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A Hybrid Haar Wavelet Collocation Method for Nonlocal Hyperbolic Partial Differential Eequations

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Abstract

In this article, we develop a hybrid Haar wavelet collocation method for the numerical solution of nonlocal hyperbolic partial differential equations. Developing an accurate and efficient numerical method to solve such problem is a difficult task due to the presence of nonlocal boundary condition. The speciality of the proposed method is to handle integral boundary condition efficiently using the given data. Because of various attractive properties of Haar wavelets such as orthogonality, compact support and closed form expression, these wavelets are efficiently used for the spatial discretization whereas second order accurate finite difference is used for temporal discretization. Stability and error estimates have been investigated in order to ensure the convergence of the method. Finally, numerical results are compared with few known results and it is shown that the results obtained by proposed method is better than few known results.

Keywords : Haar wavelet, Nonlocal condition, Finite difference, Collocation method, Stability.

Mathematics Subject Classification(2010): 65T60, 65C30, 31A30.

1. INTRODUCTION

Numerical techniques for solving nonlocal hyperbolic partial differential equations have received enormous attention over the last few years. These nonlocal hyperbolic PDEs are used to describe the dynamics of ground water (see[4,15]). Some problems in visco-elasticity and food industry are also modelled by nonlocal hyperbolic PDEs (see[3,11,16]). The nonlocal boundary conditions appear in the hyperbolic partial differential equations when we cannot measure the boundary data directly.

In this article, let us consider the non local hyperbolic problem in the rectangular domain $D = \{(x,t) : 0 \leq x \leq 1,\ 0 \leq t \leq T\}$ given by

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \phi(x,t),\ x \in (0,1),\ t \in [0,T], \tag{1.1}$$

with initial conditions

$$u(x,0) = f(x), \qquad 0 \leq x \leq 1, \tag{1.2}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 \leq x \leq 1, \tag{1.3}$$

Dirichlet boundary condition

$$u(0,t) = h(t), \qquad 0 < t \leq T, \tag{1.4}$$

and nonlocal condition

$$\int_0^1 u(x,t)dx = \nu(t),\ \ 0 < t \leq T, \tag{1.5}$$

where ϕ, f, g, h and ν are known functions. It is assumed that

$$f \in C[0,1] \cap C^2(0,1) \text{ and } g \in C[0,1] \cap C^1(0,1).$$

It should be noted that f and g satisfy the following compatibility conditions

$$f(0) = h(0), \int_0^1 f(x)dx = \nu(0),$$

$$g(0) = h^{'}(0), \int_0^1 g(x)dx = \nu^{'}(0).$$

The existence, uniqueness and stability results that combine integral as well as Neumann conditions are studied by Beilin *et al.*[2] Gordeziani *et al.*[10] and Kavalloris *et al.*[11] have also investigated hyperbolic PDEs with nonlocal boundary conditions.

Numerical techniques for the solution of nonlocal hyperbolic equations have been proposed by several researchers. Ang *et al.*[1] developed a numerical method based on local interpolating functions and integro-differential equation to solve the nonlocal hyperbolic PDEs. Dehghan and his collaborators investigated nonlocal hyperbolic PDEs using several numerical methods, e.g. cubic B-spline scaling functions based finite difference method,[7] variational iterative method,[8] meshless method using radial basis functions,[9] and Bernstein Ritz-Galerkin method.[18] A numerical method based on shifted Legendre tau technique has been proposed by Saadatmandi *et al.*[17]

In the last few years, collocation methods based on Haar wavelets[12,13] are extensively used for the numerical solution of PDEs. Because of various attractive properties of Haar wavelets such as orthogonality, compact support and closed form expression, it is widely used in various areas of science and engineering. The major drawback of Haar wavelet basis is its discontinuity. Therefore, we cannot express the solution in terms of Haar wavelet basis directly. To overcome with this difficulty, either we can regularize the Haar wavelets with interpolating splines (see[5]) or expand the highest derivatives in terms of Haar wavelet basis and integrate it out to get the desired expressions (see[6]). We have used later approach to handle the difficulty coming from discontinuity of the wavelet. Second order accurate finite difference scheme is used for the temporal discretization whereas Haar wavelet basis is used for the spatial discretization. Stability and error analysis have been rigorously studied in order to ensure the convergence of the method. The obtained numerical results are compared with the numerical results provided in the paper[9] by Dehghan *et al.* In paper,[9] authors reformulated the problem in such a way that the integral boundary condition is converted into a periodic boundary condition. We have dealt with integral boundary condition directly using the given data which is more accurate.

The article is organized as follows. We review some basics of Haar wavelets in section 2. In section 3, we propose a hybrid Haar wavelet collocation method (HHWCM) for nonlocal hyperbolic PDEs. In section 4, Stability and error analysis have been studied. Numerical results are analyzed in section 5. A brief conclusion with future work is provided in section 6.

2. BASIC BACKGROUND

In this section, we review some basics of Haar wavelet which will be used for the proposed numerical method.

