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A Variant of the Gamma Function and the Riemann Hypothesis

Darrell Cox [1]

[1]*Department of Mathematics,*
Grayson County College,
United States.

Abstract

A variant of the gamma function is shown to be directly related to non-trivial Riemann zeta function zeros. A reflection formula similar to that of the usual gamma function is derived. A linear relationship between the amplitude of this reflection formula and the imaginary part of the Riemann zeta function zeros is derived.
Keywords : Riemann zeta function, gamma function.

1. INTRODUCTION

Equation (3) in section 1.3 of Edward's [1] book is

$$\Pi(s) = \lim_{N\to\infty} \frac{1\cdot 2\cdots N}{(s+1)(s+2)\cdots(s+N)}(N+1)^s \tag{1}$$

This equation is valid for all s in the halfplane $\Re s > -1$. Let $\Pi_1(s)$ denote

$$\lim_{N\to\infty} \frac{1\cdot 2\cdots N}{(\Re s+1)(\Re s+2)\cdots(\Re s+N)}(N+1)^s \tag{2}$$

For positive integers n, $\Gamma(n) = (n-1)!$. The real part of $\Pi_1(n)$ equals $n!$. A plot of the imaginary part of $\Pi_1(4)$ for $N = 1, 2, 3, \ldots, 1000$ is

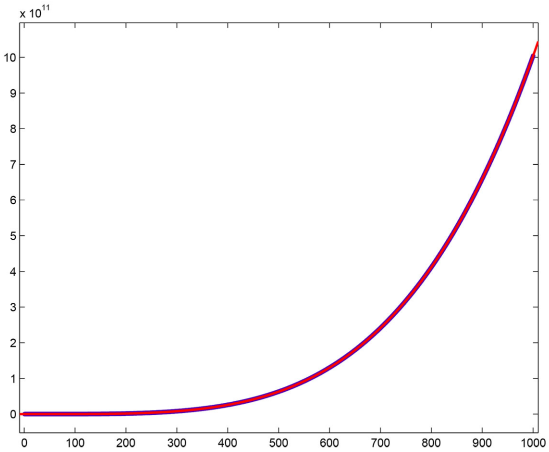

Figure 1

For a quartic least-squares fit of the curve, $p_1 = 1$ with a 95% confidence interval of (1, 1), $p_2 = 4$ with a 95% confidence interval of (4, 4), $p_3 = 6$ with a 95% confidence interval of (6, 6), $p_4 = 4$ with a 95% confidence interval of (4, 4), $p_5 = 0.9998$ with a 95% confidence interval of (0.9997, 0.9998), SSE=$5.359 \cdot 10^{-6}$, R-squared=1, and RMSE=$7.399 \cdot 10^{-5}$. As $N \to \infty$, the fifth parameter approaches 1. For $n = 1$, the imaginary part is a line with a slope of 1 and and a y-intercept of 1. For $n = 2$, the parameters are 1, 2, 1. For $n = 3$ the parameters are 1, 3, 3, 1. The parameters consist of the binary coefficients of Pascal's triangle.

Some "half-integer" properties are: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$, $\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi}$, $\Pi_1(\frac{1}{2}) = \Gamma(\frac{3}{2})$, $\Pi_1(\frac{3}{2}) = \Gamma(\frac{5}{2})$, and $\Pi_1(\frac{5}{2}) = \Gamma(\frac{7}{2})$.
A plot of the imaginary part of $\Pi_1(\frac{1}{2})$ versus $\sqrt{N}$ for $N = 1, 2, 3, \ldots, 1000$ is

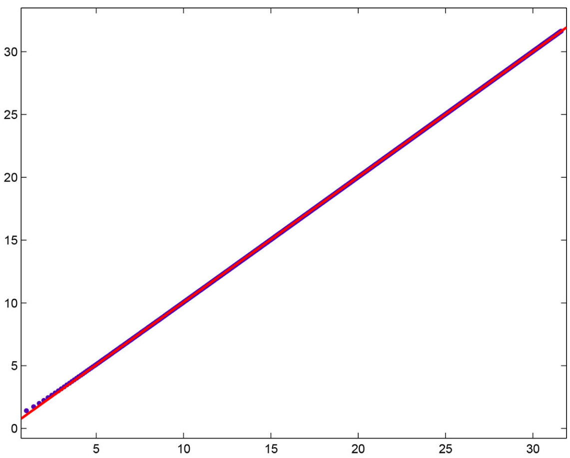

Figure 2

For a linear lease-squares fit of the curve, $p_1 = 0.99973$ with a 95% confidence interval of (0.9972, 0.9925), $p_2 = 0.08672$ with a 95% confidence interval of (0.08301, 0.09043), SSE=0.3947, R-squared=1, and RMSE=0.01989. As $N \to \infty$, the slope approaches 1.

A plot of the imaginary part of $\Pi_1(\frac{3}{2})$ versus $\sqrt{N}$ for $N = 1, 2, 3, \ldots, 1000$ is

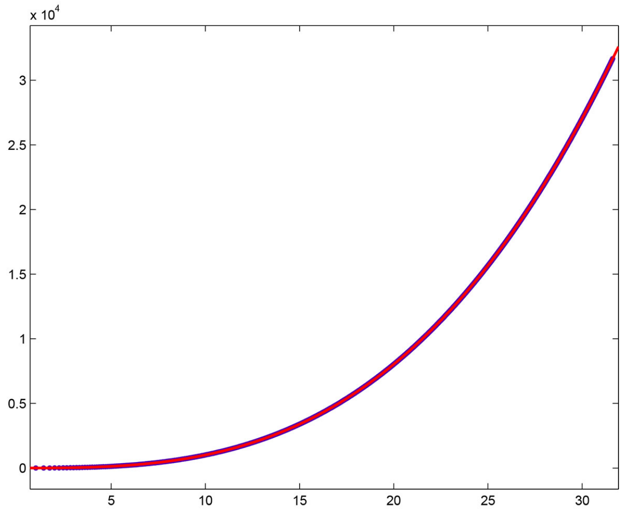

Figure 3

For a cubic least-squares fit of the curve, $p_1 = 1$ with a 95% confidence interval of (1, 1), $p_2 = 0.001134$ with a 95% confidence interval of (0.001073, 0.001195), $p_3 = 1.475$ with a 95% confidence interval of (1.475, 1.477), $p_4 = 0.1857$ with a 95% confidence interval of (0.1807, 0.1908), SSE=0.06298, R-squared=1, and RMSE=0.007952. As $N \to \infty$, the second and fourth parameters approach 0 and the third parameter approaches 1.5.

More half-integer properties are $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$, $\Gamma(-\frac{3}{2}) = \frac{4}{3}\sqrt{\pi}$, $\Gamma(-\frac{5}{2}) = -\frac{8}{15}\sqrt{\pi}$, $\Pi_1(-\frac{1}{2}) = \sqrt{\pi}$, $\Pi_1(-\frac{3}{2}) = 2\sqrt{\pi}$, and $\Pi_1(-\frac{5}{2}) = \frac{4}{3}\sqrt{\pi}$.

