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Numerical Solution of Hyperbolic Inverse Problem Based on Chebyshev Wavelet Method

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Abstract

In this paper, we study an efficient and accurate numerical method based on Chebyshev wavelets for the parameter identification in hyperbolic inverse partial differential equations. In the proposed method, integration approach is used to obtain the required unknown terms. Taylor series approximation is used to identify the parameter appeared in hyperbolic PDEs. Error estimates of the method are discussed under suitable smoothness assumptions. Various numerical examples are shown in order to demonstrate the accuracy of the method.

Keywords : Chebyshev wavelet,Taylor series, Hyperbolic inverse PDEs, Tensor product.

1. INTRODUCTION

In the recent few years, there has been increasing interest in the development of efficient and accurate numerical methods for hyperbolic inverse problem[4] with parameter. Hyperbolic inverse problem has many application in areas of science and technology e.g. earth science , geophysics, and optical devices. It is also used in e.g. seismology, medical imaging, nondestructive testing, radar and sonar, electromagnetic imaging, and structural monitoring.

In this study, we propose a numerical method based on Chebyshev wavelet to indentify parameter in the one-dimensional hyperbolic inverse problem given by

$$\frac{\partial^2 y}{\partial t^2}(x,t) = \frac{\partial^2 y}{\partial x^2}(x,t) + X(t)y(x,t) + \xi(x,t),\ x \in (0,1),\ \ 0 \leq t \leq T, \tag{1.1}$$

with initial condition

$$y(x,0) = f_1(x), \qquad \frac{\partial y}{\partial t}(x,0) = f_2(x),\ x \in (0,1), \tag{1.2}$$

and Dirichlet boundary conditions

$$y(0,t) = g_1(t), \qquad y(1,t) = g_2(t), \quad 0 \leq t \leq T, \tag{1.3}$$

subject to the overspecified condition

$$y(x^*,t) = Q(t), \quad 0 \leq t \leq T, \tag{1.4}$$

where x^* is a fixed point such that $0 < x^* < 1$ and ξ, f_1, f_2, g_1, g_2 and Q are known functions. Let us assumed that $Q(t) \neq 0$. We have to determine unknown functions $y(x,t)$ and $X(t)$ together where $y(x,t)$ stands for the solution of the given problem and $X(t)$ is the parameter.
Chebyshev wavelet method(CWM) has simple structure and recurrence relations make them easy to implement in hyperbolic inverse PDEs. It is orthogonal with respect to a weight function. This gives a high accuracy and computational efficiency. Chebyshev wavelets are nonzero only over a finite interval, which makes this wavelet efficient for representation. Chebyshev wavelets gives very fast convergence for smooth functions.

The existence, uniqueness, and regularity results for the hyperbolic inverse problem with a parameter is discussed by Priyaranjan *et al.*[9] Uniqueness of inverse problem for the wave equation has been investigated by Rakesh and Symes.[14] Wave equation has been also discussed in.[15] Gopal *et al.*[5] proposed hybrid Haar wavelet collocation method for nonlocal hyperbolic partial differential equations. Lin *et al.*[16] solved hyperbolic inverse problem by finite difference method. Numerical solution of the one dimensional wave equation has been discussed in.[7] Pulkina *et al.*[6] solved a nonlocal problem with integral condition for hyperbolic equations. Various numerical method have been proposed by Dehghan *et al.*[1–3] for hyperbolic partial differential equations. Atta *et al.*[10] used shifted first-kind Chebyshev polynomials to solve nonlinear time-fractional partial integro-differential equation with a weakly singular kernal. Farooq *et al.*[11] proposed numerical solutions of fractional delay differential equation by Chebyshev wavelet method(CWM). Baghani[12] introduced second Chebyshev wavelets(SCWs) method for solving finite time fractional linear quadratic optimal control problems. Application of wavelet collocation method have been discussed in.[13]

The content of this paper is organized as follows. In Sect. 2, some basic background of Chebyshev wavelet method is discussed. In Sect. 3, Chebyshev wavelet collocation method is implemented on the hyperbolic inverse problem. In Sect. 4, convergence analysis for the given method is discussed. In Sect. 5, numerical results are discussed on some problems. In sect. 6, conclusion and future work are discussed.

