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Reverse Derivation on Hyperrings

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Abstract

In this paper, we study reverse derivations on Krasner hyperrings and investigate their structural consequences. Several commutativity results are established for prime and 2-torsion-free hyperrings admitting non-zero reverse derivations under suitable algebraic conditions. We also obtain centrality and vanishing results for elements and idempotent reverse derivations.

Keywords: Hypergroupoid, Krasner hyperrings, Reverse derivation.

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1. INTRODUCTION

Hyperstructure theory provides a natural generalization of classical algebraic systems by allowing the composition of two elements to be a set rather than a single element. This idea was first proposed by F. Marty [11] in 1934 and has since developed into an active research area with applications across several branches of mathematics. J. Mittas [12] investigated canonical hypergroups, hyperrings, and hyperlattices, establishing several fundamental results in the theory of hyperstructures.

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The notion of a hyperring was introduced by M. Krasner [9], who defined a special class now known as Krasner hyperrings, where addition is a hyperoperation while multiplication remains a binary operation. This structure preserves many essential ring-like properties while allowing greater algebraic flexibility. Over the years, Krasner hyperrings and their variants have been extensively studied, and their algebraic properties, ideals, and homomorphic images have been explored by several researchers. B. Davvaz et al. [5] investigated hyperring theory and its applications.

Derivations form a powerful tool in understanding the internal structure of algebraic systems. In classical ring theory, derivations were introduced by E. C. Posner [15] and have been widely used to investigate commutativity, primeness, and structural constraints of rings. Bell and Kappe [3] studied rings in which derivations satisfy certain algebraic conditions. Motivated by these developments, derivations were later extended to hyperrings, notably by A. Asokkumar [1], who initiated a systematic study of derivations in Krasner hyperrings. Nikhil D. Sonone et al. [13, 14] studied semi-derivations on prime hyperrings and introduced generalized derivations on hyperrings, establishing vanishing and commutativity conditions under suitable algebraic assumptions. Utsanee Leerawat et al. [10] investigated (f, g)-semi-derivations on hyperrings and analyzed their influence on the structure and commutativity of hyperrings.

The concept of reverse derivation in ring theory was first introduced by I. N. Herstein [6] and later formalized by M. Bresar and J. Vukman [4]. C. Jaya Subba Reddy et al. [7] have studied reverse derivations on prime rings. More recently, results related to the commutativity of hyperrings under reverse derivations were obtained in [2].

The present paper aims to fill this gap by introducing and investigating reverse derivations on Krasner hyperrings. We establish fundamental properties of reverse derivations and analyze their interaction with the prime and torsion-free conditions of hyperrings. Furthermore, several commutativity theorems are proved.

2. PRELIMINARIES

Let us recall some definitions and concepts of hyperstructures which are used in the sequel. For details, we refer to Kamali Ardekani. L and Davvaz. B [8] and Asokkumar [1]. For a set H, let $\mathcal{P}(H)$ denote the power set of H, and let $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$.

Definition 2.1.[8] A mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a hyperoperation. An algebraic system $(H, \circ)$ is called a hypergroupoid. Let $(H, \circ)$ be a hypergroupoid. For any subsets $M, N \subseteq H$ and $h \in H$, we define

$$M \circ N = \bigcup_{m \in M,\, n \in N} (m \circ n), \quad M \circ h = M \circ \{h\}, \quad h \circ N = \{h\} \circ N.$$

A hypergroupoid $(H, \circ)$ is said to be commutative if $m \circ n = n \circ m$ for all $m, n \in H$.

Definition 2.2.[8] A hypergroupoid $(H, \circ)$ is called a semihypergroup, if it satisfies the associativity condition

$$(a \circ b) \circ c = a \circ (b \circ c), \quad \forall\, a, b, c \in H,$$

that is,

$$\bigcup_{u \in a \circ b} (u \circ c) = \bigcup_{v \in b \circ c} (a \circ v).$$

A hypergroupoid $(H, \circ)$ is called a quasihypergroup, if $a \circ H = H \circ a = H$ for all $a \in H$. This property is also known as the reproduction axiom.