Some reciprocal of integer properties are $\Gamma(\frac{1}{3}) \approx 2.6789385347077476337$, $\Gamma(\frac{1}{4}) \approx 3.6256099082219083119$, $\Gamma(\frac{1}{5}) \approx 4.5908437119988030532$, $\Gamma(\frac{1}{6}) \approx 5.5663160017802352043$, $3\Pi_1(\frac{1}{3}) = \Gamma(\frac{1}{3})$, $4\Pi_1(\frac{1}{4}) = \Gamma(\frac{1}{4})$, $5\Pi_1(\frac{1}{5}) = \Gamma(\frac{1}{5})$, and $6\Pi_1(\frac{1}{6}) = \Gamma(\frac{1}{6})$. Also, $\Gamma(\frac{1}{6}) = \frac{\sqrt{\frac{3}{\pi}}\Gamma(\frac{1}{3})^2}{\sqrt[3]{2}}$.

2. REFLECTION FORMULAS AND A VARIANT

$\Pi(z) = \Gamma(z+1) = z\Gamma(z) = \int_0^\infty e^{-t}t^z dt$ so that $\Pi(n) = n!$ for every non-negative integer n. The reflection formula for the pi function is $\Pi(z) = z\Pi(z-1)$. A plot of $\Pi_1(z) - z\Pi_1(z-1)$ (denoted by $R(z)$ or $R(z,N)$) for the first non-trivial zeta function zero and $N = 1, 2, 3, \ldots, 2000$ is

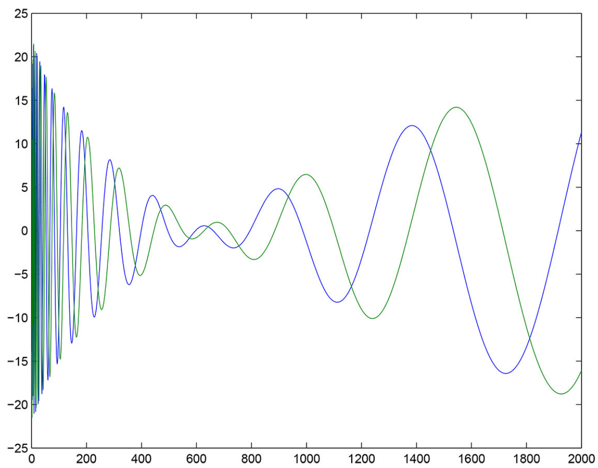

Figure 4

The function rapidly converges and then diverges. A plot of the logarithms of the N values where the imaginary component of $R(z,N)$ crosses the x-axis from above for $N \leq 10000000$ is

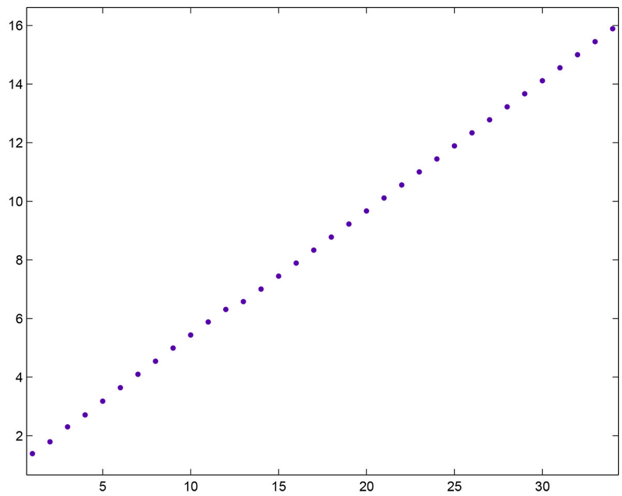

Figure 5

Except for a discontinuity at the twelveth point, the logarithm of the "wavelength" increases almost linearly. A plot of the first twelve points is

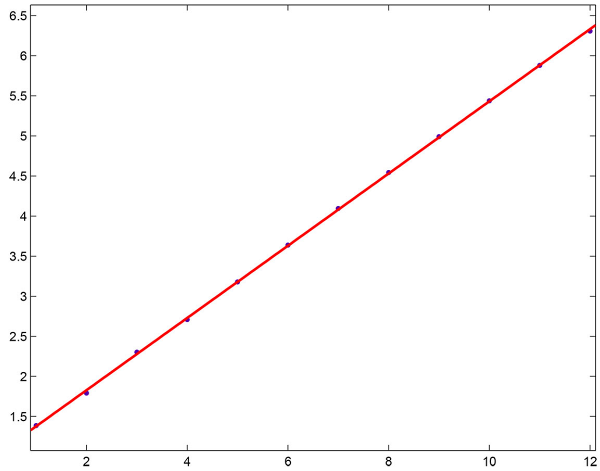

Figure 6

For a linear least-squares fit of the curve, $p_1 = .4505$ with a 95% confidence interval of (.447, .454), $p_2 = 0.9265$ with a 95% confidence interval of (0.9006, 0.9523), SSE=0.0003551, R-squared=0.9999, and RMSE=0.01885. A plot of the remaining twenty-two points is

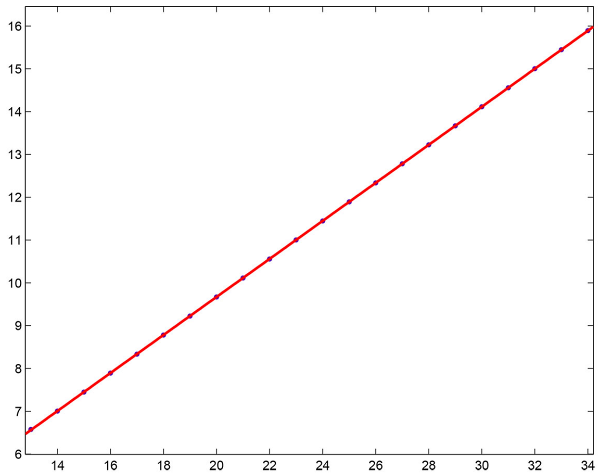

Figure 7

For a linear least-squares fit of the curve, $p_1 = 0.4442$ with a 95% confidence interval of (0.444, 0.4445), $p_2 = 0.7861$ with a 95% confidence interval of (0.7795, 0.7927), SSE=0.0002988, R-squared=1, and RMSE=0.003865.

