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An Exponential Fitted Finite Difference Method for Solving Singularly Perturbed Delay Differential Equations Two-Point Boundary Value Problems

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Abstract

This paper presents an exponentially fitted numerical method to solve a singularly perturbed delay differential equation (SPDDE) having a boundary layer at one end (left or right) point of the domain. A fitting factor is introduced in the derived scheme to improve the accuracy and to control the oscillations in the solution due to a small delay using the theory of singular perturbations. Thomas's algorithm is used to solve the tridiagonal system of equations. Convergence of the derived scheme is investigated. The scheme's applicability is shown by implementing it on four linear problems(two left-end and two right-end problems) with minimal computational effort for various values of the delay parameter δ and the perturbation parameter ε. The effect of δ (small shift) on the boundary layer(s) has also been analyzed and depicted in graphs.

Keywords: Singular perturbed; Delay Differential Equation; Stability and convergence; Finite difference method; Boundary Layer.
MSC (2010): 65L10, 65L11, 65L12, 65L20.

1. INTRODUCTION

In mathematical modeling, various practical systems are studied, including the hydrodynamics of liquid helium, the first exit time problem in modeling neuronal variability, thermo-elasticity, reaction-diffusion equations, diffusion in polymers, second-sound theory, and a variety of models for physiological processes or diseases.A SPDDE, or a singularly perturbed delay differential equation, is an ordinary differential equation that involves a tiny parameter ($0 < \varepsilon << 1$) multiplying the highest order derivative and includes at least one component with a delay.In paper [23–25] have been study of singular perturbation analysis of boundary value problems for differential-difference equations, with the presence of shift terms, which induces large amplitudes and exhibit rapid oscillations and resonances, turning point problems, boundary and interior layer behaviour. The authors [1] empolyed a non-polynomial cubic spline method to solve singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behaviour on uniform mesh. In this method, the second-order singularly perturbed delay reaction-diffusion equation transformed into an asymptotically equivalent singularly perturbed two-point boundary value problem using Taylor series expansion. Then, non-polynomial cubic spline approximations are developed into a three-term recurrence relation, solved using Thomas Algorithm. In [6] authors presents a solution for a class of singularly perturbed convection with delay problems using a parametric cubic spline method. The authors in [2] used a numerical method for solving SPDDE with a layer or oscillatory behavior for which a small shift is introduced in the reaction term on uniform mesh. In [5] presents a new exponentially integration fitted three point scheme for solving singularly perturbed delay problems with boundary layer at left (or right) end of the given domain. A good number of articles [2–4, 7–19] and high level monographs [20–22, 26–31] are in literature which describe the various numerical and analytical methods for the solution of singular perturbation problems. With this motivation, in this paper we employed an exponentially fitted numerical finite difference scheme based on uniform mesh to solve singularly perturbed delay differential equations having delay in convection term which has the boundary layer at left and right end of the given domain respectively.

The paper is arranged as follows: Formulation of the continuous problem is given in Section: 2. A brief discussion of the proposed method with left and right end boundary layer problems are presented in the subsections 2.1. In Section: 3, Convergence analysis of the proposed method is analysed. Computational illustrations with the results in terms of MAE are presented in tables in Section: 4. Conclusions and Discussions are presented in the last Section: 5. Paper ends with the references.

2. STATEMENT OF THE PROBLEMS.

Consider a class of singularly perturbed differential equations with delay in the convection term:

$$\varepsilon\, w''(u) + s(u)w'(u-\delta) + t(u)w(u) = g(u) \tag{1}$$

on $0 \leq u \leq 1,\ \ 0 < \varepsilon << 1$ subject to the interval and boundary conditions(BCs):

$$w(0) = \psi(u),\ -\delta \leq u \leq 0,\ \ w(1) = \gamma \tag{2}$$

where $s(u)$, $t(u)$, $\psi(u)$ and $g(u)$ are continuously differentiable functions in (0,1). where γ is constant. If we assume $s(u) \geq M > 0$ throughout the interval $[0,1]$, where M is positive constant, then boundary layer will be in the neighbourhood of $u = 0$ and $s(u) \leq M < 0$ throughout the interval $[0,1]$, where M is negative constant, then boundary layer will be in the neighbourhood of $u = 1$. Now using the Taylor's Series expansion of the term $w'(u-\delta)$ around u we have

$$w'(u-\delta) \approx w'(u) - \delta w''(u)$$

Substituting the value of $w'(u-\delta)$ in Eq.(2.1) we get

$$(\varepsilon - \delta s(u))w''(u) + s(u)w'(u) + t(u)w(u) = g(u)\ on\ 0 \leq u \leq 1 \tag{3}$$

$$\mu w''(u) + s(u)w'(u) + t(u)w(u) = g(u)\ on\ 0 \leq u \leq 1 \tag{4}$$

with boundary conditions (BCs)

$$w(0) = \psi,\ \ w(1) = \gamma, \tag{5}$$

where $\mu = \varepsilon - \delta s(u)$. The solution of Eq.(4) provides a best approximation to the solution of Eq.(1)