Definition 2.3.[8] A Krasner hyperring is an algebraic structure $(R, +, \cdot)$ satisfying:

1. $(R, +)$ is a canonical hypergroup:
   (a) $(R, +)$ is a semihypergroup;
   (b) $p + q = q + p$ for all $p, q \in R$;
   (c) there exists $0 \in R$ such that $0 + p = \{p\}$ for all $p \in R$;
   (d) for each $p \in R$, there exists a unique $-p \in R$ such that $0 \in p + (-p)$;
   (e) if $r \in p + q$, then $q \in -p + r$ and $p \in r - q$.
2. $(R, \cdot)$ is a semigroup with zero as a bilateral absorbing element, i.e., $p \cdot 0 = 0 \cdot p = 0$ for all $p \in R$;
3. multiplication is distributive over the hyperoperation $+$.

Throughout this paper, the term hyperring refers to a Krasner hyperring.

Example 1.[2] Let $R = \{0, p\}$. Define the hyperoperations as follows:

$\oplus$	0	p
0	$\{0\}$	$\{p\}$
p	$\{p\}$	R

$\otimes$	0	p
0	$\{0\}$	$\{0\}$
p	$\{0\}$	$\{p\}$

Table 1

Then $(R, \oplus, \otimes)$ is a hyperring.

Definition 2.4.[8] A hyperring R is said to be commutative if $(R, \cdot)$ is a commutative semigroup.

Definition 2.5.[8] The center of a hyperring R is

$$Z(R) = \{a \in R \mid ab = ba \text{ for all } b \in R\}.$$

Definition 2.6.[8] A nonempty subset I of R is called a hyperideal if:

1. $p, q \in I$ implies $p - q \subseteq I$;
2. $p \in I$ and $r \in R$ imply $rp \in I$ or $pr \in I$.

Definition 2.7.[8] A hyperring R is called prime if $aRb = 0$ implies $a = 0$ or $b = 0$. It is called semiprime if $aRa = 0$ implies $a = 0$.

Example 2 [13] Let $R = \{0, 1, 2\}$. Consider the hyperoperations $\oplus$ and $\otimes$ on R defined by the following tables:

$\oplus$	0	1	2
0	0	1	2
1	1	1	R
2	2	R	2

$\otimes$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Table 2

It is verified that $(R, \oplus, \otimes)$ satisfies all the axioms of a hyperring and hence forms a prime hyperring.

Definition 2.8.[8] Let δ be a derivation of a hyperring R. An element $a \in R$ is called a constant element associated with δ if $\delta(a) = 0$. The set of all such elements is denoted by $C_\delta(R)$.

Definition 2.9.[8] A hyperring R is said to be n-*torsion free* if

$$0 \in \underbrace{a + a + \cdots + a}_{n \text{ times}} \implies a = 0.$$

Note: Example 1 is non-trivial example for n-torsion free.

Definition 2.10.[8] A function $\delta : R \to R$ is called a derivation if for all $a, b \in R$:

1. $\delta(a + b) \subseteq \delta(a) + \delta(b)$;

2. $\delta(ab) \in \delta(a)b + a\delta(b)$.

If $\delta(a+b) = \delta(a) + \delta(b)$, then δ is called a strong derivation.

Definition 2.11.[2] A function $\delta : R \to R$ is called a reverse derivation if for all $a, b \in R$:

1. $\delta(a+b) \subseteq \delta(a) + \delta(b)$;
2. $\delta(ab) \in \delta(b)a + b\delta(a)$.

If $\delta(a+b) = \delta(a) + \delta(b)$, then δ is called a strong reverse derivation.

Example 3: Let $R = \{0, a, b\}$ and consider the following hyperoperations. Then R forms a prime hyperring.

$\oplus$	0	a	b
0	0	a	b
a	a	a	R
b	b	R	b

$\otimes$	0	a	b
0	0	0	0
a	0	a	b
b	0	b	a

Table 3

Define a map $\delta : R \to R$ such that $\delta(0) = 0, \quad \delta(a) = b, \quad \delta(b) = a$. Then the function δ is a reverse derivation on R.

Example 4: Let R be a hyperring and $M(R) = \left\{ \begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix} : p, q \in R \right\}$ be the collection of 2×2 matrices over R.