The Riemann zeta function $\zeta(s)$ for $0 < \Re(s) < 1$ can be computed from the η function;

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) \tag{3}$$

A plot of $\zeta(z, N)$ for the first zeta function zero and $N = 1, 2, 3, \ldots, 100$ is

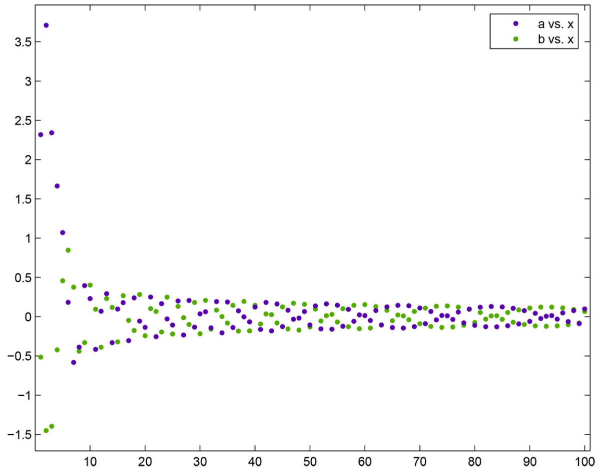

Figure 8

A plot of the logarithms of the N values where the imaginary component of $\zeta(z, N)$ crosses the x-axis from above for $N \leq 10000000$ is

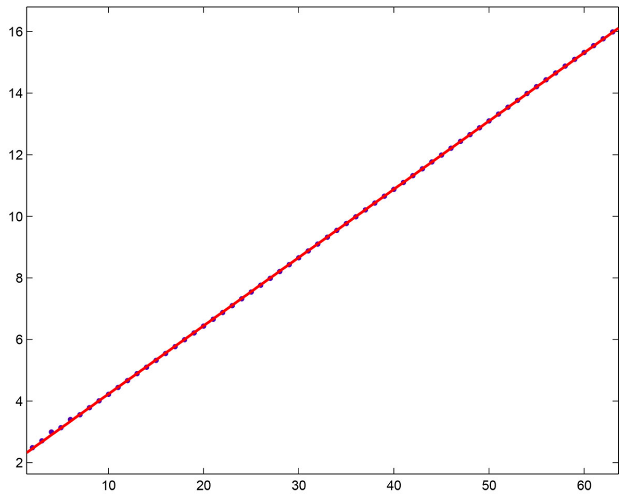

Figure 9

For a linear least-squares fit of the curve, $p_1 = 0.2216$ with a 95% confidence interval of (0.2214, 0.2219), $p_2 = 2.014$ with a 95% confidence interval of (2.005, 2.024), SSE=0.01878, R-squared=1, and RMSE=0.01769. The slope is about half of that for $R(z, N)$.

3. THE AMPLITUDE OF $R(z)$

A plot of the absolute value of the imaginary part of $R(z, N)$ for the first zeta function zero versus $\sqrt{N}$ for $N \leq 4000$ is

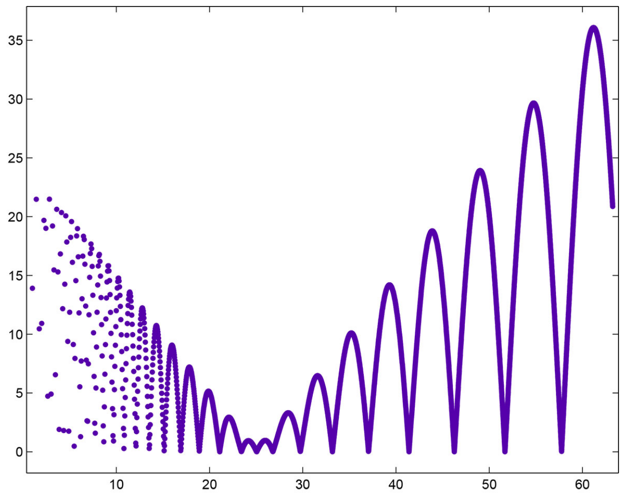

Figure 10

A plot of the decreasing amplitudes for $N \leq 10000000$ (neglecting the first seven points) is

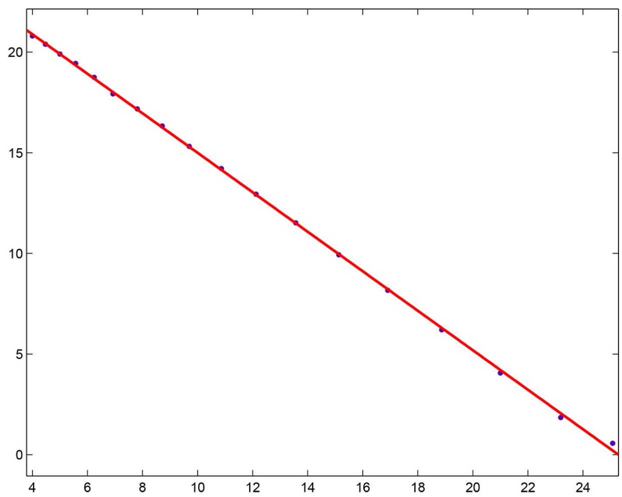

Figure 11

For a linear least-squares fit of the curve, $p_1 = -0.981$ with a 95% confidence interval of (-0.9904, -0.9705), $p_2 = 24.81$ with a 95% confidence interval of (24.68, 24.94), SSE=0.2447, R-squared=0.9997, and RMSE=0.1237.

A plot of the increasing amplitudes is

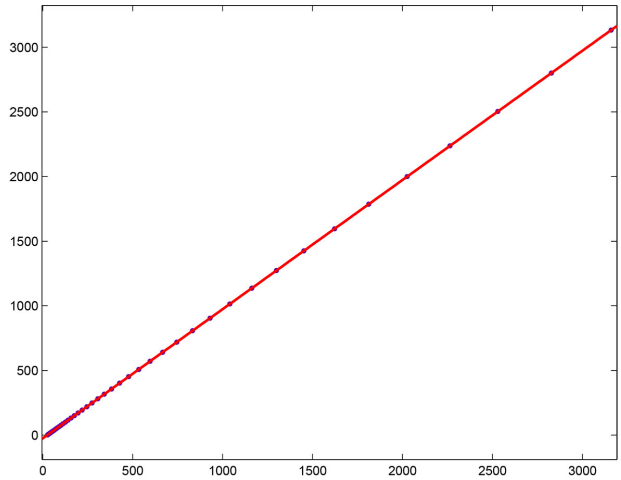

Figure 12

For a linear least-squares fit of the curve, $p_1 = 0.9994$ with a 95% confidence interval of (0.9994, 0.9994), $p_2 = -25.08$ with a 95% confidence interval of (-25.09, -25.08), SSE=0.005602, R-squared=1, and RMSE=0.01155.

Similar results are obtained for the other zeta function zeros. For a linear least-squares fit of the decreasing amplitudes for the second zeta function zero (neglecting the first twenty points), $p_1 = -0.9979$, with a 95% confidence interval of (-0.9996, -0.9962), $p_2 = 37.22$ with a 95% confidence interval of (37.19, 37.26), SSE=0.0168, R-squared=1, and RMSE=0.03055. For a linear least-squares fit of the increasing amplitudes, $p_1 = 0.9997$ with a 95% confidence interval of (0.9997, 0.9997), $p_2 = -37.28$ with a 95% confidence interval of (-37.29, -37.28), SSE=0.007108, R-squared=1, and RMSE=0.01107.