3. LEFT-END BOUNDARY LAYER PROBLEMS

To describe the proposed method, we consider the singularly perturbed boundary value problem Equation Eq. (4)-(5) with $s(u) \geq M > 0$. The theory of singular perturbation gives the solution of Eq.(4) with Eq.(5) which is of the form (cf. [27],pp.22-26):

$$w(u) = w_0(u) + \frac{s(0)}{s(u)}\left(\psi_0 - w_0(0)\right) e^{-\int_0^u \left(\frac{s(u)}{\mu} - \frac{t(u)}{s(u)}\right)du} + o(\mu), \tag{6}$$

where $w_0(u)$ denoted solution of the simplified problem:

$$s(u)w_0'(u) + t(u)w_0(u) \ = \ g(u); \ w_0(1) = \gamma. \tag{7}$$

By taking Taylor's series expansions for $s(u)$ and $t(u)$ about the point $'u = 0'$ upto their first terms only, the Eq.(2.6) becomes:

$$w(u) = w_0(u) + (\psi_0 - w_0(0))\, e^{-\left(\frac{s(0)}{\mu} - \frac{t(0)}{s(0)}\right)u} + o(\mu). \tag{8}$$

Taking the limit as $h \to 0$ and applying Eq.(6) to the point $u = u_i = ih, i = 0, 1, 2, ..., N$, we obtain

$$\lim_{h\to 0} w(ih) = w_0(0) + (\psi_0 - w_0(0))\, e^{-\left(\frac{s^2(0)-\mu t(0)}{s(0)}\right)i\rho} + o(\mu), \tag{9}$$

where $\rho = h/\mu$, the first and second order approximations have been used as below:

$$w_i' = \frac{3w_{i+1} - 2w_i - w_{i-1}}{4h} \tag{10}$$

$$w_i'' = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} \tag{11}$$

Substituting Eq.(10) and Eq.(11) in Eq.(4) we have

$$\mu\left[\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}\right] + s_i\left[\frac{3w_{i+1} - 2w_i - w_{i-1}}{4h}\right] + t_i w_i = g_i \tag{12}$$

Introducing the fitting factor $\sigma(\rho)$ into the aforementioned approach, we obtain the following result:

$$\sigma\mu\left[\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}\right] + s_i\left[\frac{3w_{i+1} - 2w_i - w_{i-1}}{4h}\right] + t_i w_i = g_i \tag{13}$$

The determination of the fitting factor $\sigma(\rho)$ aims to ensure that the solution of the difference scheme described in Eq. (13) achieves uniform convergence towards the solution of equation Eq. (4) with Eq. (5).

By multiplying Eq. (13) by h and taking the limit as $h \longrightarrow 0$, the resulting Eq. (13) is as follows:

$$\frac{\sigma}{\rho}\left[w_{i+1} - 2w_i - w_{i-1}\right] + \frac{r(0)}{4}\left[3w_{i+1} - 2w_i - w_{i-1}\right] = 0 \tag{14}$$

Let $\pi = \frac{s^2(0)-\mu t(0)}{s(0)}$. By using Eq.(9), we get

$$\lim_{h\to 0}\left(w(ih-h)+w(ih+h)-2w(ih)\right) = \left(\psi_0 - w_0(0)\right)e^{-\pi i\rho}\left(e^{\pi\rho}+e^{-\mu\rho}-2\right)$$

$$\lim_{h\to 0}\left(3w(ih+h)-2w(ih)-w(ih-h)\right) = \left(\psi_0 - w_0(0)\right)e^{-\pi i\rho}\left(3e^{-\pi\rho}-2-e^{\pi\rho}\right).$$

By using the above equations in Eq.(14), we get

$$\sigma(\rho) = \frac{s(0)\rho}{2}\coth\left(\frac{\left(s^2(0)-\mu t(0)\right)\rho}{2s(0)}\right) - \frac{s(0)\rho}{4}, \tag{15}$$

which is a required fitting factor $\sigma(\rho)$.
Finally, from the equation Eq.(14) with the value of $\sigma(\rho)$ given by equation Eq.(15), we obtain the following three-term recurrence relationship:

$$A_i w_{i-1} - B_i w_i + C_i w_{i+1} = H_i,\ (i = 1, 2, 3, ..., N-1), \tag{16}$$

where

$$\begin{aligned} A_i &= \tfrac{\sigma\mu}{h^2} - \tfrac{s_i}{4h} \\ B_i &= \tfrac{2\sigma\mu}{h^2} + \tfrac{2s_i}{4h} - t_i \\ C_i &= \tfrac{\sigma\mu}{h^2} + \tfrac{3s_i}{4h} \\ H_i &= g_i \end{aligned}$$

Using the Thomas Algorithm, the resulting diagonal dominating tridiagonal system is solved.

