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A Portrayal of Integer Solutions to Non-homogeneous Quinary Heptic Equation

$$2(x+y)(x^3-y^3)=39(z^2-w^2)P^5$$

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Abstract

This paper concerns with the problem of obtaining many non-zero distinct integer solutions to non-homogeneous quinary heptic equation given by $2(x+y)(x^3-y^3)=39(z^2-w^2)P^5$. Substitution technique and factorization method are utilized to determine the same.

Keywords: Quinary heptic equation, Non-homogeneous heptic equation, Integer solutions, Substitution technique, Factorization Method.

Introduction

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics. The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree Diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these higher power Diophantine equation in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. For the sake of clear understanding by the readers, one may refer the varieties of Cubic, Quintic and Heptic Diophantine equations with multi variables [1-30]. This paper aims at determining many integer solutions to non-homogeneous polynomial equation of degree seven with five unknowns given by $2(x+y)(x^3-y^3)=39(z^2-w^2)P^5$.

Method of analysis

The non-homogeneous quinary heptic equation to be solved is

$$2(x+y)(x^3-y^3)=39(z^2-w^2)P^5 \qquad (1)$$

Taking

$$x=u+v, y=u-v, z=u+2v, w=u-2v, u\neq v\neq 2v \qquad (2)$$

in (1),it is written as

$$v^2+3u^2=39P^5 \qquad (3)$$

Pattern 1

Observe that ,by inspection,(3) is satisfied by

$$v=39^3 V(V^2+3U^2)^2, u=39^3 U(V^2+3U^2)^2 \qquad (4)$$

and

$$P=39(V^2+3U^2) \qquad (5)$$

Using (4) in (2),one has

$$\begin{aligned} x&=39^3(U+V)(V^2+3U^2)^2, \\ y&=39^3(U-V)(V^2+3U^2)^2, \\ z&=39^3(U+2V)(V^2+3U^2)^2, \\ w&=39^3(U-2V)(V^2+3U^2)^2. \end{aligned} \qquad (6)$$

Thus,(5) & (6) satisfy (1).

Pattern 2

Let

$$P=a^2+3b^2 \qquad (7)$$

Write the integer 39 as

$$39=(6+i\sqrt{3})(6-i\sqrt{3}) \qquad (8)$$

Substituting (7) & (8) in (3) and applying factorization , consider

$$\begin{aligned} v+i\sqrt{3}\,u &= (6+i\sqrt{3})(a+i\sqrt{3}\,b)^5 \\ &= (6+i\sqrt{3})[f(a,b)+i\sqrt{3}\,g(a,b)] \end{aligned} \tag{9}$$

where

$$\begin{aligned} f(a,b) &= a^5-30a^3b^2+45ab^4, \\ g(a,b) &= 5a^4b-30a^2b^3+9b^5. \end{aligned} \tag{10}$$

On equating the real and imaginary parts in (9),we have

$$\begin{aligned} v &= 6f(a,b)-3g(a,b), \\ u &= f(a,b)+6g(a,b). \end{aligned} \tag{11}$$

In view of (2),we get

$$\begin{aligned} x &= 7f(a,b)+3g(a,b), \\ y &= -5f(a,b)+9g(a,b), \\ z &= 13f(a,b), \\ w &= -12f(a,b)+12g(a,b). \end{aligned} \tag{12}$$

Thus,(7)&(12) satisfy (1).

Pattern 3

Rewrite (3) as

$$v^2+3u^2 = 39P^5 * 1 \tag{13}$$

Assume

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \tag{14}$$

Consider

$$P = 4(a^2+3b^2) \tag{15}$$

Substituting (8),(14)&(15) in (13) and utilizing factorization ,consider

$$\begin{aligned} v+i\sqrt{3}u &= (6+i\sqrt{3}) * 2^5(a+i\sqrt{3}b)^5 * \frac{(1+i\sqrt{3})}{2} \\ &= 2^4(3+i7\sqrt{3})(f(a,b)+i\sqrt{3}\,g(a,b)) \end{aligned}$$

On equating the coefficients of corresponding terms, we have

$$\begin{aligned} v &= 2^4(3f(a,b)-21g(a,b)) \\ u &= 2^4(7f(a,b)+3g(a,b)) \end{aligned}$$

In view of (2),one has

$$\begin{aligned} x &= 2^4\,[10\,f(a,b) - 18\,g(a,b)], \\ y &= 2^4\,[4\,f(a,b) + 24\,g(a,b)], \\ z &= 2^4\,[13\,f(a,b) - 39\,g(a,b)], \\ w &= 2^4\,[\,f(a,b) + 45\,g(a,b)]. \end{aligned} \tag{16}$$

Thus,(15)&(16) satisfy (1).

Note

Apart from (14),one may consider integer 1 to be

$$1 = \frac{(3r^2 - s^2 + i\sqrt{3}\,2rs)(3r^2 - s^2 - i\sqrt{3}\,2rs)}{(3r^2 + s^2)^2}$$

$$1 = \frac{(r^2 - 3s^2 + i\sqrt{3}\,2rs)(r^2 - 3s^2 - i\sqrt{3}\,2rs)}{(r^2 + 3s^2)^2}$$

Following the above process , two more sets of integer solutions to (1) are obtained.

Pattern 4

Taking

$$u = P^2 \tag{17}$$

in (3),we have

$$v^2 = P^4(39P - 3) \tag{18}$$

After performing some algebra, the sequence of values of P, v are given by

$$P_n = 39n^2 + 12n + 1 \tag{19}$$

and

$$v_n = (39n + 6)(39n^2 + 12n + 1)^2$$

From (17),we get

$$u_n = (39n^2 + 12n + 1)^2$$

In view of (2), one obtains

$$\begin{aligned} x_n &= (39n + 7)(39n^2 + 12n + 1)^2, \\ y_n &= -(39n + 5)(39n^2 + 12n + 1)^2, \\ z_n &= (78n + 13)(39n^2 + 12n + 1)^2, \\ w_n &= -(78n + 11)(39n^2 + 12n + 1)^2. \end{aligned} \tag{20}$$

Thus,(19)&(20) satisfy (1).

Pattern 5

Taking

$$v = 6P^2 \tag{21}$$

in (3),we have

$$u^2 = P^4(13P-12) \tag{22}$$

After performing some algebra, the sequence of values of P, u are given by

$$P_n = 13n^2 + 2n + 1 \tag{23}$$

and

$$u_n = (13n+1)(13n^2+2n+1)^2$$

From (21),we get

$$v_n = 6(13n^2+2n+1)^2$$

In view of (2), one obtains

$$\left.\begin{aligned} x_n &= (13n+7)(13n^2+2n+1)^2, \\ y_n &= (13n-5)(13n^2+2n+1)^2, \\ z_n &= (13n+13)(13n^2+2n+1)^2, \\ w_n &= (13n-11)(13n^2+2n+1)^2. \end{aligned}\right\} \tag{24}$$

Thus,(23)&(24) satisfy (1).

Conclusion

In this paper, an attempt has been made to obtain many non-zero distinct integer solutions to non-homogeneous quinary heptic equation given by $2(x+y)(x^3-y^3) = 39(z^2-w^2)P^5$ through employing substitution technique and factorization method. As heptic equations with multiple variables are plenty,one may search for integer solutions to other choices of seventh degree equations with multiple variables.

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