

A Variant of the Functional Equation of the Riemann Zeta Function

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Abstract

A variant of the functional equation of the Riemann zeta function is derived. This variant maps the zeta function to another logarithmic spiral centered on (0,0) using an incomplete $\Pi(s)$ function.

Keywords : Riemann zeta function, logarithmic spirals, functional equations

1. INTRODUCTION

Equation (4) in section 1.6 of Edwards' [1] book is

$$\zeta(s) = \Pi(-s)(2\pi)^{s-1}2 \sin(s\pi/2)\zeta(1-s) \quad (1)$$

This relationship between $\zeta(s)$ and $\zeta(1-s)$ is known as the functional equation of the zeta function. Equation (5) in section 1.6 of Edwards's book is

$$\Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s) = \Pi\left(\frac{1-s}{2} - 1\right)\pi^{-(1-s)/2}\zeta(1-s). \quad (2)$$

This is considered to be the natural statement of the functional equation.

The function $\Pi(s/2) - 1)\pi^{-s/2}\zeta(s)$ which occurs in the above equation has poles at $s = 0$ and $s = 1$. Riemann multiplies it by $s(s-1)/2$ and defines

$$\xi(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s). \quad (3)$$

(Equation (1) in section 1.8 of Edwards' book.) Then $\zeta(s)$ is an analytic function of s which is defined for all values of s and the functional equation of the zeta function is equivalent to $\zeta(s) = \zeta(1 - s)$.

Equation (4) in section 1.5 of Edwards' book is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (4)$$

This is Dirichlet's function for real values of s greater than 1.

2. A VARIANT OF $\zeta(s)$

Equation (3) in section 1.3 of Edwards' book is

$$\Pi(s) = \lim_{N \rightarrow \infty} \frac{1 \cdot 2 \cdots N}{(s+1)(s+2) \cdots (s+N)} (N+1)^s \quad (5)$$

This equation is valid for all s in the halfplane $\text{Re } s > -1$. Let $\Pi_1(s)$ denote

$$\lim_{N \rightarrow \infty} \frac{1 \cdot 2 \cdots N}{(\Re s + 1)(\Re s + 2) \cdots (\Re s + N)} (N+1)^s \quad (6)$$

Let $\zeta_1(s, N)$ denote $\Pi_1(s/2, N)(s-1)\pi^{-s/2}\zeta(s, N)$. A plot of $\zeta_1(s, N)$ for the first non-trivial zeta function zero ($s = (0.5, 14.13472514173470\dots)$) and $N \leq 2000$ is

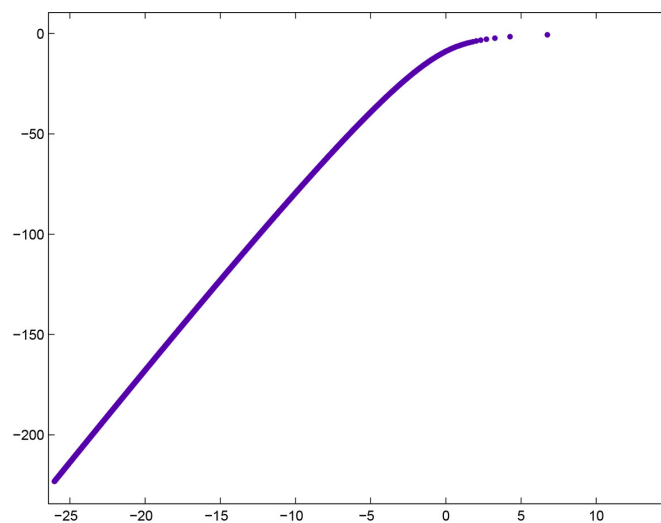


Figure 1

This curve is a line (not necessarily straight) if and only if s is a zeta function zero. A plot of the curve when $s = (0.49, 14.13472514173470\dots)$ is