3.1. Right-End Boundary Layer Problems

In this subsection, we will describe the proposed method for the solution of the problem Eq. (4)-(5) having boundary layer at right end point of the interval considered. The solution of Eq. (4) with Eq. (5) is of the following form (cf. [27], pp.22-26):

$$w(u) = w_0(u) + \frac{s(0)}{s(u)}\left(\psi_0 - w_0(1)\right)e^{-\int_0^u\left(\frac{s(u)}{\mu}-\frac{t(u)}{s(u)}\right)du} + o(\mu), \tag{17}$$

where $w_0(u)$ denoted the simplified problem's solution:

$$s(u)w_0'(u) + t(u)w_0(u) \ = \ g(u);\ w_0(1) = \beta. \tag{18}$$

By considering the Taylor series expansions of $s(u)$ and $t(u)$ around the point $u = 0$ up to their respective first terms, we can simplify Eq. (17) as follows::

$$w(u) = w_0(u) + \left(\psi_0 - w_0(0)\right)e^{-\left(\frac{s(1)}{\mu}-\frac{t(1)}{s(1)}\right)u} + o(\mu). \tag{19}$$

Taking the limit as $h \rightarrow 0$ and applying Eq.(3) to the point $u = u_i = ih, i = 0, 1, 2, ..., N$, we obtain

$$\lim_{h\rightarrow 0} w(ih) = w_0(0) + (\psi_0 - w_0(0))\, e^{-\left(\frac{s^2(1)-\mu t(1)}{s(1)}\right)i\rho} + o(\mu), \tag{20}$$

where $\rho = h/\mu$.

After multiplying Eq.(13) by h and taking the limit as $h \rightarrow 0$, the Eq. (13) converted into the following form:

$$\frac{\sigma}{\rho}\left[w_{i+1} - 2w_i - w_{i-1}\right] + \frac{s(0)}{4}\left[3w_{i+1} - 2w_i - w_{i-1}\right] = 0 \tag{21}$$

Let $\pi = \frac{s^2(0)-\mu t(0)}{s(0)}$. By using Eq.(20), we get

$$\lim_{h\rightarrow 0}\left(w(ih-h) + w(ih+h) - 2w(ih)\right) = (\psi_0 - w_0(1))\, e^{-\pi i\rho}\left(e^{\pi\rho} + e^{-\pi\rho} - 2\right)$$

$$\lim_{h\rightarrow 0}\left(3w(ih+h) - 2w(ih) - w(ih-h)\right) = (\psi_0 - w_0(1))\, e^{-\pi i\rho}\left(3e^{-\pi\rho} - 2 - e^{\pi\rho}\right).$$

By substituting the aforementioned equations into Eq.(21), we get

$$\sigma(\rho) = \frac{s(0)\rho}{2}\coth\left(\frac{\left(s^2(1) - \mu t(1)\right)\rho}{2s(1)}\right) - \frac{s(0)\rho}{4}, \tag{22}$$

which is a required fitting factor $\sigma(\rho)$ for right end boundary layer problem.

4. CONVERGENCE ANALYSIS.

In this section, we discuss the convergence analysis of the method.

Definition 3.1 (Consistency): Let

$$\tau_i[w] = L_h w(t_i) - L_\tau w(t_i),\ i = 1, 2, ..., N$$

where y represents a smooth function on $I = [0, 1]$ with L_h as the discrete difference operator.then the difference Eq.(2.13)-(2.2) bear consistency with the differential Eq.(2.1)-(2.2) if

$$|\tau_i[w]| \rightarrow 0\ as\ h \rightarrow o.$$

The quantities $\tau_i[w], i = 1, 2, 3..., N$ is called the local truncation (or local discretization) errors.

Definition 3.1 The differential Eq.(2.13)-(2.2) has local p^{th} -order accuracy if, for sufficiently smooth data, a positive constant C exists independent of h and ε such that

$$\max_{1\le i\le N} |\tau_i[w]| \le Ch^p.$$

The consistency of the differential Eq.(2.13)-(2.2) with (2.1)-(2.2) and its locally second-order accuracy is demonstrated by the following lemma.

Lemma 3.1 If $w \in C^2(I)$, then

$$|\tau_i[w]| \le \max_{u_{i-1}\le u\le u_{i+1}} \left\{ \frac{a_i h}{4} |w_i''| \right\} + O(h^2), i = 1, 2, 3, ..., N-1$$

Proof By definition

$$\begin{aligned}
\tau_i &= \sigma\varepsilon \left\{ \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} - w_i'' \right\} + \left\{ \frac{3w_{i+1} - 2w_i - w_{i-1}}{4h} \right\} \\
\tau_i &= \sigma\varepsilon \left\{ \frac{h^2}{12} w_i^{iv} + \frac{h^4}{360} w_i^{vi} + \cdots \right\} + s_i \left\{ \frac{h}{12} w_i'' + \frac{h^2}{3!} w_i''' + \cdots \right\} \\
|\tau_i| &= \max_{t_{i-1}\le t\le t_{i+1}} \left\{ \frac{\sigma\varepsilon h^2}{12} |w_i^{iv}| \right\} + \max_{t_{i-1}\le u\le u_{i+1}} \left\{ \frac{s_i h}{4} |w_i''| \right\} \\
|\tau_i| &\le \max_{u_{i-1}\le u\le u_{i+1}} \left\{ \frac{s_i h}{4} |w_i''| \right\} + O(h^2) \\
|\tau_i| &\le O(h). \quad i = 1, 2, 3, ..., N-1
\end{aligned}$$

As a result, the intended outcome is attained.