2.1. Haar wavelet

For $0 \leq j \leq J,\ 0 \leq k \leq 2^j - 1$ and set $i = 2^j + k + 1$, Haar wavelet family $\{h_i(x)\}_{i \geq 2}$ is defined as

$$h_i(x) := \begin{cases} 1 & \frac{k}{2^j} \leq x \leq \frac{k+1/2}{2^j}, \\ -1 & \frac{k+1/2}{2^j} \leq x \leq \frac{k+1}{2^j}, \\ 0 & \text{otherwise}, \end{cases}$$

where J denotes the maximum resolution level. For simplicity, we have considered $x \in [0,1]$.
Note: $i = 1$ correspond to mother wavelet $h_1(x)$ defined by

$$h_1(x) = \begin{cases} 1, & \forall x \in [0,1), \\ 0, & \text{elsewhere.} \end{cases} \tag{2.1}$$

$$P_{\beta,i}(x) = \int_0^x \int_0^x ... \int_0^x h_i(t)dt^\beta = \frac{1}{(\beta-1)!} \int_0^x (x-t)^{\beta-1} h_i(t)dt, \tag{2.2}$$

where $\beta = 1, 2, \cdots, s$ and $i = 1, 2, 3, \cdots, 2^{J+1}$.

$P_{\beta,i}(x)$ can be easily calculated using the definition of Haar wavelet.

$$P_{\beta,i}(x) = \begin{cases} 0, & x < \frac{k}{2^j}, \\ \frac{1}{\beta!}\left(x - \frac{k}{2^j}\right)^\beta, & x \in \left[\frac{k}{2^j}, \frac{k+0.5}{2^j}\right), \\ \frac{1}{\beta!}\left[\left(x - \frac{k}{2^j}\right)^\beta - 2\left(x - \frac{k+0.5}{2^j}\right)^\beta\right], & x \in \left[\frac{k+0.5}{2^j}, \frac{k+1}{2^j}\right), \\ \frac{1}{\beta!}\left[\left(x - \frac{k}{2^j}\right)^\beta - 2\left(x - \frac{k+0.5}{2^j}\right)^\beta + \left(x - \frac{k+1}{2^j}\right)^\beta\right], & x \geq \frac{k+1}{2^j}. \end{cases} \tag{2.3}$$

In particular, for $\beta = 1$ and 2, we get

$$P_{1,i}(1) = \begin{cases} 1, & \text{for } i = 1, \\ 0, & \text{for } i \neq 1. \end{cases} \tag{2.4}$$

and

$$P_{2,i}(1) = \begin{cases} 0.5, & \text{for } i = 1, \\ \dfrac{1}{2^{2j+2}}, & \text{for } i \neq 1. \end{cases} \tag{2.5}$$

Let us define

$$C_{1,i} = \int_0^1 P_{1,i}(x)dx = \begin{cases} 0.5, & \text{for } i = 1, \\ \dfrac{1}{2^{2j+2}}, & \text{for } i \neq 1. \end{cases} \tag{2.6}$$

and

$$C_{2,i} = \int_0^1 P_{2,i}(x)dx = \begin{cases} \frac{1}{6}, & \text{for } i = 1, \\ \dfrac{2^j + 1 - 2k - 1}{2^{3j+3}}, & \text{for } i \neq 1. \end{cases} \tag{2.7}$$

The grid point y_l and collocation point x_l are given by

$$y_l = \frac{l}{2^{J+1}},\ l = 0, 1, 2, \cdots, 2^{J+1}.$$

and

$$x_l = \frac{y_l + y_{l-1}}{2},\ l = 1, 2, \cdots, 2^{J+1}.$$

Note: For the spatial discreization, we have used Haar wavelet family whereas second order accurate finite difference scheme is used for temporal discretization.
The function approximation of u in terms of Haar wavelet is given by

$$u_J(x) = \sum_{i=1}^{2^{J+1}} a_i h_i(x) \tag{2.8}$$

where

$$a_i = 2^J \int_0^1 u(x) h_i(x) dx \tag{2.9}$$

3. A HYBRID HAAR WAVELET COLLOCATION METHOD FOR NONLOCAL HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION

We propose a hybrid collocation method based on Haar wavelets and second order accurate finite difference scheme to solve the problem (1.1)–(1.5). We assume that u_J be the wavelet approximation of u.

Let us assume

$$\frac{\partial^2 u_J}{\partial x^2}(x,t) = \sum_{i=1}^{2^{J+1}} a_i(t) h_i(x) \tag{3.1}$$

Integrating equation (3.1) from 0 to x, we get

$$\frac{\partial u_J}{\partial x}(x,t) = \sum_{i=1}^{2^{J+1}} a_i(t) P_{1,i}(x) + \frac{\partial u_J}{\partial x}(0,t) \tag{3.2}$$

Integrating equation (3.2) from 0 to 1, we get

$$u_J(1,t) - u_J(0,t) = \sum_{i=1}^{2^{J+1}} a_i(t) C_{1,i} + \frac{\partial u_J}{\partial x}(0,t) \tag{3.3}$$

where

$$C_{1,i} = \int_0^1 P_{1,i}(x) dx.$$

Hence,

$$\frac{\partial u_J}{\partial x}(x,t) = \sum_{i=1}^{2^{J+1}} a_i(t) \Big(P_{1,i}(x) - C_{1,i} \Big) + u_J(1,t) - u_J(0,t) \tag{3.4}$$