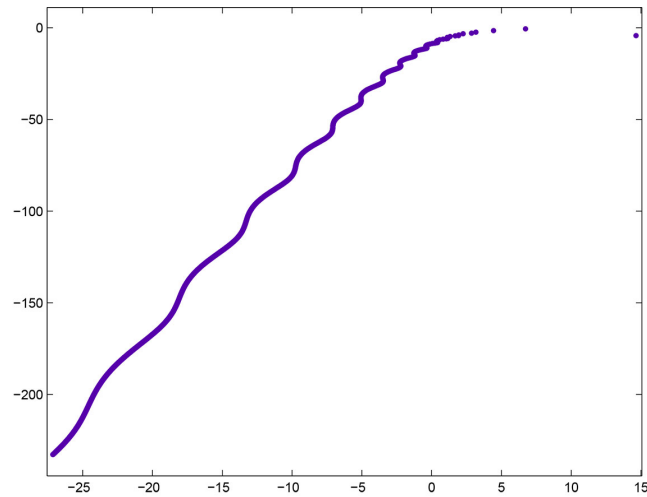


Figure 2

A plot of the curve when $s = (0.50, 14.0)$ is

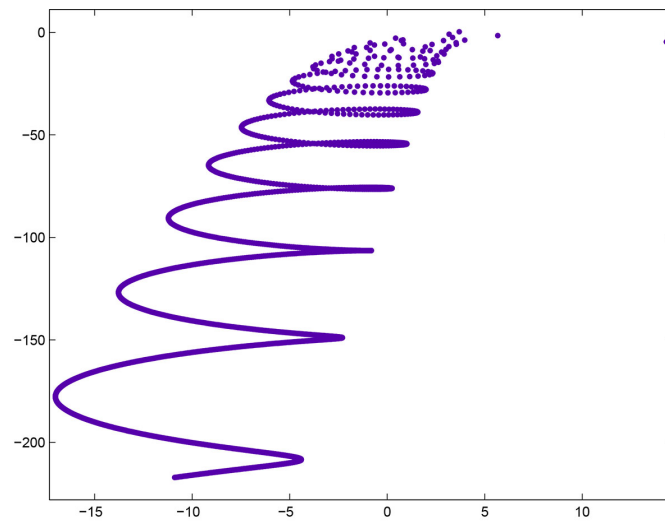


Figure 3

Further progress can be made by using an incomplete $\Pi_1(s)$ function. An example of the utility of an incomplete gamma function is given in the next section.

3. AN INCOMPLETE GAMMA FUNCTION

The incomplete gamma function is $\Gamma(s + 1, x) = s\Gamma(s, x) + x^s e^{-x}$. $\Pi(s) = \Gamma(s + 1) = s\Gamma(s) = \int_0^\infty e^{-t} t^s dt$. Let $G(s, x)$ denote $s\Pi_1(s, x) + x^s e^{-x}$ where $s - 1$ is substituted for s (to convert from $\Gamma(s)$ to $\Pi_1(s)$). C code for computing $G(x, s)$ is given in the Methods section. A plot of the modulus of $G(s, x)$ for the twentieth non-trivial zeta function zero ((0.5, 77.14484006887480...)) versus \sqrt{x} for $x \leq 2000$ is

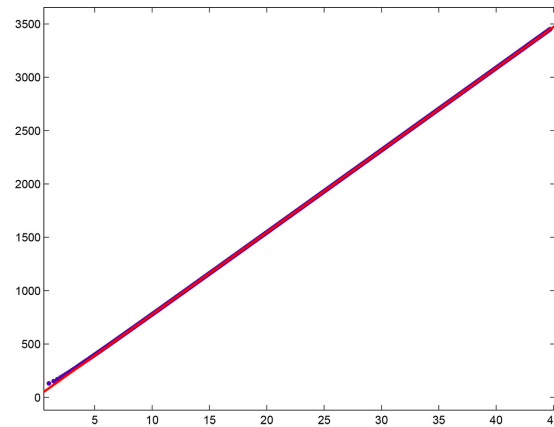


Figure 4

For a linear least-squares fit of the curve, $p_1 = 76.96$ with a 95% confidence interval of (76.95, 76.97), $p_2 = 8.619$ with a 95% confidence interval of (8.334, 8.903), SSE=9321, R-squared=1, and RMSE=2.16. A plot of the imaginary part versus the real part is.

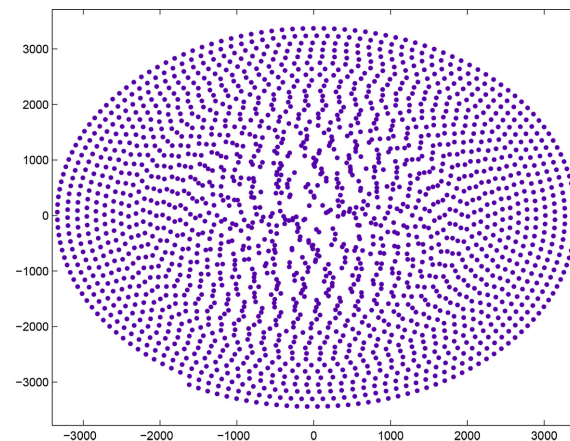


Figure 5

For the first twenty zeta function zeros, the slopes are 14.11, 20.98, 24.95, 30.35, 32.86, 37.5, 40.82, 43.22, 47.89, 49.65, 52.84, 56.31, 59.2, 60.69, 64.96, 66.92, 69.38, 71.89, 75.52, and 76.96 respectively. The y -intercepts are 1.579, 2.351, 2.795, 3.399, 3.679,

4.201, 4.57, 4.841, 5.369, 5.561, 5.92, 6.309, 6.63, 6.796, 7.276, 7.495, 7.769, 8.052, 8.458, and 8.616 respectively. The SSE values are 309.4, 698.9, 982.1, 1449, 1689, 2231, 2609, 2940, 3621, 3876, 4417, 5028, 5508, 5795, 6669, 7062, 7564, 8158, 8989, and 9321 respectively A plot of imaginary part of the zeta function zeros versus the slopes is