The y-intercepts of the least-squares fits of the increasing amplitudes for the first ten zeta function zeros are -25.08, -37.28, -44.35, -53.94, -58.39, -66.63, -72.54, -76.8, -85.09, and -88.23. A plot of these values versus the imaginary parts of the corresponding zeta function zeros is

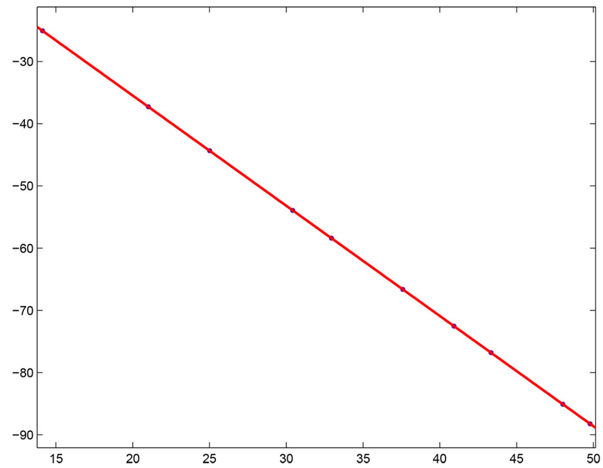

Figure 13

For a linear least-squares fit of the curve, $p_1 = -1.771$ with a 95% confidence interval of $(-1.772, -1.771)$, $p_2 = -0.04806$ with a 95% confidence interval of $(-0.05957, -0.03656)$, SSE=0.000191, R-squared=1, and RMSE=0.004886. The slope is approximately equal to $-\sqrt{\pi}$. All imaginary parts of z where $\Re(z) = 1/2$ map to this line.

A plot of the absolute value of the imaginary part of $R(z, N)$ for $z = (0.4, 15)$ versus $\sqrt{N}$ for $N \leq 4000$ is

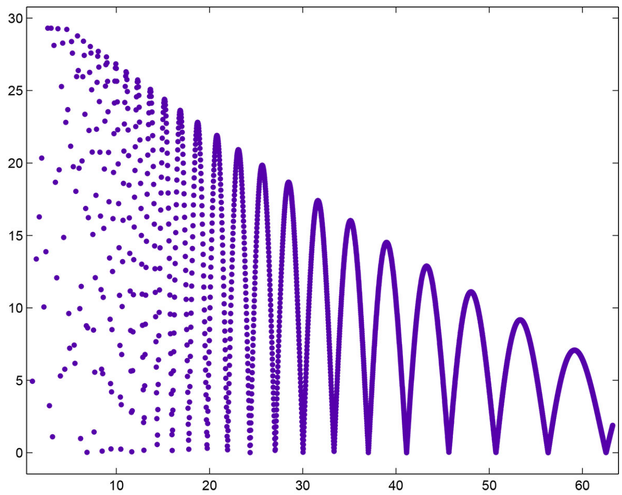

Figure 14

The amplitudes are not linear. A real part of 0.5 is unique in this regard.

4. THE LEGENDRE RELATION

The Legendre relation for the pi function is $\Pi(z) = 2^z\Pi(\frac{z}{2})\Pi(\frac{z-1}{2})\pi^{-1/2}$. Let $L(z)$ denote $\Pi_1(z) - 2^z\Pi_1(\frac{z}{2})\Pi_1(\frac{z-1}{2})\pi^{-1/2}$. A plot of the imaginary parts of $L(z, N)$ versus the real parts for the first zeta function zero and $N \leq 1000$ is

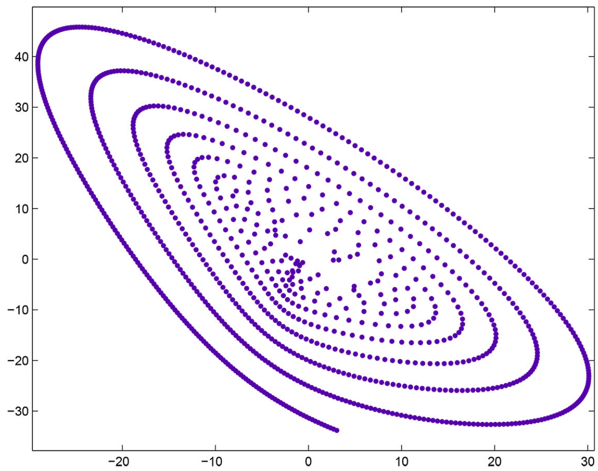

Figure 15

A plot of the amplitudes of the oscillations of the imaginary parts versus $\sqrt{N}$ for $N \leq 10000000$ is

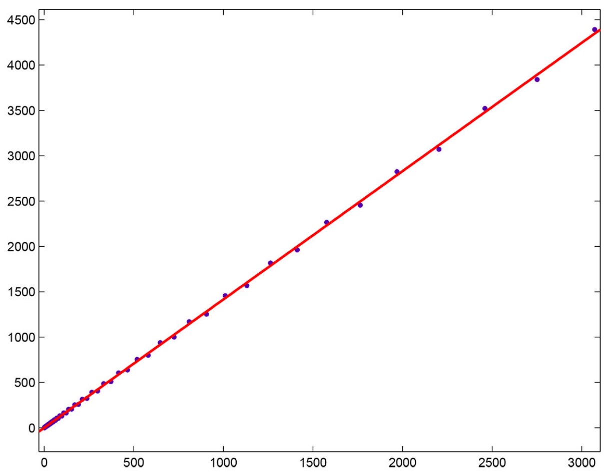

Figure 16

For a linear least-squares fit of the curve, $p_1 = 1.415$ with a 95% confidence interval of (1.409, 1.422), $p_2 = -0.2451$ with a 95% confidence interval of (-5.551, 5.061), SSE=$2.414 \cdot 10^4$, R-squared=0.9997, and RMSE=18.98. The slope is about $\sqrt{2}$ for all imaginary parts when the real part of z is 1/2.

A plot of the amplitudes of the oscillations of the real parts versus $\sqrt{N}$ for $N \leq 10000000$ is

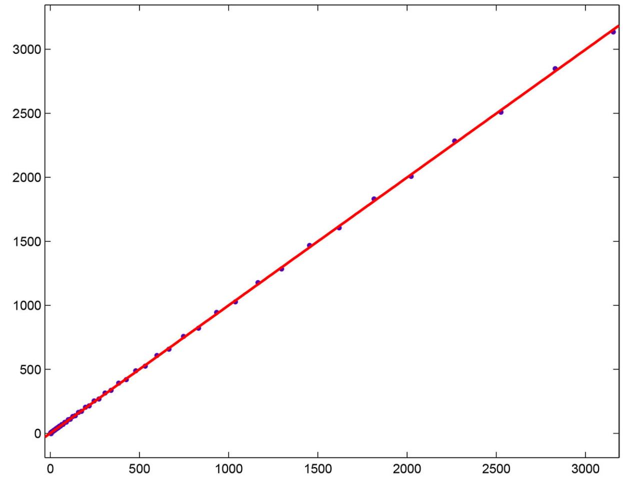

Figure 17

For a linear least-squares fit of the curve, $p_1 = 0.9992$ with a 95% confidence interval of (0.9967, 1.002), $p_2 = 0.917$ with a 95% confidence interval of $(-1.089, 2.923)$, SSE=3951, R-squared=0.9999, and RMSE=7.46. The slope is about 1 for all real parts when the real part of z is 1/2.