A hyperaddition $\oplus$ is defined on $M(R)$ as follows. Let $A = \begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix}, \quad B = \begin{pmatrix} 0 & r \\ 0 & s \end{pmatrix}$ be two matrices in $M(R)$. Then

$$A \oplus B = \left\{ \begin{pmatrix} 0 & l \\ 0 & m \end{pmatrix} : l \in p + r, \ m \in q + s \right\},$$

for all $A, B \in M(R)$. Multiplication $\otimes$ on $M(R)$ is defined by $A \otimes B = \begin{pmatrix} 0 & ps \\ 0 & qs \end{pmatrix}$, for all $A, B \in M(R)$. Clearly, hyperaddition and multiplication are well defined, and $(M(R), \oplus, \otimes)$ forms a Krasner hyperring.

Now define a function δ on $M(R)$ by $\delta \begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix} = \begin{pmatrix} 0 & -q \\ 0 & 0 \end{pmatrix}$. Here δ is well defined.

Now we show that δ is a strong reverse derivation.
For $A, B \in M(R)$, we have $\delta(A \oplus B) = \delta(A) \oplus \delta(B)$ and both sets are equal to

$$\left\{ \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} : t \in -(q+s) \right\}.$$

Also,

$$\delta(A \otimes B) = \delta \begin{pmatrix} 0 & ps \\ 0 & qs \end{pmatrix} = \begin{pmatrix} 0 & -qs \\ 0 & 0 \end{pmatrix} = \delta(B) \otimes A \oplus B \otimes \delta(A).$$

Thus δ is a strong reverse derivation on $M(R)$.

3. LEMMAS

Lemma 3.1[[8, Lemma 1.3]] Let R be a hyperring and $[a, b] = ab - ba$ for all $a, b \in R$. Then for all $a, b, c \in R$, we have

1. $[a + b, c] = [a, c] + [b, c]$,
2. $[ab, c] = a[b, c] + [a, c]b$,
3. If $a \in Z(R)$, then $[ab, c] = a[b, c]$.

Lemma 3.2[[10, Lemma 1.3]] Let R be a hyperring and $[a, b] = ab - ba$ for all $a, b \in R$. The symbol (a, b) represents the skew commutator $ab + ba$. Then for all $a, b, c \in R$, the following conditions hold:

1. $[a + b, c] = [a, c] + [b, c]$,
2. $[ab, c] \subseteq a[b, c] + [a, c]b$,
3. $(a + b, c) = (a, c) + (b, c)$,
4. $(ab, c) \subseteq (a, c)b + a[b, c] = a(b, c) - [a, c]b$.

Lemma 3.3[[8, Lemma 2.5]] Let U be a non-zero hyperideal of a prime hyperring R. Then for all $a, b \in R$, the following hold:

1. If $Ua = 0$ or $aU = 0$, then $a = 0$,

2. If $aUb = 0$, then $a = 0$ or $b = 0$,
3. If $a \in Z(R)$ and $ab = 0$, then $a = 0$ or $b = 0$,
4. If $a \in R$ such that $[U, a] = 0$, then $a \in Z(R)$,
5. If $a \in Z(R)$ and $ab \in Z(R)$, then $a = 0$ or $b \in Z(R)$.

Lemma 3.4[[2, Lemma 3.1]] Let U be a non-zero hyperideal of a prime hyperring R and let δ be a reverse derivation on R. Then for every $a \in R$,

1. If $\delta(U) = 0$, then $\delta = 0$,
2. If $a\delta(U) = 0$ or $\delta(U)a = 0$, then $a = 0$ or $\delta = 0$,
3. If $a\delta(R) = 0$ or $\delta(R)a = 0$, then $a = 0$ or $\delta = 0$.

Lemma 3.5[[2, Theorem 3.3]] Let R be a two torsion-free hyperring with a non-zero hyperideal U and let δ be a reverse derivation of R. If $\delta^2(u) = 0$ for all $u \in U$, then $\delta = 0$.

Lemma 3.6[[2, Theorem 3.4]] Let R be a prime hyperring such that the center $Z(R)$ is a ring and let U be a non-zero hyperideal of R. Then R is commutative in each of the following cases:

1. If δ is a reverse derivation with $\delta^2 \neq 0$ and $\delta(R) \subseteq Z(R)$,
2. Suppose R is a two-torsion free hyperring. If there exists a non-zero reverse derivation δ such that $\delta(U) \subseteq Z(R)$.