Integrating equation (3.4) from 0 to x, we get

$$u_J(x,t) = \sum_{i=1}^{2^{J+1}} a_i(t)\Big(P_{2,i}(x) - xC_{1,i}\Big) + x[u_J(1,t) - u_J(0,t)] + u_J(0,t) \tag{3.5}$$

Using nonlocal condition (1.5), we obtain

$$\sum_{i=1}^{2^{J+1}} a_i(t)\Big(C_{2,i} - \frac{1}{2}C_{1,i}\Big) + \frac{1}{2}[u_J(1,t) - u_J(0,t)] + u_J(0,t) = \nu(t) \tag{3.6}$$

It is to be noted that u_J is chosen in such a way that u_J satisfies (1.4) and (1.5). After simplification, we get

$$u_J(1,t) = \sum_{i=1}^{2^{J+1}} a_i(t)\Big(C_{1,i} - 2C_{2,i}\Big) + 2\nu(t) - h(t) \tag{3.7}$$

Thus, from equation (3.5), we get

$$u_J(x,t) = \sum_{i=1}^{2^{J+1}} a_i(t)\Big(P_{2,i}(x) - 2xC_{2,i}\Big) + 2x[\nu(t) - h(t)] + h(t) \tag{3.8}$$

Using second order finite difference scheme for temporal discretization and Haar wavelets for spatial discretization in equation (1.1), we obtain

$$\frac{u_J(x,t_{n+1}) - 2u_J(x,t_n) + u_J(x,t_{n-1})}{\Delta t^2} = \sum_{i=1}^{2^{J+1}} a_i h_i(x) + \phi(x,t_n) \tag{3.9}$$

From the given boundary condition (1.3) and using central difference formula, we get

$$\frac{u_J(x,t_1) - u_J(x,t_{-1})}{2\Delta t} = g(x) \tag{3.10}$$

This implies

$$u_J(x,t_{-1}) = u_J(x,t_1) - 2\Delta t g(x) \tag{3.11}$$

Using (3.9) and (3.11), we obtain the following equation at $t_0 = 0$

$$u_J(x,t_1) = u_J(x,t_0) + \Delta t g(x) + \frac{(\Delta t)^2}{2}\sum_{i=1}^{2^{J+1}} a_i h_i(x) + \frac{(\Delta t)^2}{2}\phi(x,t_0) \tag{3.12}$$

Using equation (3.8) and (3.12), we obtain

$$\sum_{i=1}^{2^{J+1}} a_i\Big(P_{2,i}(x) - \frac{\Delta t^2}{2}h_i(x) - 2xC_{2,i}\Big) + 2x[\nu(t_1) - h(t_1)] + h(t_1)$$
$$= u_J(x,t_0) + \Delta t g(x) + \frac{\Delta t^2}{2}\phi(x,t_0) \tag{3.13}$$

At $t = t_n$, we obtain the following discretized scheme,

$$\sum_{i=1}^{2^{J+1}} a_i\Big(P_{2,i}(x) - \Delta t^2 h_i(x) - 2xC_{2,i}\Big) + 2x[\nu(t_{n+1}) - h(t_{n+1})] + h(t_{n+1})$$
$$= \Delta t^2\phi(x, t_n) + 2u_J(x, t_n) - u_J(x, t_{n-1}) \tag{3.14}$$

Equation (3.14) at the collocation points $x_l, l = 1, 2, 3, \ldots, 2^{J+1}$ is given by

$$\sum_{i=1}^{2^{J+1}} a_i\Big(P_{2,i}(x_l) - \Delta t^2 h_i(x_l) - 2x_lC_{2,i}\Big) + 2x_l[\nu(t_{n+1}) - h(t_{n+1})] + h(t_{n+1})$$
$$= \Delta t^2\phi(x_l, t_n) + 2u_J(x_l, t_n) - u_J(x_l, t_{n-1}) \tag{3.15}$$

Finally, we obtain a matrix system at $t = t_n$

$$\mathbf{Ca} = \mathbf{b}$$

where $\mathbf{C} = \{c_{li},\ 1 \le l, i \le 2^{J+1}\}$ and $\mathbf{b} = \{b_l,\ 1 \le l \le 2^{J+1}\}$. The expressions for c_{li} and b_l are given by

$$c_{li} = \Big(P_{2,i}(x_l) - \Delta t^2 h_i(x_l) - 2x_lC_{2,i}\Big)$$

and

$$b_l = \Delta t^2\phi(x_l, t_n) + 2u_J(x_l, t_n) - u_J(x_l, t_{n-1}) - 2x_l[\nu(t_{n+1}) - h(t_{n+1})] - h(t_{n+1})$$

Here, $\mathbf{a} = \{a_i,\ 1 \le i \le 2^{J+1}\}$ is the set of wavelet coefficients. We compute wavelet coefficients at each time step and obtain the required solution.