There are seventy-three points in this curve. A plot of the first thirty is

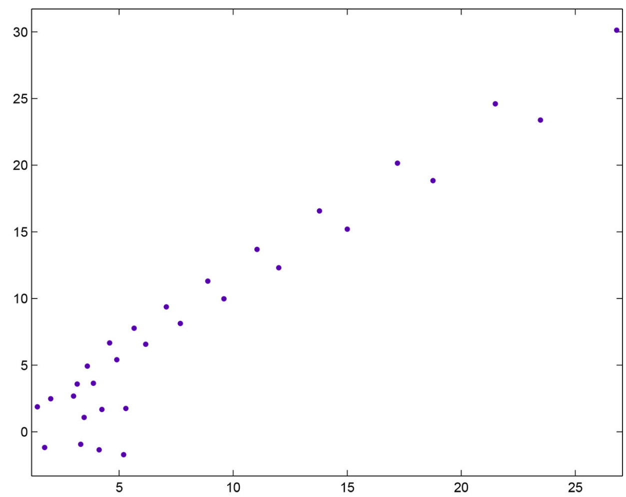

Figure 18

The zigzag values in the upper portion of the curves account for the large SSE value in the linear least-squares fit.

A plot of the amplitude the oscillations of the imaginary part of $L(z, N)$ versus $\sqrt{N}$ when $z = (0.45, 38.5)$ and $N \leq 10000000$ is

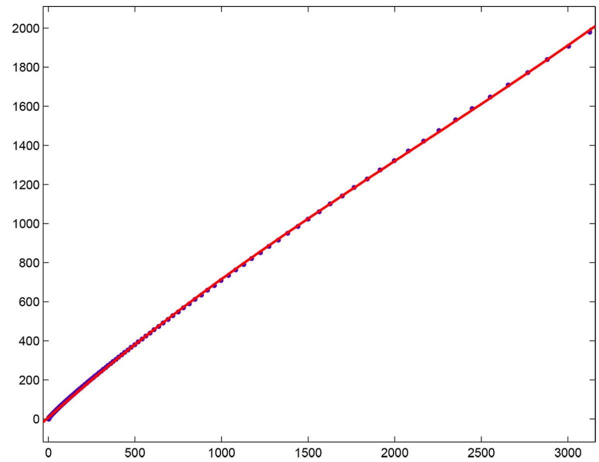

Figure 19

The curve is cubic.

A generalization of the Legendre relation is $\frac{\Pi(z)}{n^z \Pi(\frac{z}{n})\Pi(\frac{z-1}{n})\cdots\Pi(\frac{z-n+1}{n})} = \sqrt{\frac{2\pi n}{(2\pi)^n}}$ where n is a natural number greater than 2. Let $L(z, N, n)$ denote $\Pi_1(z) - n^z \Pi_1(\frac{z}{n})\Pi_1(\frac{z-1}{n}) \cdots \Pi_1(\frac{z-n+1}{n}) \cdot \sqrt{\frac{2\pi n}{(2\pi)^n}}$. A plot of $L(z, N, 3)$ for the second zeta function and $N \leq 1000$ is

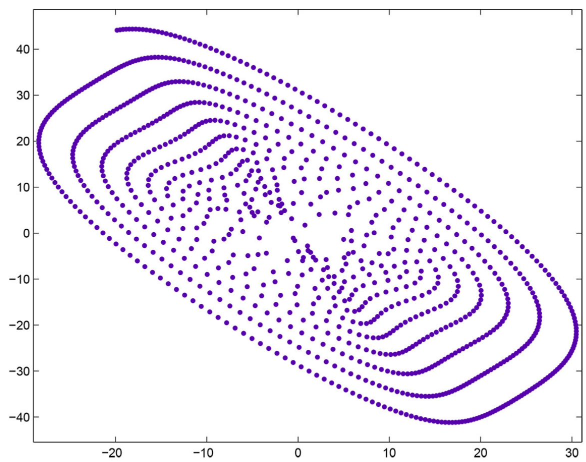

Figure 20

A plot of the amplitudes of the oscillations of the imaginary parts versus $\sqrt{N}$ for $N \leq 10000000$ is

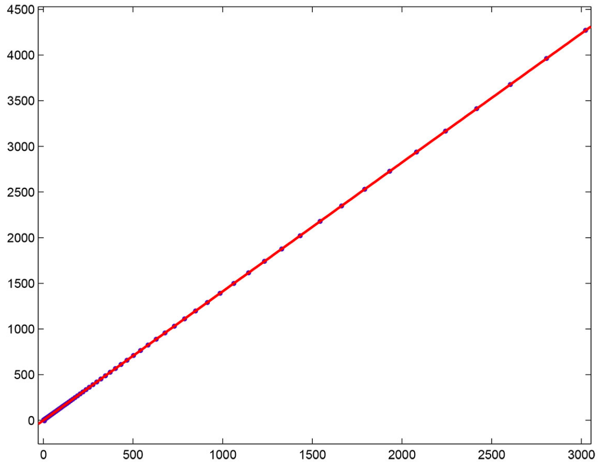

Figure 21

For a linear least-squares fit of the curve, $p_1 = 1.412$ with a 95% confidence interval of (1.412, 1.413), $p_2 = -0.7244$ with a 95% confidence interval of $(-1.065, -0.3837)$, SSE=216.7, R-squared=1, and RMSE=1.48.

A plot of the amplitudes of the oscillations of the real parts versus $\sqrt{N}$ for $N \leq 10000000$ is

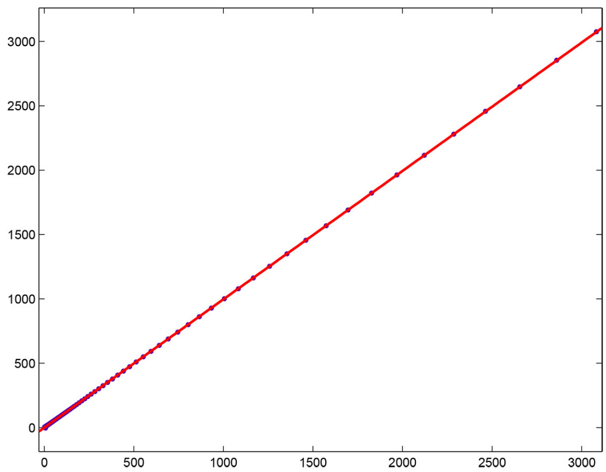

Figure 22

For a linear least-squares fit of the curve, $p_1 = 0.9973$ with a 95% confidence interval of (0.9968, 0.9978), $p_2 = -0.547$ with a 95% confidence interval of $(-0.9682, -0.1258)$, SSE=347, R-squared=1, and RMSE=1.853.