4. MAIN RESULTS

Theorem 4.1: Let R be a prime hyperring and δ be a non-zero reverse derivation on R such that $\delta([x, y]) = 0$ for all $x, y \in R$. Then R is a commutative hyperring.

Proof. Assume that $\delta([x, y]) = 0$ for all $x, y \in R$. Replacing x by xy, we obtain $0 = \delta([xy, y]) = \delta([x, y]y)$. Using the reverse derivation property, this gives $0 \in \delta(y)[x, y]$. Hence, $\delta(y)xy = \delta(x)yx$ for all $x, y \in R$. Now replace x by zx. Then $0 \in \delta(y)zxy = \delta(y)yzx$, which implies $0 \in \delta(y)x[z, y]$ for all $x, z, y \in R$. Since δ is non-zero, it follows that $0 \in [z, y]$ for all $z, y \in R$. Therefore, R is a commutative hyperring. □

Theorem 4.2: Let R be a 2-torsion free prime hyperring and $a \in R$. Suppose δ is a non-zero reverse derivation on R such that $0 = [a, \delta(R)]$. Then $a \in Z(R)$.

Proof. Assume that $0 = [a, \delta(x)]$ for all $x \in R$. Replacing x by xy, we obtain $0 = [a, \delta(xy)]$ for all $x, y \in R$. Since δ is a reverse derivation, we have $\delta(xy) \in \delta(y)x + y\delta(x)$. Thus, $0 \in [a, \delta(y)x + y\delta(x)]$. Using Lemma 3.2, we obtain $0 \subseteq [a, \delta(y)x] + [a, y\delta(x)]$. Hence, $0 \subseteq [a, \delta(y)]x + \delta(y)[a, x] + [a, y]\delta(x) + y[a, \delta(x)]$. Since $[a, \delta(x)] = 0$, we obtain $0 \in \delta(y)[a, x] + [a, y]\delta(x)$ for all $x, y \in R$. Now replace x by $\delta(x)$. Then $0 \in \delta(y)[a, \delta(x)] + [a, y]\delta^2(x)$. Since $[a, \delta(x)] = 0$, this reduces to $0 \in [a, y]\delta^2(x)$. Because R is a prime hyperring, either $\delta^2 = 0$ or $0 \in [a, y]$. Since $\delta \neq 0$, it follows that $0 \in [a, y]$ for all $y \in R$. Therefore $a \in Z(R)$. □

Theorem 4.3: Let R be a 2-torsion free prime hyperring and let $\delta : R \to R$ be an idempotent reverse derivation of R. Suppose that $r \in R \setminus Z(R)$ and $[\delta(x), r] = \{0\}$ for all $x \in R$. Then $\delta(x) = 0$ for all $x \in R$.

Proof. Let $r \in R \setminus Z(R)$ such that $[\delta(x), r] = \{0\}$ for every $x \in R$. (1) Replacing x by $x\delta(x)$ in (1), we obtain $0 = [\delta(x\delta(x)), r]$. Since δ is a reverse derivation, $\delta(x\delta(x)) \in \delta^2(x)x + \delta(x)\delta(x)$. Hence $0 \in [\delta^2(x)x + \delta(x)\delta(x), r] \subseteq [\delta^2(x)x, r] + [\delta(x)\delta(x), r]$. Thus $0 \in \delta^2(x)[x, r] + [\delta^2(x), r]x$. Since δ is idempotent, $\delta^2 = \delta$, and therefore $0 \in \delta(x)[x, r]$ for all $x \in R$. If $[x, r] = 0$ for all $x \in R$, then $r \in Z(R)$, which contradicts the hypothesis. Hence $\delta(x) = 0$ for all $x \in R$. □

Theorem 4.4: Let R be a prime hyperring and δ be a reverse derivation of R. Suppose there exists $r \in R$ such that $r \notin Z(R)$ and $(\delta(x), r) = \{0\}$ for all $x \in R$. Then $\delta((x, r)) = \{0\}$ for all $x \in R$.