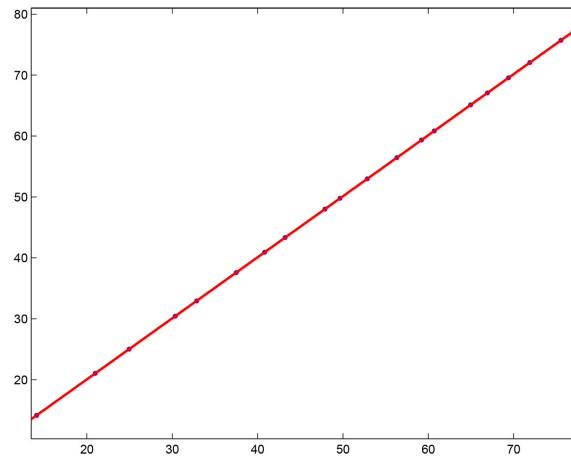


Figure 6

For a linear least-squares fit of the curve, $p_1 = 1.003$ with a 95% confidence interval of (1.002, 1.003), $p_2 = -0.005615$ with a 95% confidence interval of (-0.01057, -0.0006633), SSE=0.002381, R-squared=1, and RMSE=0.003637.

A plot of the imaginary part of the zeta function zeros versus the y -intercepts is

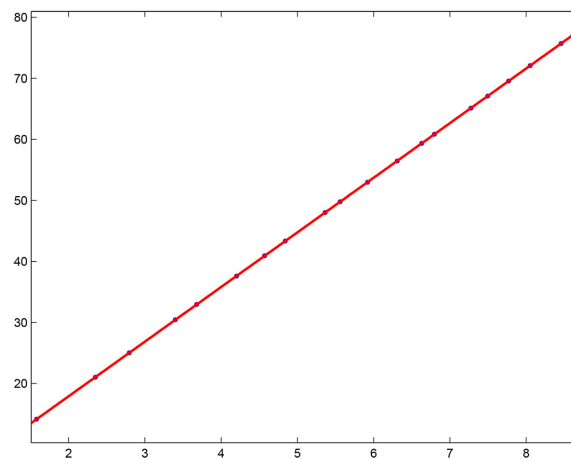


Figure 7

For a linear least-squares fit of the curve, $p_1 = 8.952$ with a 95% confidence interval of (8.949, 8.955), $p_2 = -0.01034$ with a 95% confidence interval of (-0.02654, 0.00586), SSE=0.002548, R-squared=1, and RMSE=0.0119.

A plot of the imaginary part of the zeta function zeros versus the square-root of the SSEs is

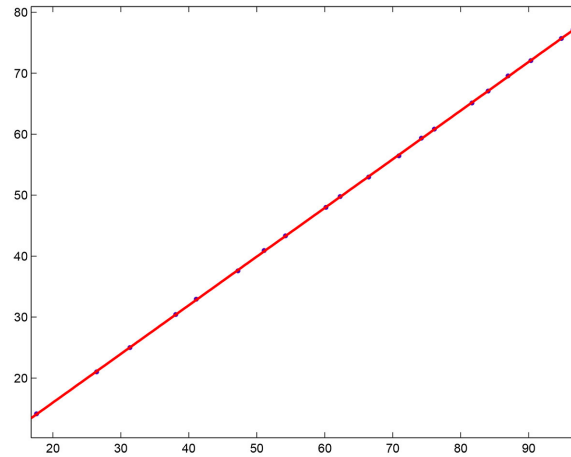


Figure 8

For a linear least-squares fit of the curve, $p_1 = 0.7982$ with a 95% confidence interval of (0.7964, 0.8), $p_2 = 0.02055$ with a 95% confidence interval of (-0.102, 0.1431), SSE=0.1459, R-squared=1, and RMSE=0.09002.

A plot of the modulus of $G(s, x)$ for $s = (0.45, 14.13472514173470\dots)$ (the imaginary part equal to that of the first zeta function zero) versus \sqrt{x} for $x \leq 2000$ is

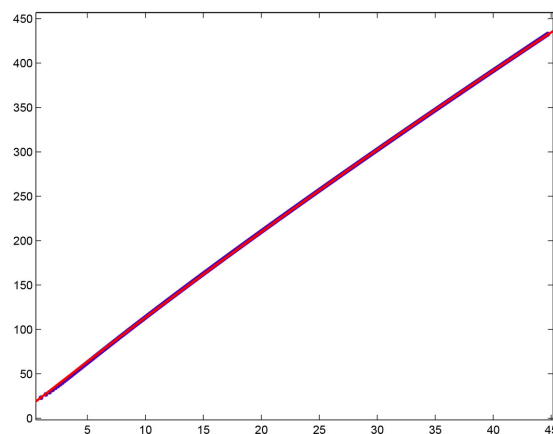


Figure 9

For a quadratic least-squares fit of the curve, $p_1 = -0.01883$ with a 95% confidence interval of (-0.01896, -0.0187), $p_2 = 10.21$ with a 95% confidence interval of (10.2, 10.22), $p_3 = 13.17$ with a 95% confidence interval of (13.08, 13.26), SSE=228.1, R-squared=1, and RMSE=0.3379.