4. STABILITY AND ERROR ANALYSIS

4.1. Stability analysis

In this subsection, we will study stability analysis for the proposed method.
Equation (1.1) can be written as follows:

$$\frac{\partial^2 u}{\partial t^2}(x, t) = \mathcal{L}u(x, t) + \phi(x, t) \tag{4.1}$$

where $\mathcal{L} = \frac{\partial^2}{\partial x^2}$ is the differential operator. Following the temporal discretization using finite difference technique, we obtain

$$u^{n+1} - 2u^n + u^{n-1} = (\Delta t)^2\mathcal{H}u^{n+1} + (\Delta t)^2\phi(x, t_n) \tag{4.2}$$

$$\implies u^{n+1} = 2(I-(\Delta t)^2\mathcal{H})^{-1}u^n-(I-(\Delta t)^2\mathcal{H})^{-1}u^{n-1}+(I-(\Delta t)^2\mathcal{H})^{-1}(\Delta t)^2\phi(x,t_n) \tag{4.3}$$

where I is the identity matrix and $\mathcal{H}$ is the Haar matrix corresponding to the differential operator $\mathcal{L}$.

Since equation involves two time levels, we add one identity equation in order to make single time level. We proceed as follows:

$$\begin{aligned} u^{n+1} &= 2Bu^n - Bu^{n-1} \\ u^n &= u^n \end{aligned} \tag{4.4}$$

where $B = (I-(\Delta t)^2\mathcal{H})^{-1}$. Equation (4.4) can be written in the matrix form as follows

$$\begin{bmatrix} 2B & -B \\ I_{2^{J+1}} & \mathbf{0}_{2^{J+1}} \end{bmatrix} \begin{bmatrix} u^n \\ u^{n-1} \end{bmatrix} = \begin{bmatrix} u^{n+1} \\ u^n \end{bmatrix} \tag{4.5}$$

We know that the eigenvalues of identity matrix is always 1. The stability of the numerical scheme will depend upon the eigenvalues of the matrix X where

$$X = \begin{bmatrix} 2B & -B \\ I_{2^{J+1}} & \mathbf{0}_{2^{J+1}} \end{bmatrix} \tag{4.6}$$

The proposed numerical scheme will be stable if all the eigenvalues of the matrix X is less than or equal to 1.

The eigenvalues of the matrix X for hybrid Haar wavelet collocation method for different Δt and J are given below

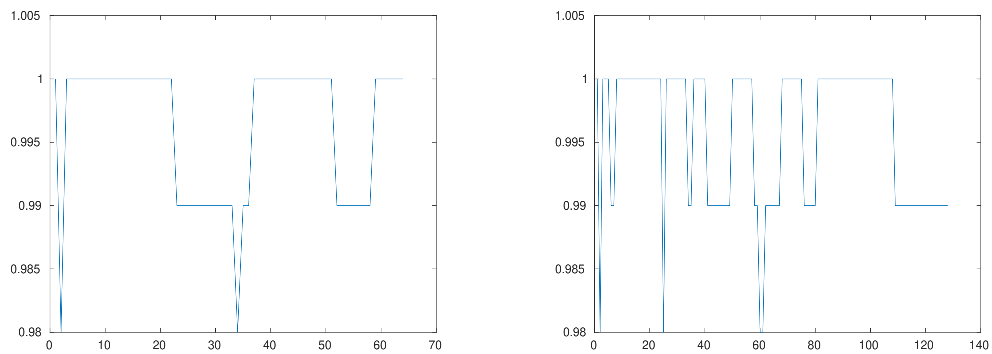

Figure 1: Eigenvalues of B at $\Delta t = 10^{-2}$ and $J = 4, 5$.

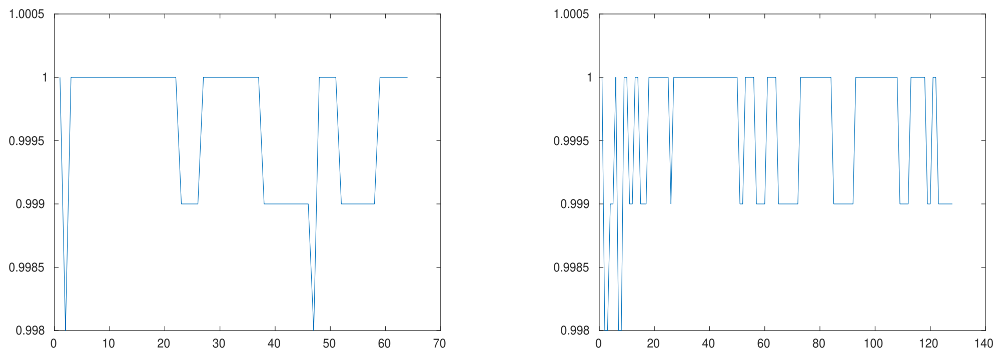

Figure 2: Eigenvalues of B at $\Delta t = 10^{-3}$ and $J = 4, 5$.

From the above figures, it can be easily guaranteed that the proposed method is stable.

