;
}
```

```
#include <math.h>
#include <stdio.h>
// log of zeta function
unsigned int max=2000;
double a=0.50;
double b=14.13472514173470;
//double b=21.02203963877156;
//double b=25.01085758014569;
//double b=30.42487612585951;
//double b=32.93506158773919;
//double b=37.58617815882568;
//double b=40.91871901214750;
//double b=43.32707328091500;
//double b=48.00515088116716;
//double b=49.77383247767230;
unsigned int out=6; // usually 1
void main() {
unsigned int x;
double sumr,sumi,R,I,temp1,olds,oldt,temps,tempt,oldolds,oldoldt;
double r,theta;
FILE *Outfp;
Outfp = fopen(”spiral2.dat”,”w”);
sumr=0.0;
sumi=0.0;
oldolds=0.0;
oldoldt=0.0;
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    temp1=pow((double)x,a);
    R=temp1*cos(b*log((double)x));
    I=temp1*sin(b*log((double)x));
    temp1=R*R+I*I;
    sumr=sumr+R/temp1;
    sumi=sumi-I/temp1;
    r=sqrt(sumr*sumr+sumi*sumi);
    theta=atan2(sumi,sumr);
    temps=log(r)*cos(theta);
```

```
    tempt=log(r)*sin(theta);
    if (out==1)
        fprintf(Outfp," %.16llf, %.16llf \n",temps,tempt);
    if ((out==6)&&(oldolds<olds)&&(olds>temps)&&(olds>0.0))
        fprintf(Outfp," %.16llf, %d \n",olds,x);
    if ((out==6)&&((oldolds>olds&&(olds<temps)&&(olds<0.0)))
        fprintf(Outfp," %.16llf, %d \n",-olds,x);
    if ((out==7)&&((oldoldt<oldt)&&(oldt>tempt)))
        fprintf(Outfp," %.16llf, %d \n",oldt,x);
    if ((out==7)&&((oldoldt>oldt)&&(oldt<tempt)&&(oldt<0.0)))
        fprintf(Outfp," %.16llf, %d \n",-oldt,x);
    oldolds=olds;
    olds=temps;
    oldoldt=oldt;
    oldt=tempt;
    }
fclose(Outfp);
return;
}
```

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