Proof. Let $r \notin Z(R)$ such that $(\delta(x), r) = \{0\}$ for all $x \in R$. (2)

Replacing x by xy in (2), we get $0 = (\delta(xy), r)$. Since δ is a reverse derivation, $\delta(xy) \in \delta(y)x + y\delta(x)$. Thus $0 \in (\delta(y)x + y\delta(x), r) \subseteq (\delta(y)x, r) + (y\delta(x), r)$. Hence $0 \in (\delta(y), r)x + \delta(y)[x, r] + y(\delta(x), r) - [y, r]\delta(x)$. Since $(\delta(x), r) = 0$, we obtain $0 \in \delta(y)[x, r] - [y, r]\delta(x)$.
Therefore $0 \in [y, r]\delta(x)$. (3)
Replacing y by yz in (3), we obtain $0 \in [yz, r]\delta(x) \subseteq y[z, r]\delta(x) + [y, r]z\delta(x)$. Hence $0 \in [y, r]z\delta(x)$ for all $y, z \in R$. Since R is prime, either $[y, r] = 0$ or $\delta(x) = 0$. If $[y, r] = 0$, then $r \in Z(R)$, a contradiction. Hence $\delta(x) = 0$.
Finally, for all $x \in R$, $\delta((r, x)) = \delta(rx + xr) \subseteq \delta(rx) + \delta(xr) \in \delta(x)r + x\delta(r) + \delta(r)x + r\delta(x)$. Since $\delta(x) = 0$, we obtain $\delta((r, x)) = 0$. □

Theorem 4.5: Let R be a 2-torsion free prime hyperring and let $\delta : R \to R$ be a non-zero reverse derivation of R. Then R is a commutative hyperring under any one of the following conditions:

1. $\delta(R) \subseteq Z(R)$, where $Z(R)$ is a ring;
2. $[\delta(R), \delta(R)] = \{0\}$ and $Z(R)$ is a ring;
3. $\delta([x, y]) = \{0\}$ for all $x, y \in R$.

Proof. (1) Assume that $\delta(R) \subseteq Z(R)$. Then for all $x, y \in R$, $[\delta(x), y] = \{0\}$. (4)
Replacing x by xz in (4), we obtain $0 = [\delta(xz), y]$. Since δ is a reverse derivation, $\delta(xz) \in \delta(z)x + z\delta(x)$. Thus $0 \in [\delta(z)x + z\delta(x), y] \subseteq \delta(z)[x, y] + [x, y]\delta(z)$. (5)
Putting $z = x$ in (5), we get $0 \in \delta(x)[x, y] + [x, y]\delta(x)$ which implies $0 \in (\delta(x), [x, y])$. Since $\delta \neq 0$, it follows that $[x, y] = 0$ for all $x, y \in R$. Hence R is commutative.

(2) Let $x, y, z \in R$. By assumption, $[\delta(x), \delta(y\delta(z))] \in [\delta(x), \delta^2(z)y + \delta(z)\delta(y)]$.
Thus $\subseteq [\delta(x), \delta^2(z)y] + [\delta(x), \delta(z)\delta(y)]$.
$= [\delta(x), \delta^2(z)]y + \delta^2(z)[\delta(x), y] + [\delta(x), \delta(z)]\delta(y) + \delta(z)[\delta(x), \delta(y)]$.
Since $[\delta(R), \delta(R)] = 0$, we obtain $= \delta^2(z)[\delta(x), y]$.
By primeness of R, either $\delta^2(z) = 0$ or $[\delta(x), y] = 0$. Since δ is non-zero, we conclude that $\delta(R) \subseteq Z(R)$.
Thus by part (1), R is commutative.

(3) Assume that $\delta([x, y]) = \{0\}$ for all $x, y \in R$. (6)
Replacing y by yx in (6), we obtain $0 = \delta([x, y]x) \in \delta(x)[x, y] + x\delta([x, y])$.
Hence $xy\delta(x) = yx\delta(x)$. (7)
Replacing y by zy in (7), we obtain $0 \in xzy\delta(x) - zxy\delta(x)$
which implies $0 \in [x, z]R\delta(x)$.
Since R is prime and $\delta \neq 0$, we obtain $[x, z] = 0$ for all $x, z \in R$.
Hence R is a commutative hyperring. □

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