A plot of the modulus of $G(s, x)$ for $s = (0.55, 14.13472514173470\dots)$ versus \sqrt{x} for $x \leq 2000$ is

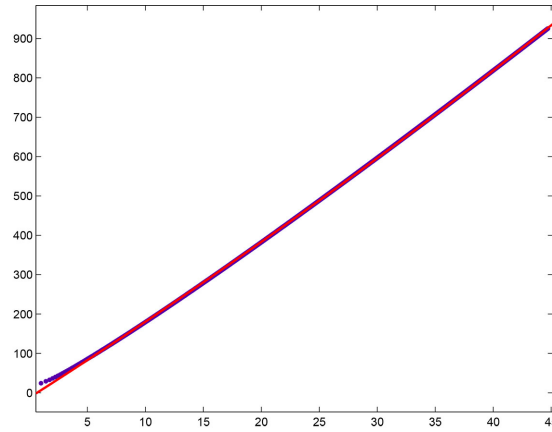


Figure 10

For a quadratic least-squares fit of the curve, $p_1 = 0.04942$ with a 95% confidence interval of $(0.04895, 0.04989)$, $p_2 = 18.79$ with a 95% confidence interval of $(18.77, 18.82)$, $p_3 = -12.28$ with a 95% confidence interval of $(-12.6, -11.95)$, $SSE=3003$, $R\text{-squared}=1$, and $RMSE=1.226$.

The curve appears to be linear only for a real part of $1/2$.

4. $\zeta_2(s)$ USING AN INCOMPLETE $\Pi_1(s)$ FUNCTION

Let $\zeta_2(s, N)$ denote $(\Pi_1(s/2, N) + N^s e^{-N})(s - 1)\pi^{-s/2}\zeta(s, N)$. A plot of $\zeta_2(s, N)$ for the first zeta function zero and $N \leq 2000$ is

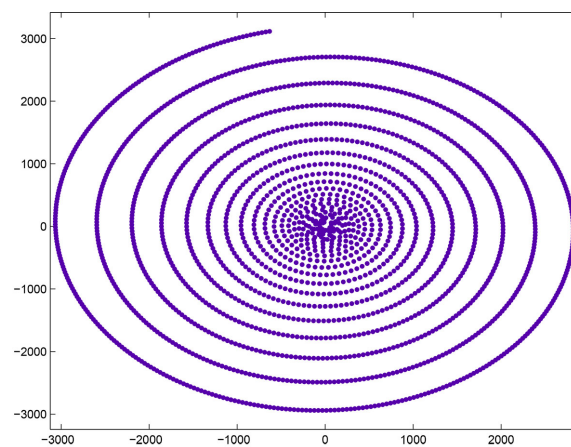


Figure 11

The function maps $\zeta(s, N)$ to another logarithmic spiral centered on $(0,0)$.

A plot of the real part of $\xi_2(s, N)$ for the third zeta function zero ($s = (0.5, 25.01085758014569\dots)$) and $N \leq 2000$ is

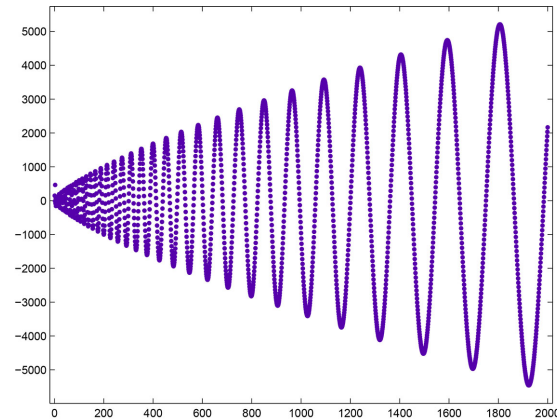


Figure 12

A plot of the logarithm of the N values where the absolute values of the amplitudes crest is