A plot of $L(z, N, 4)$ for the third zeta function and $N \leq 1000$ is

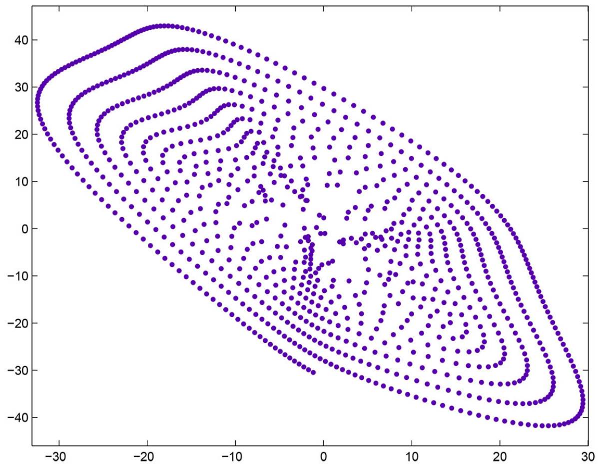

Figure 23

A plot of the amplitudes of the oscillations of the imaginary parts versus $\sqrt{N}$ for $N \leq 10000000$ is

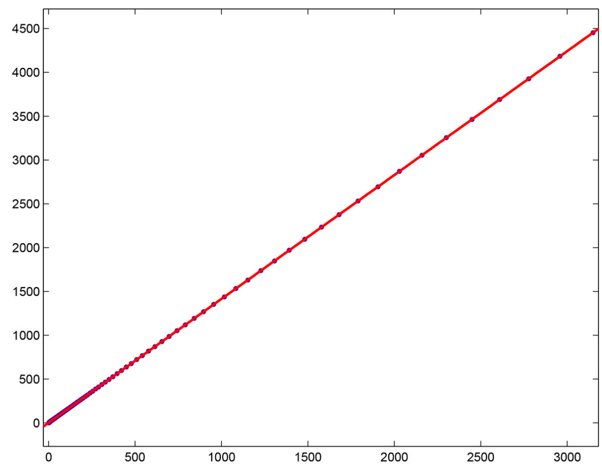

Figure 24

For a linear least-squares fit of the curve, $p_1 = 1.414$ with a 95% confidence interval of (1.414, 1.414), $p_2 = 0.3494$ with a 95% confidence interval of (0.1302, 0.5685), SSE=135.1, R-squared=1, and RMSE=1.057

A plot of the amplitudes of the oscillations of the real parts versus $\sqrt{N}$ for $N \leq 10000000$ is

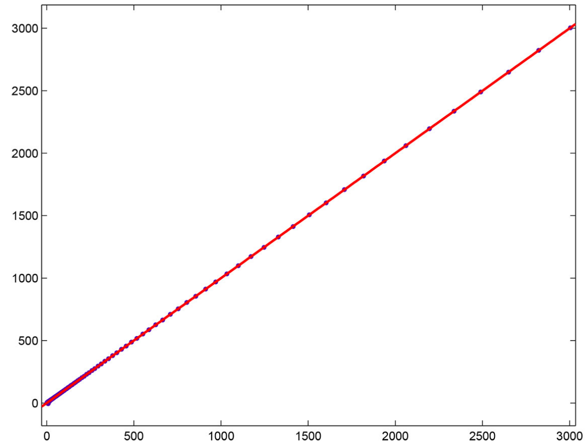

Figure 25

For a linear least-squares fit of the curve, $p_1 = 0.9999$ with a 95% confidence interval of (0.9993, 1.0), $p_2 = 0.07819$ with a 95% confidence interval of (-0.3583, 0.5147), SSE=610.6, R-squared=1, and RMSE=2.184.

5. THE GAMMA FUNCTION AND THE RIEMANN ZETA FUNCTION

A relationship between the gamma function and the Riemann zeta function is $\pi^{-z/2}\Gamma(\frac{z}{2})\zeta(z) = \pi^{-(1-z)/2}\Gamma(\frac{1-z}{2})\zeta(1-z)$. Let $G(z)$ denote $\pi^{-z/2}\Pi_1(\frac{z}{2})\zeta(z) - \pi^{-(1-z)/2}\Pi_1(\frac{1-z}{2})\zeta(-z)$ where $z-1$ is substituted for z (to convert from $\Gamma(z)$ to $\Pi(z)$). Note that $\zeta(-z)$ is used instead of $\zeta(1-z)$ in the right-hand side of the equation. The zeta function is defined in the critical strip. A plot of the imaginary part of $G(z, N)$ versus the real part for the third zeta function zero (z=(0.5, 25.01085758014569...)) and $N \leq 10000$ is

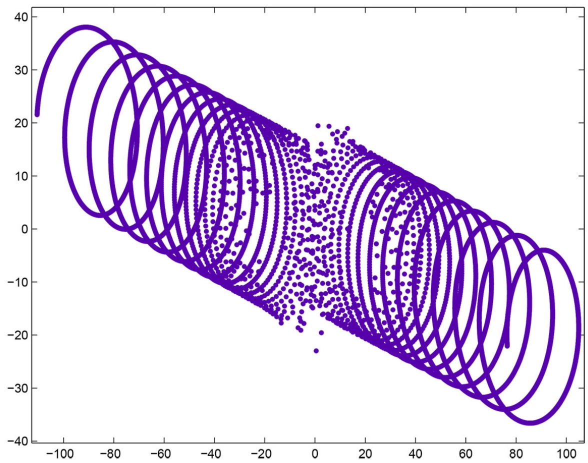

Figure 26

A plot of the amplitude of the oscillations of the real part of $G(z, N)$ versus $\sqrt{N}$ is

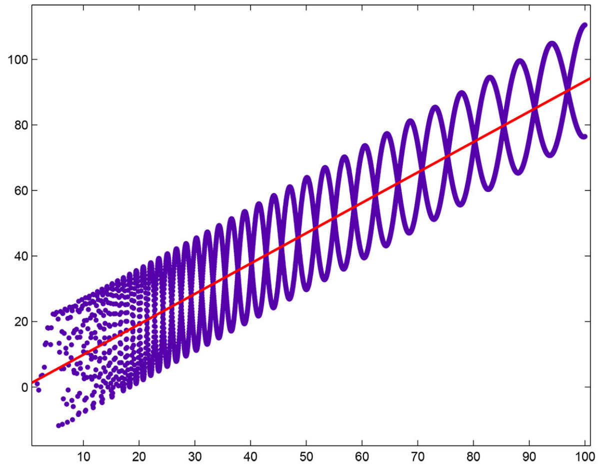

Figure 27

For a linear least-squares fit of the curve, $p_1 = 0.9272$ with a 95% confidence interval of (0.917, 0.9374), $p_2 = 0.6178$ with a 95% confidence interval of (-0.1045, 1.34), SSE=$1.448 \cdot 10^6$, R-squared=0.7618, and RMSE=12.06.