4.2. Error Analysis

In this subsection, we will study error analysis for the proposed numerical method. From equation (3.8), we have the approximate representation of function u given by

$$u_J(x,t) = \sum_{i=1}^{2^{J+1}} a_i \Big(P_{2,i}(x) - 2xC_{2,i} \Big) + 2x(\nu(t) - h(t)) + h(t) \tag{4.7}$$

and the exact representation of u is given by

$$u(x,t) = \sum_{i=1}^{\infty} a_i \Big(P_{2,i}(x) - 2xC_{2,i} \Big) + 2x(\nu(t) - h(t)) + h(t) \tag{4.8}$$

Hence, the error term is given by

$$E_J = u - u_J = \sum_{i=2^{J+1}+1}^{\infty} a_i \Big(P_{2,i}(x) - 2xC_{2,i} \Big) \tag{4.9}$$

Equation (4.9) can be written as

$$E_J = \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k+1} \Big(P_{2,2^j+k+1}(x) - 2xC_{2,2^j+k+1} \Big) \tag{4.10}$$

Lemma 4.2.1. [12] Let u be Lipschitz continuous on the unit interval. Then,

$$|a_{2^j+k+1}| \leq \frac{L}{2^{j+1}} \tag{4.11}$$

where L denote the Lipschitz constant and

$$a_{2^j+k+1} = 2^j \int_0^1 u(x) h_{2^j+k+1}(x) dx \tag{4.12}$$

Lemma 4.2.2. Let u be the Lipschitz continuous on the unit interval. Then, for fixed t, the proposed method is convergent and order of convergence is 2 in spatial variable i.e.

$$||E_J||_2 = ||u - u_J||_2 = \mathcal{O}\Big[\Big(\frac{1}{2^J+1}\Big)^2\Big] \tag{4.13}$$

Proof From equation (4.10), we get

$$\begin{aligned}
\|E_J\|^2 &= \int_0^1 \Big| \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k+1}\Big(P_{2,2^j+k+1}(x) - 2xC_{2,2^j+k+1}\Big)\Big|^2 dx \\
&= \int_0^1 \Big| \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{r=J+1}^{\infty} \sum_{s=0}^{2^r-1} a_{2^j+k+1} a_{2^r+s+1}\Big(P_{2,2^j+k+1}(x) - 2xC_{2,2^j+k+1}\Big)\times \\
&\qquad \Big(P_{2,2^r+s+1}(x) - 2xC_{2,2^r+s+1}\Big)\Big| dx \\
&\le \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{r=J+1}^{\infty} \sum_{s=0}^{2^r-1} |a_{2^j+k+1}||a_{2^r+s+1}| \Big| \int_0^1 \Big(P_{2,2^j+k+1}(x) - 2xC_{2,2^j+k+1}\Big)\times \\
&\qquad \Big(P_{2,2^r+s+1}(x) - 2xC_{2,2^r+s+1}\Big)\Big| dx \\
&\le \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{r=J+1}^{\infty} \sum_{s=0}^{2^r-1} |a_{2^j+k+1}||a_{2^r+s+1}| \Big| \int_0^1 \Big(P_{2,2^j+k+1}(x) P_{2,2^r+s+1}(x) \\
&\quad - 2xC_{2,2^j+k+1}P_{2,2^r+s+1}(x) - 2xC_{2,2^r+s+1}P_{2,2^j+k+1}(x) + 4x^2 C_{2,2^{j+k+1}}C_{2,2^r+s+1}\Big) dx\Big|
\end{aligned} \tag{4.14}$$

From lemma 4.2.1, we have

$$|a_{2^j+k+1}| \le \frac{A_1}{2^{j+1}}, \tag{4.15}$$

and

$$|a_{2^r+s+1}| \le \frac{A_2}{2^{r+1}}. \tag{4.16}$$

$$\Big| \int_0^1 P_{2,2^j+k+1}(x) P_{2,2^r+s+1}(x) dx \Big| \le \|P_{2,2^j+k+1}\|_{L^2} \|P_{2,2^r+s+1}\|_{L^2}$$

Since $P_{2,2^j+k+1}(x)$ is monotonically increasing in $[0, 1]$, we have

$$P_{2,2^j+k+1}(x) \le P_{2,2^j+k+1}(1) = \Big(\frac{1}{2^{j+1}}\Big)^2$$

Using the fact that our domain is of finite measure, we get

$$\|P_{2,2^j+k+1}\|_{L^2} \leq \left(\frac{1}{2^{j+1}}\right)^2$$

Hence,

$$\left|\int_0^1 P_{2,2^j+k+1}(x)P_{2,2^r+s+1}(x)dx\right| \leq \left(\frac{1}{2^{j+1}}\right)^2\left(\frac{1}{2^{r+1}}\right)^2 \tag{4.17}$$

Similarly

$$\left|\int_0^1 2xC_{2,2^j+k+1}P_{2,2^r+s+1}(x)dx\right| \leq \left(\frac{1}{2^{j+1}}\right)^2\left(\frac{1}{2^{r+1}}\right)^2 \tag{4.18}$$

$$\left|\int_0^1 2xC_{2,2^r+s+1}P_{2,2^j+k+1}(x)dx\right| \leq \left(\frac{1}{2^{j+1}}\right)^2\left(\frac{1}{2^{r+1}}\right)^2 \tag{4.19}$$

and

$$\left|\int_0^1 4x^2C_{2,2^j+k+1}C_{2,2^r+s+1}dx\right| \leq \frac{4}{3}\left(\frac{1}{2^{j+1}}\right)^2\left(\frac{1}{2^{r+1}}\right)^2 \tag{4.20}$$