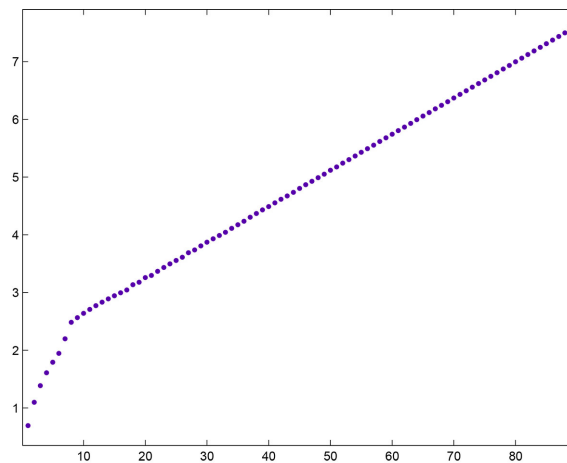


Figure 13

There are 89 points. A plot of the curve excluding the first 17 points is

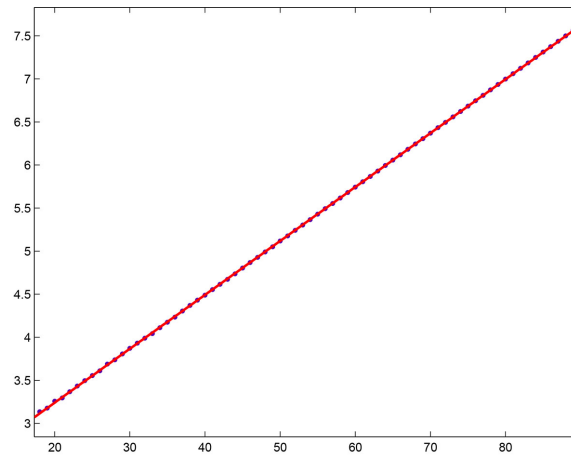


Figure 14

For a linear least-squares fit of the curve, $p_1 = 0.06258$ with a 95% confidence interval of (0.06253, 0.06264), $p_2 = 1.989$ with a 95% confidence interval of (1.986, 1.991), SSE=0.001585, R-squared=1, and RMSE=0.004758.

A plot of the logarithm of the absolute value of the amplitudes at these points is

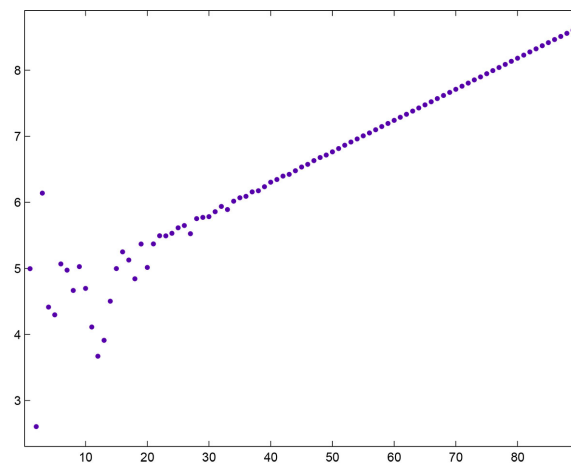


Figure 15

A plot of the curve excluding the first 39 points is

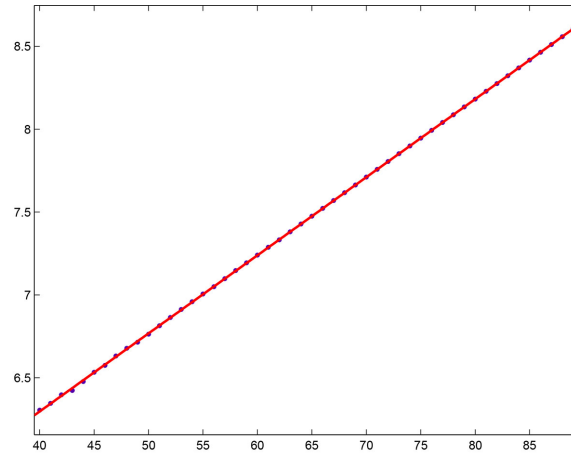


Figure 16

For a linear least-squares fit of the curve, $p_1 = 0.04712$ with a 95% confidence interval of (0.0476, 0.04719), $p_2 = 4.412$ with a 95% confidence interval of (4.407, 4.416), $SSE=0.0005422$, $R\text{-squared}=1$, and $RMSE=0.003364$. The slope is about 3/4 of that of the curve of the logarithm of the N values.

A plot of the real part of $\zeta(s, N)$ for the third zeta function zero and $N \leq 2000$ is

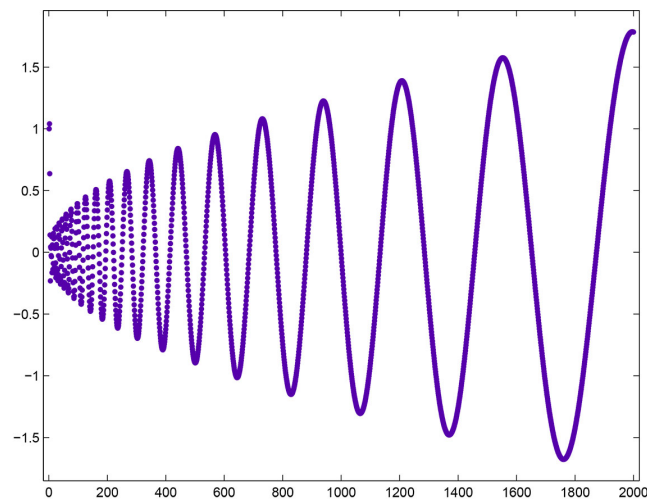


Figure 17

A plot of the logarithm of the N values where the absolute values of the amplitudes crest is