A plot of the amplitude of the oscillations of the real part of $G(z, n)$ versus $\sqrt{N}$ where the real part is set to 0.49 (and the imaginary part remains the same) is

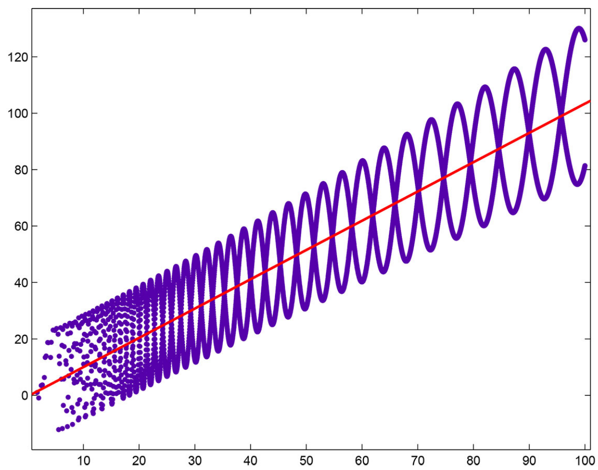

Figure 28

The amplitudes do not increase linearly.

A plot of the amplitude of the oscillations of the real part of $G(z,n)$ versus $\sqrt{N}$ where $z=(0.5,25.0)$ (roughly equal to the third zeta function zero) is

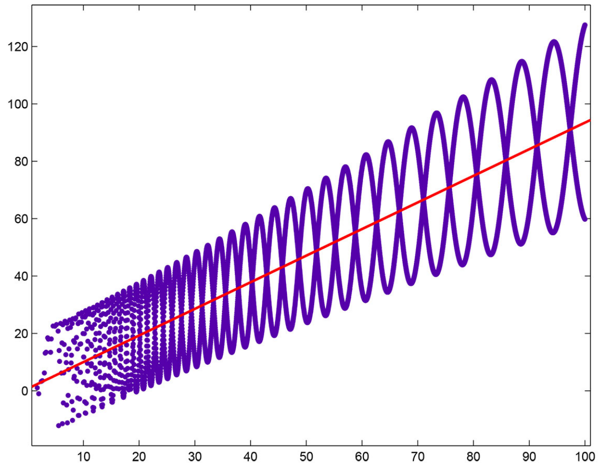

Figure 29

The amplitudes do not increase linearly.

A plot of the amplitude of the oscillations of the real part of $G(z,N)$ for the first zeta function zero (z=(0.5,14.13472514173470...))versus $\sqrt{N}$ for $N \leq 200000$ is

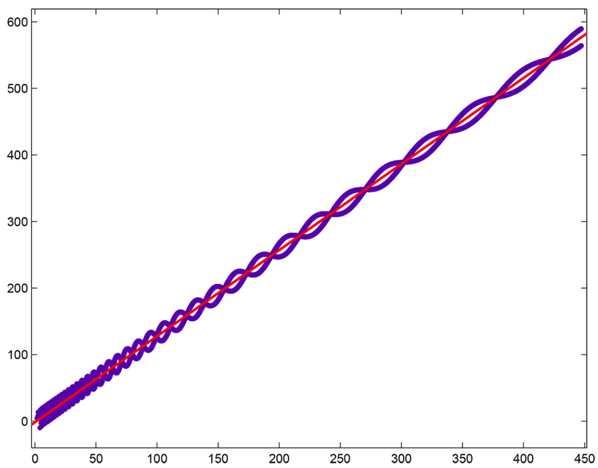

Figure 30

For a linear least-squares fit of the curve, $p_1 = 1.293$ with a 95% confidence interval of (1.293, 1.294), $p_2 = -1.673$ with a 95% confidence interval of $(-1.792, -1.555)$, SSE=$1.623 \cdot 10^7$, R-squared=0.9956, and RMSE=9.008.

The slopes of the curves for the first ten zeta function zeros range from 1.326 to 0.2128. The slopes of the curves for the imaginary part range from 0.9875 to 0.101.