Substituting the above estimates (4.15–4.20) in (4.14), we get

$$\begin{aligned}
\|E_J\|^2 &\leq \frac{13}{3}A_1A_2\sum_{j=J+1}^{\infty}\sum_{r=J+1}^{\infty}\left(\frac{1}{2^{j+1}}\right)^3\left(\frac{1}{2^{r+1}}\right)^3 2^j2^r \\
&\leq A\sum_{j=J+1}^{\infty}\sum_{r=J+1}^{\infty}\left(\frac{1}{2^{j+1}}\right)^2\left(\frac{1}{2^{r+1}}\right)^2, \quad \text{where} \quad (A = \frac{13}{12}A_1A_2) \\
&\leq A\left[\sum_{j=J+1}^{\infty}\left(\frac{1}{2^{j+1}}\right)\right]^4 \\
&\leq A\frac{1}{2^4}\left[\frac{1}{2^{J+1}}\sum_{j=0}^{\infty}\left(\frac{1}{2^j}\right)\right]^4 \\
&\leq A\left(\frac{1}{2^{J+1}}\right)^4 \quad (\text{since} \sum_{j=0}^{\infty}\left(\frac{1}{2^j}\right) = 2.)
\end{aligned}$$

Hence

$$\|E_J\|_{L_2} = \mathcal{O}\left[\left(\frac{1}{2^{J+1}}\right)^2\right]$$

For the fully discretized Haar wavelet method (Haar wavelet for spatial discretization and second order finite difference for temporal discretization), the L_2 error E is given by

$$\|E\|_{L_2} = \|u(x,t) - u_J(x,t)\|_{L_2}$$

Theorem 4.2.3. *Let* $\frac{\partial^2 u}{\partial t^2}$ *and* $\frac{\partial^2 u}{\partial x^2}$ *exist and bounded in* $[0,1]\times[0,T]$. *Then the error estimate for the fully discretized hybrid Haar wavelet collocation method is*

$$\|E\|_{L_2} = \mathcal{O}\left[\left(\frac{1}{2^{J+1}}\right)^2 + \Delta t^2\right].$$

Proof. From the above lemma

$$||E_J||_2 = \mathcal{O}\Big[\Big(\frac{1}{2^J+1}\Big)^2\Big]$$

As we have used second order finite difference method for the temporal discretization, the error estimate for the fully discretized numerical method is given by

$$\|E\|_{L_2} = \mathcal{O}\left[\left(\frac{1}{2^{J+1}}\right)^2 + \Delta t^2\right].$$

5. RESULTS OF NUMERICAL EXPERIMENTS

Following the hybrid Haar wavelet collocation method proposed in section 3, we solve the problem (1.1)– (1.5) on MATLAB. We present various numerical examples and compare it with few existing results. Our numerical results are better than the existing results.[9]

Example 1.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \Big(\frac{1}{4}+\pi^2\Big)e^{-\frac{t}{2}}\sin(\pi x), \quad 0 < x < 1, 0 < t \le T, \tag{5.1}$$

with initial conditions

$$u(x,0) = \sin(\pi x), \qquad 0 \le x \le 1, \tag{5.2}$$

$$\frac{\partial u}{\partial t}(x,0) = -\frac{1}{2}\sin(\pi x), \quad 0 \le x \le 1, \tag{5.3}$$

and Dirichlet boundary condition

$$u(0,t) = 0, \qquad 0 < t \le T, \tag{5.4}$$

with nonlocal condition

$$\int_0^1 u(x,t)dx = \frac{2}{\pi}e^{-\frac{t}{2}}, \qquad 0 < t \le T. \tag{5.5}$$

The exact solution of (5.1 – 5.5) is

$$u(x,t) = e^{-\frac{t}{2}}\sin(\pi x).$$

Figure 3 presents the exact and approximate solutions by the proposed method at different spatial and temporal points. Point wise absolute error at time $T = 1$ and max norm error at different time steps are reported in Table 1 and Table 2 respectively.