We will now examine the proposed method's convergence across the entire interval range $0 \le u \le 1$. We write the tridiagonal system (2.13) in matrix-vector form using the method outlined in [39]:

$$WY = D \tag{23}$$

where $W = (c_{ij}),\ 1 \le i, j \le N-1$ is a tridiagonal matrix of order N-1 with

$$\begin{aligned}
c_{i,i-1} &= \sigma\mu - \frac{s_i h}{4} \\
c_{i,i} &= 2\sigma\mu + \frac{2hs_i}{4} - t_i h^2 \\
c_{i,i+1} &= \sigma\mu\mu + \frac{3hs_i}{4} \\
d_i &= h^2 g_i
\end{aligned}$$

and $D = (d_i)$ is a column vector with $d_i = h^2 f_i$ for $i = 1, 2, 3, ...N - 1$ with local truncation error τ_i:

$$|\tau_i| \leq O(h) \tag{24}$$

we also have

$$W\bar{Y} - \tau(h) = D \tag{25}$$

where $\bar{Y} = (\bar{y}_0, \bar{y}_1, \bar{y}_2, \bar{y}_3, ..., \bar{y}_N)^t$ and $\tau(h) = (\tau_1(h), \tau_2(h), \tau_3(h), ..., \tau_N(h),)^t$ stands for the local truncation error and the real solution, respectively. Eqs.(3.1) and (3.3) give us

$$W(\bar{Y} - Y) = \tau(h) \tag{26}$$

Thus the error equation is

$$WE = \tau(h), \tag{27}$$

where $E = \bar{Y} - Y = (e_0, e_1, e_2, ..., e_N)^t$. If S_i^* is the total of the components in the i^{th} row of W, then we have

$$S_1^* = \sum_{j=1}^{N-1} c_{1,j} = \sigma\varepsilon - \frac{s_i h}{4}$$

$$S_{N-1}^* = \sum_{j=1}^{N-1} c_{N-1,j} = 2\sigma\varepsilon + \frac{2hs_i}{4} - t_i h^2$$

$$S_i^* = \sum_{j=1}^{N-1} c_{i,j} = h^2 g_i = p_i = P_{i0}$$

where $P_{i0} = p_i = h^2 g_i$

Since $0 < \varepsilon << 1$, The matrix W is irreducible and monotone for sufficiently small h. As a result, W^{-1} must exist and contain non-negative elements. Therefore, we have from Eq. (3.5)

$$E = W^{-1}\tau(h) \tag{28}$$

$$\|E\| \leq \|W^{-1}\| \|\tau(h)\| \tag{29}$$

Let $\bar{c}_{ki}$ represent the $(ki)^{-th}$ components of Y^{-1}.Since $\bar{c}_{ki} \geq 0$, we have from the operations on matrices:

$$\sum_{j=1}^{N-1} \bar{c}_{ki} S_i^* = 1; k = 1, 2, ..., N - 1 \tag{30}$$

Therefore, its follows

$$\sum_{j=1}^{N-1} \bar{c}_{ki} \leq \frac{1}{\min\limits_{0\leq i\leq N-1} S_i^*} = \frac{1}{P_{i0}} \leq \frac{1}{|P_{i0}|} \tag{31}$$

for some i_0 between 1 and $N-1$, and $P_{i0} = p_i$.
Therefore, from Eqs.(3.2),(3.6)and (3.8) we get

$$e_j = \sum_{i=1}^{N-1} \bar{c}_{ki}\tau_i(h); \;\; j = 1(1)N-1 \tag{32}$$

which implies

$$e_j \leq \frac{O(h)}{|p_i|}; \;\; j = 1(1)N-1 \tag{33}$$

Consequently, by applying the definitions and Eq.(3.10), we obtain:

$$\|E\| = O(h)$$

This suggests that the method is first-order convergence rate. We can quickly determine that the technique for right end boundary value problems is of first order convergence by using a similar procedure.

5. NUMERICAL ILLUSTRATIONS.

To demonstrate the accuracy and efficiency of the proposed method we have chosen two linear singular perturbation problems with left-end boundary layer and two linear singular perturbation problems with right-end boundary layer which are widely discussed in literature.

Example-1 Consider the Homogeneous Singular perturbation Problem with left end boundary layer:

$$\varepsilon w''(u) + w'(u-\delta) - w(u) = 0; \;\; u \in [0,1]$$

with boundary condition $w(0) = 1$ and $w(1) = 1$.
The exact solution is given by

$$w(u) = \frac{(1-e^{\lambda_2})e^{\lambda_1 u} + (e^{\lambda_1}-1)e^{\lambda_2 u}}{e^{\lambda_2} - e^{\lambda_1}}$$

where $\lambda_1 = \dfrac{(-1-\sqrt{1+4(\varepsilon-\delta)})}{2(\varepsilon-\delta)}$ and $\lambda_2 = \dfrac{(-1+\sqrt{1+4(\varepsilon-\delta)})}{2(\varepsilon-\delta)}$

Example-2 consider singularly perturbed delay differential equation with variable coefficient left end boundary layer

$$\varepsilon w''(u) + e^{-0.5u} w'(u-\delta) - w(u) = 0;\ u \in [0,1]$$

with boundary condition $w(0) = 1$ and $w(1) = 1$. The exact solution of this example is not given.