6. METHODS

```
#include <math.h>
#include <stdio.h>
//
// Pi(z)-z*Pi(z-1)
//
unsigned int max=10000000;
double s=0.50;
//double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
double t=49.77383247767230;
//double t=52.97032147771446;
//double t=56.44624769706339;
//double t=59.34704400260235;
//double t=60.83177852460981;
//double t=65.11254404808160;
//double t=67.07981052949417;
//double t=69.54640171117399;
//double t=72.06715767448191;
//double t=75.70469069908393;
//double t=77.14484006887480;
double pi=3.14159265359;
unsigned int out=6; // 2 for top, 3 for left, 4 for bottom, 5 for right
void main() {
unsigned int x;
double temp1,temps,tempt,prods,prods1,a,b,c,d,olds,oldt,s1,oldolds,oldoldt;
FILE *Outfp;
Outfp = fopen("test4b.dat","w");
```

```
oldoldt=0.0;
oldt=0.0;
oldolds=0.0;
olds=0.0;
prods=1.0;
prods1=1.0;
for (x=1; x<=max; x++) {
    prods=prods*(double)x/((double)x+s);
    if (s>=0.0)
        temp1=pow((double)(x+1),s);
    else {
        temp1=pow((double)(x+1),-s);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x+1)));
    tempt=temp1*(sin(t*log(x+1)));
    a=prods*temps-tempt;
    b=prods*tempt+temps;
    s1=s-1;
    prods1=prods1*(double)x/((double)x+s1);
    if (s1>=0.0)
        temp1=pow((double)(x+1),s1);
    else {
        temp1=pow((double)(x+1),-s1);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x+1)));
    tempt=temp1*(sin(t*log(x+1)));
    c=prods1*temps-tempt;
    d=prods1*tempt+temps;
    temps=c*s-d*t;
    tempt=c*t+d*s;
    temps=temps-a;
    tempt=tempt-b;
    if (out==1)
        fprintf(Outfp," %.16llf %.16llf \n",temps,tempt);
    if ((out==2)&&(olds>0.0)&&(temps<=0.0)))
        fprintf(Outfp," %d \n",x);
    if ((out==3)&&((oldt>0.0)&&(tempt<=0.0)))
```

```
        fprintf(Outfp," %d \n",x);
    if ((out==4)&&((olds<0.0)&&(temps>=0.0)))
        fprintf(Outfp," %d \n",x);
    if ((out==5)&&((oldt<0.0)&&(tempt>=0.0)))
        fprintf(Outfp," %d \n",x);
    if ((out==6)&&((oldolds<olds)&&(olds>temps)))
        fprintf(Outfp," %d %.16llf \n",x,olds);
    if ((out==6)&&((oldolds>olds)&&(olds<temps)&&(olds<0.0)))
        fprintf(Outfp," %d %.16llf \n",x,-olds);
    if ((out==7)&&((oldoldt<oldt)&&(oldt>tempt)))
        fprintf(Outfp," %d %.16llf \n",x,oldt);
    if ((out==7)&&((oldoldt>oldt)&&(oldt<tempt)&&(oldt<0.0)))
        fprintf(Outfp," %d %.16llf \n",x,-oldt);
    oldolds=olds;
    olds=temps;
    oldoldt=oldt;
    oldt=tempt;
    }
fclose(Outfp);
return;
}

#include <math.h>
#include <stdio.h>
// Pi and zeta function
// second zeta function value is -z instead of 1-z
unsigned int max=200000;
double s=0.50;
//double t=38.5;
//double t=14.13472514173470;
//double t=21.02203963877156;
double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
```

```
//double t=52.97032147771446;
//double t=56.44624769706339;
//double t=59.34704400260235;
//double t=60.83177852460981;
//double t=65.11254404808160;
//double t=67.07981052949417;
//double t=69.54640171117399;
//double t=72.06715767448191;
//double t=75.70469069908393;
//double t=77.14484006887480;
double pi=3.14159265359;
unsigned int out=7; // 2 for top, 3 for left, 4 for bottom, 5 for right
unsigned int nosub=0;
void main() {
unsigned int x;
double temp1,temps,tempt,prods,prods1,prods2,a,b,c,d,e,f,g,h,olds,oldt,s1;
double sumr,sumi,temp,y,i,j,sumr2,sumi2,e2,f2,k,l;
double oldolds,oldoldt;
FILE *Outfp;
Outfp = fopen("test9.dat","w");
printf(" pi=%.16llf \n",pi);
s1=-(s-1.0)/2.0; // first Pi
if (s1>=0.0)
    temp1=pow((double)pi,s1);
else {
    temp1=pow((double)pi,-s1);
    temp1=1.0/temp1;
    }
g=temp1*(cos(t*log(pi)));
h=temp1*(sin(t*log(pi)));
s1=-(2.0-s)/2.0; // second Pi
if (s1>=0.0)
    temp1=pow((double)pi,s1);
else {
    temp1=pow((double)pi,-s1);
    temp1=1.0/temp1;
    }
k=temp1*(cos(t*log(pi)));
l=temp1*(sin(t*log(pi)));
```

```
y=s-1.0; // first zeta function
if (y>=0.0)
    temp1=pow((double)2,y);
else {
    temp1=pow((double)2,-y);
    temp1=1.0/temp1;
    }
e=temp1*(cos(t*log(2)));
f=temp1*(sin(t*log(2)));
e=1.0-e;
f=-f;
y=1.0-s; // second zeta function, normally 2.0-s
if (y>=0.0)
    temp1=pow((double)2,y);
else {
    temp1=pow((double)2,-y);
    temp1=1.0/temp1;
    }
e2=temp1*(cos(t*log(2)));
f2=temp1*(sin(t*log(2)));
e2=1.0-e2;
f2=-f2;
olds=0.0;
oldolds=0.0;
oldt=0.0;
oldoldt=0.0;
prods=1.0;
prods1=1.0;
prods2=1.0;
sumr=0.0;
sumi=0.0;
sumr2=0.0;
sumi2=0.0;
for (x=1; x<=max; x++) {
    s1=s-1.0; // first zeta function
    if (s1>=0.0)
        temp1=pow((double)x,s1);
    else {
        temp1=pow((double)x,-s1);
```

```
        temp1=1.0/temp1;
        }
    a=temp1*(cos(t*log(x)));
    b=temp1*(sin(t*log(x)));
    temp1=a*a+b*b;
    if (x!=(x/2)*2) {
        sumr=sumr+a/temp1;
        sumi=sumi-b/temp1;
        }
    else {
        sumr=sumr-a/temp1;
        sumi=sumi+b/temp1;
        }
    a=sumr*e+sumi*f;
    b=sumr*f-sumi*e;
    b=-b;
    s1=(s-1)/2; // first Pi
    prods1=prods1*(double)x/((double)x+s1);
    if (s1>=0.0)
        temp1=pow((double)(x+1),s1);
    else {
        temp1=pow((double)(x+1),-s1);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x+1)));
    tempt=temp1*(sin(t*log(x+1)));
    c=prods1*temps-tempt;
    d=prods1*tempt+temps;
    temp=a*c-b*d; // Pi*zeta
    b=a*d+b*c;
    a=temp;
    c=a*g-b*h; // product times power of pi
    d=a*h+b*g;
    s1=1.0-s; // second zeta function, normally 2.0-s
    if (s1>=0.0)
        temp1=pow((double)x,s1);
    else {
        temp1=pow((double)x,-s1);
        temp1=1.0/temp1;
```

```
    }
a=temp1*(cos(t*log(x)));
b=temp1*(sin(t*log(x)));
temp1=a*a+b*b;
if (x!=(x/2)*2) {
    sumr2=sumr2+a/temp1;
    sumi2=sumi2-b/temp1;
    }
else {
    sumr2=sumr2-a/temp1;
    sumi2=sumi2+b/temp1;
    }
a=sumr2*e2+sumi2*f2;
b=sumr2*f2-sumi2*e2;
b=-b;
s1=(2.0-s)/2.0; // second Pi function
prods2=prods2*(double)x/((double)x+s1);
if (s1>=0.0)
    temp1=pow((double)(x+1),s1);
else {
    temp1=pow((double)(x+1),-s1);
    temp1=1.0/temp1;
    }
temps=temp1*(cos(t*log(x+1)));
tempt=temp1*(sin(t*log(x+1)));
i=prods2*temps-tempt;
j=prods2*tempt+temps;
temp=a*i-b*j; // second zeta times second Pi
b=a*j+b*i;
a=temp;
temp=a*k-b*l; // product times power of pi
b=a*l+b*k;
a=temp;
if (nosub==0) {
    temps=c-a;
    tempt=d-b;
    }
if (nosub==1) {
    temps=c;
```

```
        tempt=d;
        }
    if (nosub==2) {
        temps=a;
        tempt=b;
        }
    if (out==1)
        fprintf(Outfp," %.16llf %.16llf \n",temps,tempt);
    if ((out==2)&&(olds>0.0)&&(temps<=0.0)))
        fprintf(Outfp," %d \n",x);
    if ((out==3)&&((oldt>0.0)&&(tempt<=0.0)))
        fprintf(Outfp," %d \n",x);
    if ((out==4)&&((olds<0.0)&&(temps>=0.0)))
        fprintf(Outfp," %d \n",x);
    if ((out==5)&&((oldt<0.0)&&(tempt>=0.0)))
        fprintf(Outfp," %d \n",x);
    if ((out==6)&&((oldolds<olds)&&(olds>temps)))
        fprintf(Outfp," %d %.16llf \n",x,olds);
    if ((out==6)&&((oldolds>olds)&&(olds<temps)&&(olds<0.0)))
        fprintf(Outfp," %d %.16llf \n",x,-olds);
    if ((out==7)&&((oldoldt<oldt)&&(oldt>tempt)))
        fprintf(Outfp," %d %.16llf \n",x,oldt);
    if ((out==7)&&((oldoldt>oldt)&&(oldt<tempt)&&(oldt<0.0)))
        fprintf(Outfp," %d %.16llf \n",x,-oldt);
    oldolds=olds;
    olds=temps;
    oldoldt=oldt;
    oldt=tempt;
    }
fclose(Outfp);
return;
}
```

REFERENCES

[1] H. M. Edwards, *Riemann's Zeta Function*, Dover, (1974)