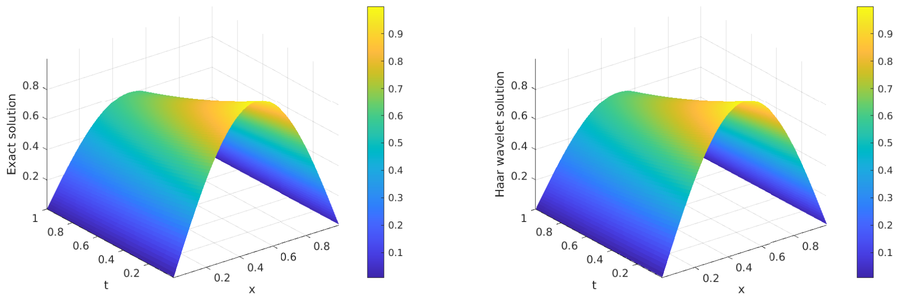

Figure 3: Exact and HHWCM based approximate solutions at $J = 6, T = 1$ and $\Delta t = 10^{-4}$.

x	Exact u	Absolute error
0.1	0.18742828	1.6×10^{-6}
0.2	0.35650978	2.6×10^{-6}
0.3	0.49069361	3.0×10^{-6}
0.4	0.57684494	3.0×10^{-6}
0.5	0.60653066	2.5×10^{-6}
0.6	0.57684494	1.5×10^{-6}
0.7	0.49069361	3.0×10^{-7}
0.8	0.35650978	2.9×10^{-6}
0.9	0.18742828	6.4×10^{-6}
1.0	0.00000000	8.7×10^{-6}

Table 1: Pointwise absolute error at $T = 1, \Delta t = 10^{-4}$ and J = 6.

T	HHWCM	TPS-RBF[9]	MQ-RBF[9]	CS-RBF[9]
0.5	3.4×10^{-5}	3.8×10^{-3}	1.3×10^{-3}	2.8×10^{-2}
1.0	1.0×10^{-5}	6.8×10^{-3}	2.4×10^{-3}	5.1×10^{-2}

Table 2: Comparison of maximum error using various numerical methods.

From the above results, we observe that a very good accuracy is achieved at very less J. It is also noticed that maximum absolute error decreases significantly with reducing Δt

size. Comparison table shows that the proposed method is better than various meshless method developed by Dehghan *et al.*[9] in terms of maximum error.

Example 2.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, 0 < t \leq T, \tag{5.6}$$

with initial conditions

$$u(x,0) = \cos(\pi x), \qquad 0 \leq x \leq 1, \tag{5.7}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 \leq x \leq 1, \tag{5.8}$$

and Dirichlet boundary condition

$$u(0,t) = \cos(\pi t), \qquad 0 < t \leq T, \tag{5.9}$$

with nonlocal condition

$$\int_0^1 u(x,t)dx = 0, \qquad 0 < t \leq T. \tag{5.10}$$

The exact solution of (5.6 – 5.10) is

$$u(x,t) = \frac{1}{2}[\cos\pi(x+t) + \cos\pi(x-t)].$$

Figure 4 presents the exact and approximate solutions by the proposed method at different spatial and temporal points. Point wise absolute error at time $T = 0.25$ and maximum absolute error at different time steps are reported in Table 3 and Table 4 respectively.

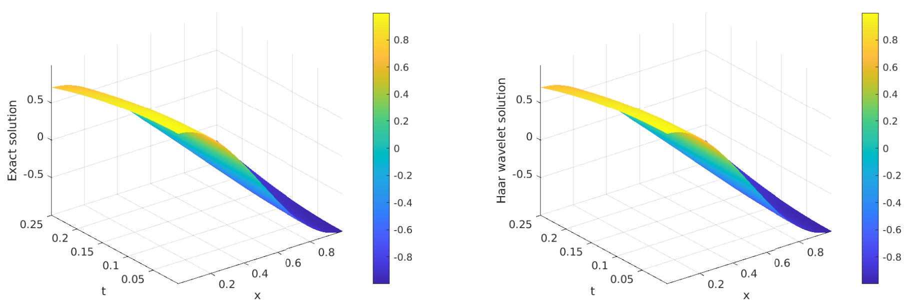

Figure 4: Exact and HHWCM based approximate solutions at $J = 6, T = 0.25$ and $\Delta t = 10^{-4}$.

x	Exact u	HHWCM error	TPS-RBF[9] error	MQ-RBF[99] error	CS-RBF[9] error	Optimal explicit[9] error
0.1	0.67249851	1.3×10^{-5}	1.5×10^{-5}	1.9×10^{-5}	1.6×10^{-4}	5.2×10^{-5}
0.2	0.57206140	1.4×10^{-5}	1.5×10^{-5}	2.2×10^{-5}	1.9×10^{-4}	5.1×10^{-5}
0.3	0.41562694	1.0×10^{-6}	2.2×10^{-6}	2.3×10^{-6}	1.8×10^{-5}	5.1×10^{-5}
0.4	0.21850801	5.5×10^{-6}	2.2×10^{-6}	6.7×10^{-7}	8.0×10^{-6}	5.3×10^{-5}
0.5	0.00000000	1.3×10^{-7}	1.3×10^{-8}	2.7×10^{-7}	8.7×10^{-10}	5.0×10^{-5}
0.6	-0.21850801	5.2×10^{-6}	1.3×10^{-6}	1.2×10^{-6}	8.0×10^{-6}	5.2×10^{-5}
0.7	-0.41562694	1.0×10^{-6}	2.2×10^{-6}	2.7×10^{-6}	1.8×10^{-5}	5.4×10^{-5}
0.8	-0.57206140	1.4×10^{-5}	1.5×10^{-5}	2.2×10^{-5}	1.9×10^{-4}	5.3×10^{-5}
0.9	-0.67249851	1.3×10^{-5}	1.5×10^{-5}	1.7×10^{-5}	1.6×10^{-4}	5.5×10^{-5}
1.0	-0.70710678	2.4×10^{-7}	4.3×10^{-9}	2.1×10^{-6}	2.8×10^{-9}	5.4×10^{-5}

Table 3: Point wise absolute error at $T = 0.25$, $\Delta t = 10^{-4}$ and J = 6.