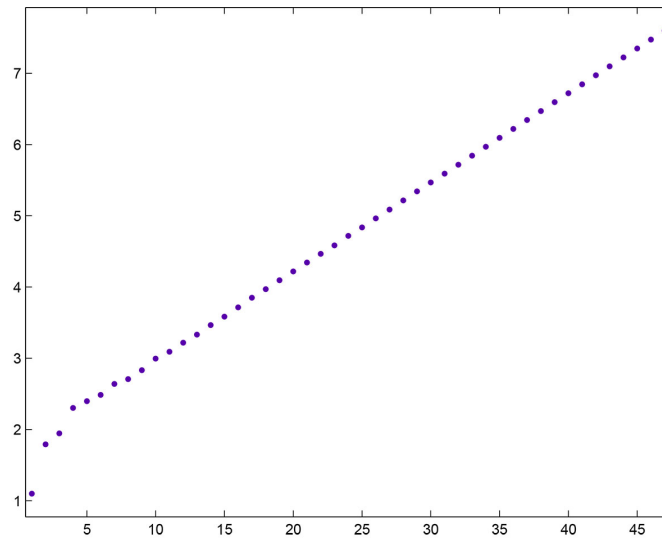


Figure 18

There are 47 points. A plot of the curve excluding the first 10 points is

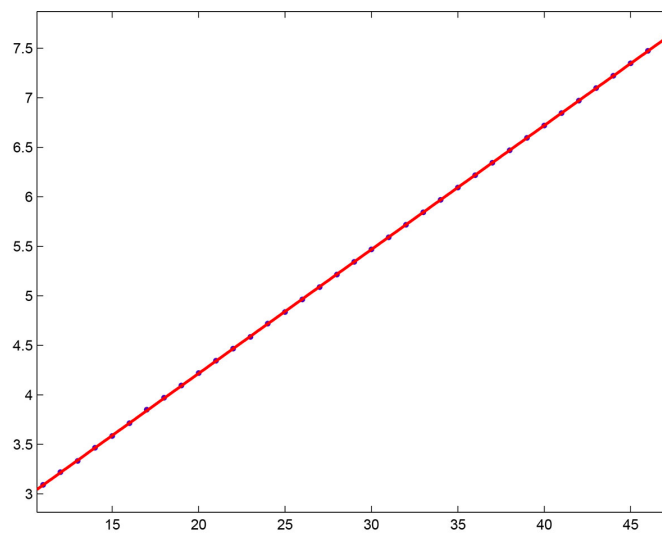


Figure 19

For a linear least-squares fit of the curve, $p_1 = 0.1252$ with a 95% confidence interval of (0.1251, 0.1253), $p_2 = 1.711$ with a 95% confidence interval of (1.708, 1.714), $SSE=0.0004387$, $R\text{-squared}=1$, and $RMSE=0.00354$. The slope is about twice that of the corresponding curve for $\zeta_2(s, N)$.

A plot of the logarithm of the absolute value of the amplitudes at these points is