Example-3 consider singularly perturbed delay differential equation with variable coefficient right end boundary layer

$$\varepsilon w''(u) - e^{u} w'(u-\delta) - w(u) = 0;\ u \in [0,1]$$

with boundary condition $w(0) = 1$ and $w(1) = 1$. The exact solution of this example is not given.

Example-4 consider singularly perturbed delay differential equation with right end boundary layer

$$\varepsilon w''(u) - w'(u-\delta) - w(u) = 0;\ u \in [0,1]$$

with boundary condition $w(0) = 1$ and $w(1) = -1$.
The exact solution is given by

$$w(u) = \frac{(e^{\lambda_2}+1)e^{\lambda_1 u} - (1+e^{\lambda_1})e^{\lambda_2 u}}{e^{\lambda_2} - e^{\lambda_1}}$$

where $\lambda_1 = \dfrac{(1-\sqrt{1+4(\varepsilon+\delta)})}{2(\varepsilon+\delta)}$ and $\lambda_2 = \dfrac{(1+\sqrt{1+4(\varepsilon+\delta)})}{2(\varepsilon+\delta)}$

The solution of the first two examples (i.e of Example-1 and Example-2) exhibits the boundary layer behavior at left-end points of the interval $[0,1]$ while of last two example (i.e of Example-3 and Example-4) exhibits the boundary layer behavior at right-end point of the interval $[0,1]$. Further, from the graphs (Figures 1 to 2) plotted for the solutions for all the examples 1 to 2 for varying δ, we easily observe that as δ increases, the thickness of the boundary layer decreases while from the graphs (Figures 3 to 4) plotted for the solutions for all the examples 3 to 4 for varying δ, we easily observe that as δ decreases, the thickness of the boundary layer increases.

Table 1: MAE of Example 1 with $\delta = 0.001$, $\varepsilon = 10^{-2}$ and $h = 10^{-2}$

u	Exact values	Computational values	Absolute Error
0.00	1.000000000000000	1.000000000000000	0.0000e-00
0.01	0.579815892739947	0.582117270435849	2.3013e-03
0.02	0.445387207353913	0.447056288748595	1.6690e-03
0.04	0.393254632654488	0.393896917597684	6.4228e-04
0.06	0.394641990435684	0.395044292010474	4.0230e-04
0.08	0.401854022753968	0.402216129889963	3.6211e-04
0.10	0.409826314973749	0.410181913932218	3.5560e-04
0.30	0.499668244835438	0.500004582856141	3.3634e-04
0.50	0.609217932516785	0.609510817500019	2.9288e-04
0.70	0.742785824667047	0.750393774477031	2.1424e-04
0.90	0.896705821516804	0.905724850202464	8.7061e-04
1.00	1.000000000000000	1.000000000000000	0.0000e-00

Table 2: MAE of Example 1 with $\delta = 0.003$, $\varepsilon = 10^{-2}$ and $h = 10^{-2}$

u	Exact values	Computational values	Absolute Error
0.00	1.000000000000000	1.000000000000000	0.0000e-00
0.01	0.523511634094260	0.525345688922314	1.8340e-03
0.02	0.413305805769191	0.414432878145795	1.1270e-03
0.04	0.387434721226649	0.387949650008273	5.1493e-04
0.06	0.393283410846356	0.393724580877500	4.4117e-04
0.08	0.401064571828557	0.401498671719792	4.3410e-04
0.10	0.409104025003127	0.409536717310256	4.3269e-04
0.30	0.498990898664473	0.499401296974819	4.1040e-04
0.50	0.608627924765192	0.608985432766953	3.5751e-04
0.70	0.742354122681231	0.742615727209560	2.6160e-04
0.90	0.905462304698606	0.905568653413133	1.0635e-04
1.00	1.000000000000000	1.000000000000000	0.0000e-00

Table 3: MAE of Example 1 with $\delta = 0.0003$, $\varepsilon = 10^{-3}$ and $h = 10^{-2}$

u	Exact values	Computational values	Absolute Error
0.00	1.000000000000000	1.000000000000000	0.0000e-00
0.01	0.371834313700486	0.373408720610892	1.5744e-03
0.02	0.375568290529659	0.377142409983964	1.5741e-03
0.04	0.383149916630014	0.384722972376783	1.5730e-03
0.06	0.390884593602526	0.392455904072993	1.5713e-03
0.08	0.398775411096730	0.400344267695320	1.5688e-03
0.10	0.406825521133404	0.408391187425460	1.5656e-03
0.30	0.496828349986014	0.498314858682435	1.4865e-03
0.50	0.606742685813173	0.608038826569976	1.2961e-03
0.70	0.740973591378484	0.741922919164250	9.4933e-04
0.90	0.904900670346751	0.905286955253102	3.8628e-04
1.00	1.000000000000000	1.000000000000000	0.0000e-00