Δt	HHWCM	TPS-RBF[9]	MQ-RBF[9]	CS-RBF[9]
10^{-3}	1.9×10^{-5}	6.8×10^{-5}	7.3×10^{-5}	2.3×10^{-4}
10^{-4}	1.4×10^{-5}	1.5×10^{-5}	2.2×10^{-5}	1.8×10^{-4}

Table 4: Comparison of maximum error using various numerical methods at different time steps.

6. CONCLUSION

In this article, we have developed a hybrid Haar wavelet collocation method for the numerical solution of nonlocal hyperbolic partial differential equations. Instead of reformulating the original problem into periodic problem, we dealt with the integral boundary condition directly using the given data which is more accurate. For the spatial discretization, Haar wavelets are used whereas second order finite difference is used for temporal discretization. Stability analysis based on eigenvalue properties is carried out. We have derived error estimate for the proposed method. Finally, numerical results are presented and it is shown that our method is better than few existing method. This method can easily be generalized to higher dimensional problems.

DECLARATION

The authors declare that they have no conflict of interest. It is also declared that there is no associated data in this manuscript. We thank the reviewers for their thoughtful comment.

REFERENCES

[1] W. T. Ang, A numerical method for the wave equation subject to a non local conservation condition, *Appl. Numer. Math.*, 56:1054–1060, 2006.

[2] S. A. Beilin, Existence of solutions for one-dimensional wave equations with nonlocal conditions, *Electron. J. Diff. Eq.*, 76:1–8, 2001.

[3] A. Bouziani, Initial boundary value problem with a nonlocal condition for a viscosity equation, *Int. J. Math. Math. Sci.*, 30:327–338, 2002.

[4] A. Bouziani, Mixed problem for certain nonclassical equations containing a small parameter, *Acad. Roy. Belg.*, 6:389–400, 1994.

[5] C. Cattani, Harmonic wavelets toward the solution of nonlinear PDE, *Comput. Math. Appl.*, 50:1191–1210, 2005.

[6] C. Chen and C.H. Hsiao, Haar wavelet method for solving lumped and distributed parameter systems, *IEE Proc. Control Theory Appl.*, 144:87–94, 1997.

[7] M. Dehghan and M. Lakestani, The use of cubic B-spline scaling functions for solving the one-dimensional hyperbolic equation with a nonlocal conservation condition, *Numer. Methods Partial Differ. Equ.*, 23:1277–1289, 2007.

[8] M. Dehghan and A. Saadatmandi, Variational iteration method for solving the wave equation subject to an integral conservation condition, *Chaos, Solitons and Fractals*, 41:1448–1453, 2009.

[9] M. Dehghan and A. Shokri, A meshless method for numerical solution of the one-dimensional wave equation with an integral condition using radial basis functions, *Numer. Algor.*, 52:461–477, 2009.

[10] I. S. Gordeziani and G. A. Avalishvili, On the constructing of solutions of the nonlocal initial boundary problems for one-dimensional medium oscillation equations, *Matem. Modelirovani*, 12:94–103, 2000.

[11] N. I. Kavalloris and D. S. Tzanetis, Behaviour of critical solutions of a nonlocal hyperbolic problem in ohmic heating of foods, *Appl. Math. E-Notes.*, 2:59–65, 2002.

[12] G. Priyadarshi and B.V. Rathish Kumar, Reconstruction of the parameter in parabolic partial differential equations using Haar wavelet method, *Engineering Computations*, 38(5), 2020.

[13] G. Priyadarshi and B. V. Rathish Kumar. Parameter identification in multidimensional hyperbolic partial differential equations using wavelet collocation method, *Mathematical Methods in the Applied Sciences* 44: 9079–9095, 2021.

[14] G. Priyadarshi and A. Halim. A hybrid Haar wavelet collocation method for nonlocal hyperbolic partial differential equations. arXiv preprint arXiv:2211.07249. 2022

[15] L. S. Pulkina, A nonlocal problem with integral conditions for hyperbolic equations, *Electron. J. Diff. Eq.*, 45:1–6, 1999.

[16] M. Renardy, W. Hrusa and J. A. Nohel, Mathematical problems in viscoelasticity, *Longman Science and Technology,* England, 1987.

[17] A. Saadatmandi and M. Dehghan, Numerical solution of the one dimensional wave equation with an integral condition, *Numer. Methods Partial Differ. Equ.*, 23:282–292, 2007.

[18] S. A. Yousefi, Z. Barikbin and M. Dehghan, Bernstein Ritz-Galerkin method for solving an initial-boundary value problem that combines Neumann and integral condition for the wave equation, *Numer. Methods Partial Differ. Equ.*, 26:1236–1246, 2010.