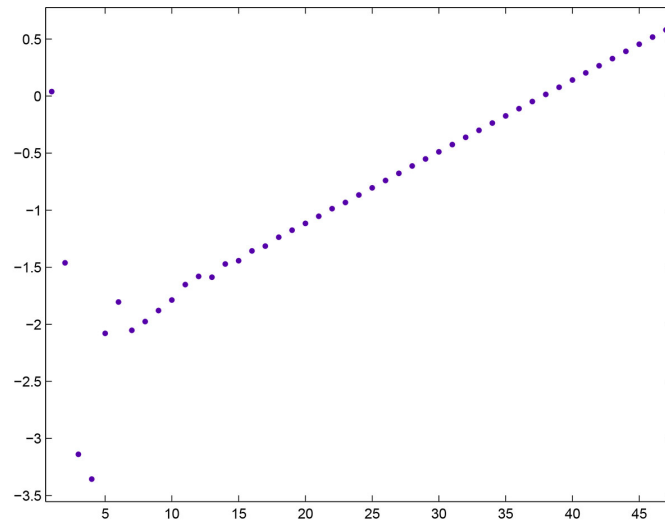


Figure 20

A plot of the curve excluding the first 17 points is

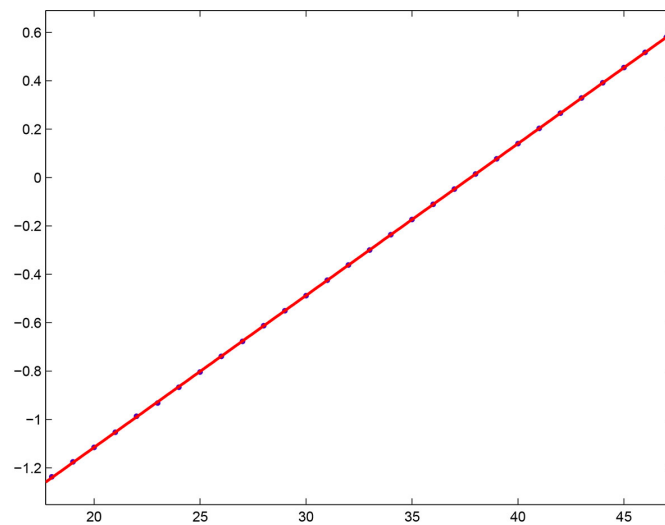


Figure 21

For a linear least-squares fit of the curve, $p_1 = 0.06278$ with a 95% confidence interval of $(0.06271, 0.06284)$, $p_2 = -2.371$ with a 95% confidence interval of $(-2.373, -2.369)$, $SSE = 7.191 \cdot 10^{-5}$, $R\text{-squared} = 1$, and $RMSE = 0.001603$. The slope is about half that of the curve of the logarithm of the N values.

A plot of the real parts of $\zeta_2(s, N)$ for $s = (0.40, 25.01085758014569\dots)$ and $s = (0.60, 25.01085758014569\dots)$ and $N \leq 2000$ is

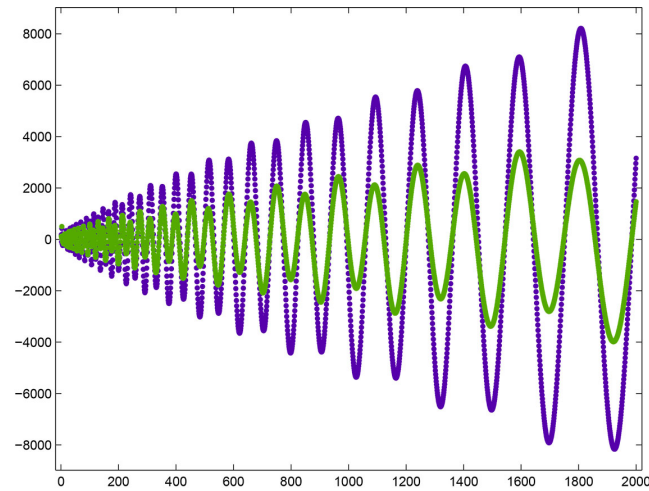


Figure 22

Except for the amplitudes, there is a functional relationship.

5. METHODS

```

#include <math.h>
#include <stdio.h>
// xi function
unsigned int max=2000;
double s=0.50;
double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
//double t=52.97032147771446;
//double t=56.44624769706339;
//double t=59.34704400260235;
//double t=60.83177852460981;
//double t=65.11254404808160;

```

```

//double t=67.07981052949417;
//double t=69.54640171117399;
//double t=72.06715767448191;
//double t=75.70469069908393;
//double t=77.14484006887480;
unsigned int out=1; // 2 for top, 3 for left, 4 for bottom, 5 for right
unsigned int scale=0; // if set, multiply by s(s-1)/2
unsigned int mod=1; // set for incomplete
double pi=3.14159265;
void main() {
unsigned int temp,x;
double temp1,temps,tempt,prods,a,b,c,d,e,f,olds,oldt,sums,sumt;
double g,h,j,k,temp2,oldolds,oldoldt;
FILE *Outfp;
Outfp = fopen("transe4.dat","w");
if (s>=0.0)
    temp1=pow(pi,s/2.0);
else {
    temp1=pow(pi,-s/2.0);
    temp1=1.0/temp1;
}
temps=temp1*(cos(t*log(pi/2.0)));
tempt=temp1*(sin(t*log(pi/2.0)));
a=temps*temps+tempt*tempt;
e=temps/a;
f=-tempt/a;
oldolds=0.0;
olds=0.0;
oldoldt=0.0;
oldt=0.0;
prods=1.0;
sums=0.0;
sumt=0.0;
for (x=1; x<=max; x++) {
    temp=x;
    prods=prods*(double)temp/((double)temp+s);
    if (s>=0.0)
        temp1=pow((double)(x+1),s/2.0);

```

```

else {
    temp1=pow((double)(x+1),-s/2.0);
    temp1=1.0/temp1;
}
temps=temp1*(cos(t*log(x+1)));
tempt=temp1*(sin(t*log(x+1)));
a=prods*temps-tempt;
b=prods*tempt+temps;
if (mod!=0) {
    j=a*s+b*t;
    k=a*t-b*s;
    if (s>=0.0)
        temp1=pow((double)x,s);
    else {
        temp1=pow((double)x,-s);
        temp1=1.0/temp1;
    }
    g=temp1*(cos(t*log(x)));
    h=temp1*(sin(t*log(x)));
    temp2=1.0/exp((double)x);
    temps=g*temp2;
    tempt=h*temp2;
    a=temps+j;
    b=tempt+k;
}
if (s>=0.0)
    temp1=pow((double)x,s);
else {
    temp1=pow((double)x,-s);
    temp1=1.0/temp1;
}
temps=temp1*(cos(t*log(x)));
tempt=temp1*(sin(t*log(x)));
temp1=temps*temps+tempt*tempt;
c=temps/temp1;
d=tempt/temp1;
sums=sums+c;
sumt=sumt-d;