Table 4: MAE of Example 1 with $\delta = 0.0008$, $\varepsilon = 10^{-3}$ and $h = 10^{-2}$

u	Exact values	Computational values	Absolute Error
0.00	1.000000000000000	1.000000000000000	0.0000e-00
0.01	0.371650241071041	0.373408324452410	1.7580e-03
0.02	0.375384637620808	0.377142407696934	1.7577e-03
0.04	0.382966379070068	0.384722970091643	1.7565e-03
0.06	0.390701250929160	0.392455901790485	1.7546e-03
0.08	0.398592346012917	0.400344265416473	1.7519e-03
0.10	0.406642819602559	0.408391185151345	1.7483e-03
0.30	0.496654802801967	0.498314856524217	1.6600e-03
0.50	0.606591291559862	0.608038824688949	1.4475e-03
0.70	0.740862653336662	0.741922917787124	1.0602e-03
0.90	0.904855507730071	0.905286954692983	4.3145e-04
1.00	1.000000000000000	1.000000000000000	0.0000e-00

Table 5: Computational values of Example 2 with $\varepsilon = 10^{-2}$ and $h = 10^{-2}$

u	$\delta = 0.1\varepsilon$	$\delta = 0.3\varepsilon$	$\delta = 0.5\varepsilon$	$\delta = 0.7\varepsilon$	$\delta = 0.9\varepsilon$
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.10	0.3083073	0.3102507	0.3090734	0.3079698	0.3070320
0.20	0.3463024	0.3450560	0.3438416	0.3426922	0.3416894
0.40	0.4350592	0.4337858	0.4325371	0.4313382	0.4302476
0.60	0.5593709	0.5581851	0.5570162	0.5558815	0.5548193
0.80	0.7376493	0.7367985	0.7359557	0.7351305	0.7343427
0.90	0.8557760	0.8552608	0.8547493	0.8542467	0.8537632
0.92	0.8823349	0.8819063	0.8814805	0.8810619	0.8806586
0.94	0.9099826	0.9096481	0.9093158	0.9089888	0.9086733
0.96	0.9387717	0.9385396	0.9383090	0.9380819	0.9378625
0.98	0.9687579	0.9686371	0.9685170	0.9683987	0.9682843
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 6: Computational values of Example 2 for $\varepsilon = 10^{-3}$ and $h = 10^{-2}$

u	$\delta = 0.1\varepsilon$	$\delta = 0.3\varepsilon$	$\delta = 0.5\varepsilon$	$\delta = 0.7\varepsilon$	$\delta = 0.9\varepsilon$
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.10	0.3060669	0.3060645	0.3060637	0.3060635	0.3060635
0.20	0.3406174	0.3406148	0.3406138	0.3406136	0.3406136
0.40	0.4289359	0.4289327	0.4289316	0.4289313	0.4289313
0.60	0.5533261	0.5533224	0.5533210	0.5533207	0.5533207
0.80	0.7330245	0.7330211	0.7330197	0.7330194	0.7330193
0.90	0.8528780	0.8528756	0.8528746	0.8528743	0.8528743
0.92	0.8799068	0.8799048	0.8799039	0.8799036	0.8799036
0.94	0.9080746	0.9080730	0.9080723	0.9080720	0.9080720
0.96	0.9374385	0.9374374	0.9374369	0.9374367	0.9374367
0.98	0.9680590	0.9680584	0.9680581	0.9680580	0.9680580
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 7: Computational values of Example 3 for $\varepsilon = 10^{-2}$ and $h = 10^{-2}$

u	$\delta = 0.1\varepsilon$	$\delta = 0.3\varepsilon$	$\delta = 0.5\varepsilon$	$\delta = 0.7\varepsilon$
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.10	0.9110738	0.9113715	0.9116715	0.9119727
0.20	0.8372703	0.8377786	0.8382930	0.8388097
0.40	0.7235112	0.7242746	0.7250514	0.7258345
0.60	0.6417440	0.6426329	0.6435430	0.6444630
0.80	0.5816001	0.5825473	0.5835230	0.5845124
0.90	0.5574832	0.5584448	0.5594443	0.5604958
0.92	0.5530538	0.5540212	0.5550693	0.5563199
0.94	0.5487475	0.5497809	0.5512256	0.5535379
0.96	0.5446620	0.5467926	0.5513966	0.5588723
0.98	0.5472918	0.5652946	0.5894113	0.6152350
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Table 8: Computational values of Example 3 for $\varepsilon = 10^{-3}$ and $h = 10^{-2}$

u	$\delta = 0.1\varepsilon$	$\delta = 0.3\varepsilon$	$\delta = 0.5\varepsilon$	$\delta = 0.7\varepsilon$
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.10	0.9100629	0.9100633	0.9100644	0.9100666
0.20	0.8356435	0.8356441	0.8356457	0.8356490
0.40	0.7213154	0.7213160	0.7213179	0.7213220
0.60	0.6394061	0.6394067	0.6394085	0.6394125
0.80	0.5792874	0.5792879	0.5792895	0.5792934
0.90	0.5552082	0.5552087	0.5552103	0.5552140
0.92	0.5507873	0.5507878	0.5507894	0.5507930
0.94	0.5464879	0.5464884	0.5464900	0.5464937
0.96	0.5423061	0.5423066	0.5423082	0.5423120
0.98	0.5382380	0.5382385	0.5382400	0.5382438
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Table 9: MAE of Example 4 for $\varepsilon = 10^{-2}$, $\delta = 0.002$ and $h = 10^{-2}$

u	Exact values	Computational values	Absolute Error
0.00	1.000000000000000	1.000000000000000	0.0000000e-00
0.20	0.820652167961327	0.820783627567682	1.3145961e-04
0.40	0.673469980779626	0.673685763283163	2.1578250e-04
0.60	0.552684599783670	0.552950244628253	2.6564484e-04
0.80	0.453561749947365	0.453852428151133	2.9067820e-04
0.90	0.410582186184883	0.410846929984373	2.6474379e-04
0.91	0.406145971559920	0.406376974975513	2.3100341e-04
0.93	0.395128166674729	0.395152387809951	2.4221136e-05
0.95	0.370822262573560	0.370076253773518	7.4600880e-04
0.97	0.274069466965660	0.271010108380651	3.0593586e-03
1.00	-1.000000000000000	-1.000000000000000	0.0000000e-04

Table 10: MAE of Example 4 for $\varepsilon = h = 10^{-2}$, $\delta = 0.003$ and $h = 10^{-2}$

u	Exact values	Computational values	Absolute Error
0.00	1.000000000000000	1.000000000000000	0.0000000e-00
0.20	0.820808474468390	0.812876207556769	1.2688860e-04
0.40	0.673726551759126	0.673934870329988	2.0831857e-04
0.60	0.553000463158217	0.553256967456175	2.5650430e-04
0.80	0.453907231947887	0.454187922804397	2.8069086e-04
0.90	0.410665817660460	0.410892111595946	2.2629394e-04
0.91	0.405956256958618	0.406125898142616	1.6964118e-04
0.93	0.393356200857104	0.393215614410588	1.4058645e-04
0.95	0.363516920723106	0.362366252196371	1.1506685e-03
0.97	0.251199254128356	0.247417807046709	3.7814471e-03
0.98	0.091057209843930	0.085441663147335	5.6155467e-03
1.00	-1.000000000000000	-1.000000000000000	0.0000000e-00

Table 11: MAE of Example 4 for $\varepsilon = 10^{-3}$, $\delta = 0.008$ and $h = 10^{-2}$

u	Exact values	Computational values	Absolute Error
0.00	1.000000000000000	1.000000000000000	0.000000e-00
0.20	0.820179804575385	0.820337504274276	1.576996e-04
0.40	0.672694911833316	0.672953620918947	2.587090e-04
0.60	0.551730781326305	0.546609694239221	3.183125e-04
0.80	0.452518444155242	0.452866575600977	3.481314e-04
0.90	0.409799244847079	0.410151993939460	3.527490e-04
0.91	0.405718985979057	0.406068716232837	3.497302e-04
0.93	0.397275433502006	0.397592184721254	3.167512e-04
0.95	0.384958674795312	0.385054724591144	9.604979e-05
0.97	0.334866924504487	0.333763441050392	1.103483e-03
0.98	0.232905252073583	0.230277745581238	1.103489e-03
1.00	-1.000000000000000	-1.00000000000000	0.000000e-00

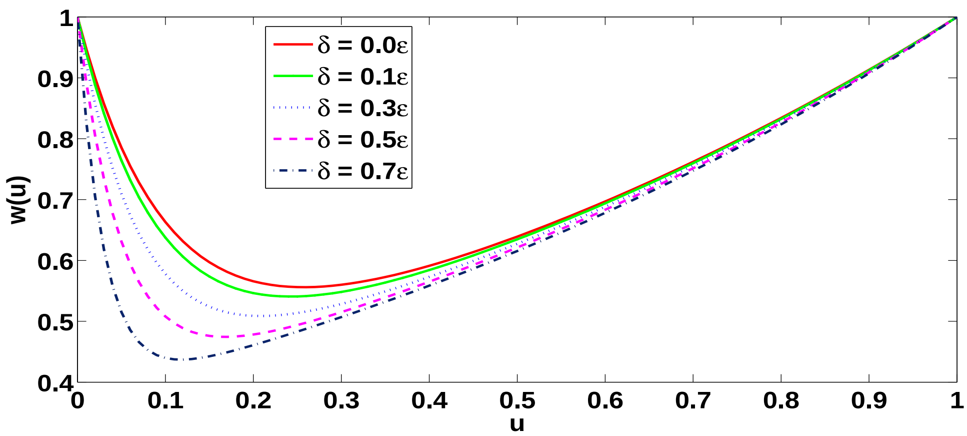

Figure 1: Computational solution of example 1 for value of $h = 10^{-2}$ and $\varepsilon = 10^{-1}$