```

```

temps=a*sums-b*sumt;
tempt=a*sumt+b*sums;
c=temps*e-tempt*f;
d=temps*f+tempt*e;
temps=c*(s-1.0)-d*t;
tempt=c*t+d*(s-1.0);
if (scale!=0) {
    a=s*(s-1.0)/2.0;
    temps=temps*a;
    tempt=tempt*a;
}
if (out==1)
    fprintf(Outfp," %.10lf %.10lf \n",temps,tempt);
if ((out==2)&&((olds>0.0)&&(temps<0.0)))
    fprintf(Outfp," %d %.10lf %.10lf \n",x,temps,tempt);
if ((out==3)&&((oldt>0.0)&&(tempt<0.0)))
    fprintf(Outfp," %d %.10lf %.10lf \n",x,temps,tempt);
if ((out==4)&&((olds<0.0)&&(temps>0.0)))
    fprintf(Outfp," %d %.10lf %.10lf \n",x,temps,tempt);
if ((out==5)&&((oldt<0.0)&&(tempt>0.0)))
    fprintf(Outfp," %d %.10lf %.10lf \n",x,temps,tempt);
if ((out==6)&&(oldolds<olds)&&(olds>temps)&&(olds>0.0))
    fprintf(Outfp," %.16lf, %d \n",olds,x);
if ((out==6)&&((oldolds>olds)&&(olds<temps)&&(olds<0.0)))
    fprintf(Outfp," %.16lf, %d \n",-olds,x);
if ((out==7)&&((oldoldt<oldt)&&(oldt>tempt)))
    fprintf(Outfp," %.16lf, %d \n",oldt,x);
if ((out==7)&&((oldoldt>oldt)&&(oldt<tempt)&&(oldt<0.0)))
    fprintf(Outfp," %.16lf, %d \n",-oldt,x);
oldolds=olds;
olds=temps;
oldoldt=oldt;
oldt=tempt;
}
fclose(Outfp);
return;
}

```

```
#include <math.h>
#include <stdio.h>
// Gamma function
unsigned int max=2000;
double s=0.55;
double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
//double t=52.97032147771446;
//double t=56.44624769706339;
//double t=59.34704400260235;
//double t=60.83177852460981;
//double t=65.11254404808160;
//double t=67.07981052949417;
//double t=69.54640171117399;
//double t=72.06715767448191;
//double t=75.70469069908393;
//double t=77.14484006887480;
unsigned int out=1; // 2 for top, 3 for left, 4 for bottom, 5 for right
unsigned int mod=1;
void main() {
unsigned int x;
double temp1,temps,tempt,prods,a,b,olds,oldt,temp2;
double g,h,j,k;
FILE *Outfp;
Outfp = fopen("gamma.dat","w");
prods=1.0;
for (x=1; x<=max; x++) {
    prods=prods*(double)x/((double)x+s);
    if (s>=0.0)
        temp1=pow((double)(x+1),s);
```

```

else {
    temp1=pow((double)(x+1),-s);
    temp1=1.0/temp1;
}
temps=temp1*(cos(t*log(x+1)));
tempt=temp1*(sin(t*log(x+1)));
a=prods*temps-tempt;
b=prods*tempt+temps;
j=a*s+b*t;
k=a*t-b*s;
if (s>=0.0)
    temp1=pow((double)x,s);
else {
    temp1=pow((double)x,-s);
    temp1=1.0/temp1;
}
g=temp1*(cos(t*log(x)));
h=temp1*(sin(t*log(x)));
temp2=1.0/exp((double)x);
temps=g*temp2;
tempt=h*temp2;
temps=temps+j;
tempt=tempt+k;
if (mod!=0) {
    temp2=sqrt(temps*temps+tempt*tempt);
    fprintf(Outfp, "%.16lf \n",temp2);
}
if ((out==1)&&(mod==0))
    fprintf(Outfp, " %.10lf %.10lf \n",temps,tempt);
if ((out==2)&&((olds>0.0)&&(temps<0.0)))
    fprintf(Outfp, " %d %.10lf %.10lf \n",x,temps,tempt);
if ((out==3)&&((oldt>0.0)&&(tempt<0.0)))
    fprintf(Outfp, " %d %.10lf %.10lf \n",x,temps,tempt);
if ((out==4)&&((olds<0.0)&&(temps>0.0)))
    fprintf(Outfp, " %d %.10lf %.10lf \n",x,temps,tempt);
if ((out==5)&&((oldt<0.0)&&(tempt>0.0)))
    fprintf(Outfp, " %d %.10lf %.10lf \n",x,temps,tempt);
olds=temps;

```

```
    oldt=tempt;
  }
fclose(Outfp);
return;
}
```

REFERENCES

- [1] H. M. Edwards, *Riemann's Zeta Function*, Dover, (1974)