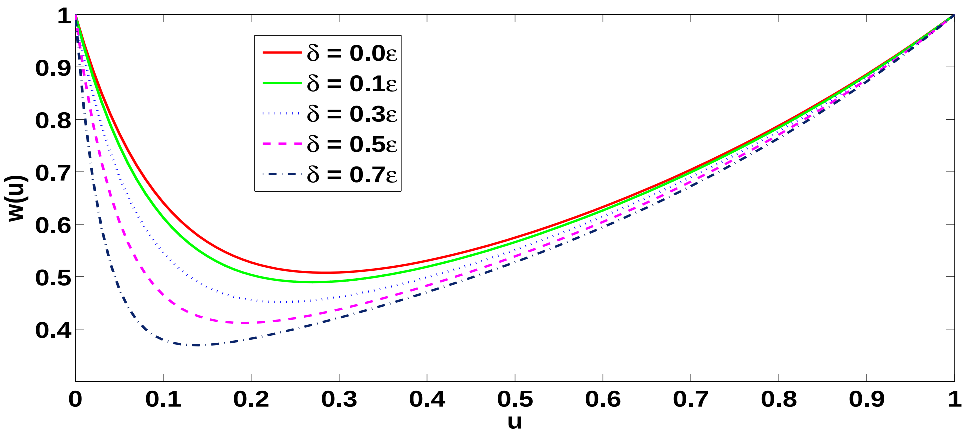

Figure 2: Computational solution of example 2 for value of $h = 10^{-2}$ and $\varepsilon = 10^{-1}$

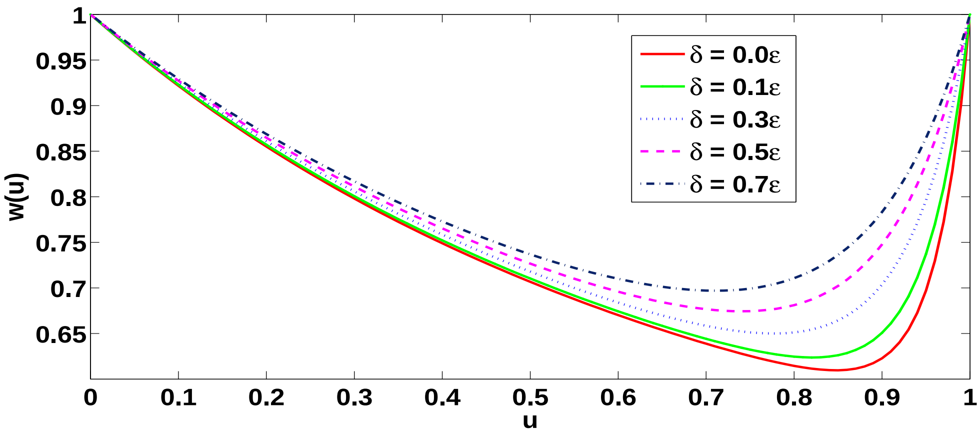

Figure 3: Computational solution of example 3 for value of $h = 10^{-2}$ and $\varepsilon = 10^{-2}$

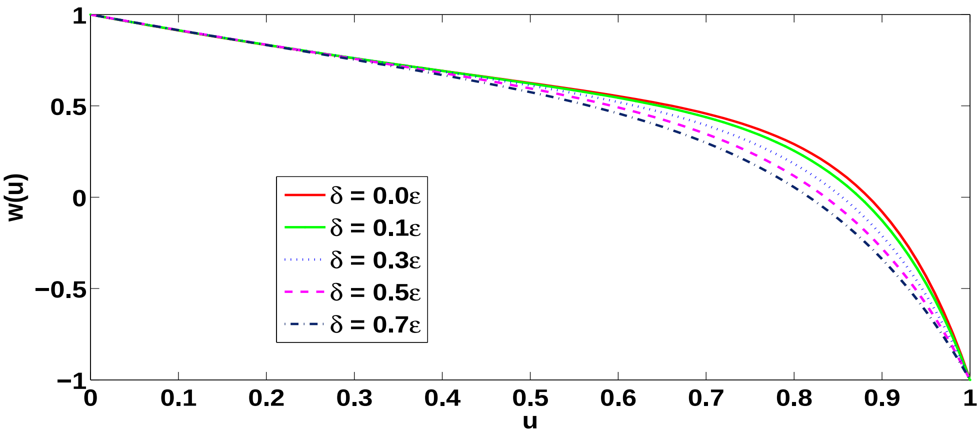

Figure 4: Computational solution of example 4 for value of $h = 10^{-2}$ and $\varepsilon = 10^{-2}$

6. CONCLUSIONS

We presented a fitted approach to solve SPDDEs of second order with boundary at one (left or right) end of the interval. In this method, the SPDDE is converted into a singularly perturbed differential equation by utilising Taylor series expansion to make an approximation of the term that contain minor negative shifts. In the newly introduced singularly perturbed problems, we implemented a fitting parameter and used the theory of singular perturbation to calculate the value of the fitting parameter. In the end, we are presented with a three-term recurrence relation that can be solved using the Thomas algorithm. Here, to test the applicability of this method, we considered four linear problems (two for left layer and two for right layer). We presented tables of values and the point wise error for uniform meshes of the problems by taking different values of the perturbation parameter ε, the delay parameter δ. The results demonstrate that the sugessted method produced good approximation to the exact solution. Also, from the numerical experiments considered in the paper, we observe that the small shift affects both the boundary layer solutions in similar fashion but reversely, i.e., as δ increases the thickness of the left boundary layer decreases while that of the right boundary layer increases. This method does not depend on the asymptotic expansion as well as on the matching of coefficients. Thus we have devised an alternative technique of solving singularly perturbed differential-difference equations, which can be easily implemented on computer.

7. DECLARATIONS

Conflict of interest The author declare that he has no competing interests.

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