2. BASIC BACKGROUND

In this section, we will describe some basic definitions and properties of Chebyshev wavelets and their first and second integral forms of this wavelets which will be used throughout in this article.
Before presenting the method, we introdue the notations that have been used. Let $P_{nm}(x)$ denote the wavelet method for Chebyshev wavelets. Let $Q_{nm}(x)$ and $R_{nm}(x)$ are used to denote the first and second integral of the Chebyshev wavelet.[8]
The Chebyshev wavelets are defined as follows:

$$P_{nm}(x) = \begin{cases} \rho_m 2^{\frac{k-1}{2}} C_m\left(2^k x - 2n + 1\right), & \text{if } \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}; \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

where

$$\rho_m = \begin{cases} \frac{\sqrt{2}}{\sqrt{\pi}}, & \text{if } m = 0; \\ \frac{2}{\sqrt{\pi}}, & \text{if } m \neq 0, \end{cases} \tag{2.2}$$

for $n = 1, 2, \ldots, 2^{k-1}$, $\quad k \in \mathbb{Z}^+$ and $m = 0, 1, \ldots, M-1$. Here $C_m(x)$ represents the Chebyshev polynomials of the first kind of degree m. Chebyshev wavelets are orthogonal weighted by $\alpha(2^k x - 2n + 1) = \frac{1}{\sqrt{1-(2^k x - 2n+1)^2}}$.
From the above mentioned definitions, any function $f(x) \in L^2_\alpha[0,1]$ can be expanded in terms of Chebyshev wavelet as follows:

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} s_{nm} P_{nm}(x), \tag{2.3}$$

where s_{nm} are the wavelet coefficients.

$$s_{nm} = \langle f(x), I_{nm}(x) \rangle = \left\{ \int_0^1 f(x) P_{nm}(x) \alpha(x) dx, \quad \text{for the Chebyshev wavelets,} \right. \tag{2.4}$$

The approximation of $f(x)$ is given by

$$f(x) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} s_{nm} P_{nm}(x) = \boldsymbol{S}\boldsymbol{P}(x), \tag{2.5}$$

where $\boldsymbol{S}$ is a $1 \times 2^{k-1}M$ vector given by

$$\boldsymbol{S} = \left[s_{10},\ s_{11}, \ldots, s_{1(M-1)},\ s_{20},\ s_{21}, \ldots, s_{2(M-1)}, \ldots, s_{2^{k-1}0},\ s_{2^{k-1}1}, \ldots, s_{2^{k-1}(M-1)}\right], \tag{2.6}$$

and

$$\boldsymbol{P}(x) = \left[P_{10}(x)\dots P_{1(M-1)}(x), P_{20}(x)\dots P_{2(M-1)}(x),\dots, P_{2^{k-1}0}(x)\dots P_{2^{k-1}(M-1)}(x)\right]^T. \quad (2.7)$$

Furthermore, the first and the second integral forms of the Chebyshev wavelets[8] are

$$Q_{n0}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{n-1}{2^{k-1}}; \\ \rho_0\, 2^{-\frac{(k-1)}{2}-1}\left[C_1(\theta) + C_0(\theta)\right], & \text{if } \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}; \\ \rho_0\, 2^{-\frac{(k-1)}{2}} C_0(\theta), & \text{if } \frac{n}{2^{k-1}} \le x \le 1, \end{cases} \quad (2.8)$$

$$Q_{n1}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{n-1}{2^{k-1}}; \\ \rho_1\, 2^{-\frac{(k-1)}{2}-3}\left[C_2(\phi) - C_0(\phi)\right], & \text{if } \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}; \\ 0, & \text{if } \frac{n}{2^{k-1}} \le x \le 1, \end{cases} \quad (2.9)$$

$$Q_{nm}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{n-1}{2^{k-1}}; \\ \rho_m\, 2^{-\frac{(k-1)}{2}-2}\left[\frac{C_{m+1}(\phi)}{m+1} - \frac{C_{m-1}(\phi)}{m-1} + \lambda_m\right], & \text{if } \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}; \\ \rho_m\, 2^{-\frac{(k-1)}{2}-2}\gamma_m, & \text{if } \frac{n}{2^{k-1}} \le x \le 1, \end{cases} \quad (2.10)$$

and

$$R_{n0}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{n-1}{2^{k-1}}; \\ \rho_0\, 2^{-\frac{3(k-1)}{2}-4}\left[C_2(\phi) + 4C_1(\phi) + 3C_0(\phi)\right], & \text{if } \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}; \\ \rho_0\, 2^{-\frac{(k-1)}{2}}\left(\frac{1}{2^k} + x - \frac{n}{2^{k-1}}\right), & \text{if } \frac{n}{2^{k-1}} \le x \le 1, \end{cases} \quad (2.11)$$

$$R_{n1}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{n-1}{2^{k-1}}; \\ \rho_1\, 2^{-\frac{3(k-1)}{2}-4}\left[\frac{C_3(\phi)}{6} - \frac{3C_1(\phi)}{2} - \frac{4C_0(\phi)}{3}\right], & \text{if } \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}; \\ \rho_1\, \frac{2^{-\frac{3(k-1)}{2}-1}}{-3}, & \text{if } \frac{n}{2^{k-1}} \le x \le 1, \end{cases} \quad (2.12)$$

$$R_{n2}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{n-1}{2^{k-1}}; \\ \rho_2\, 2^{-\frac{3(k-1)}{2}-3}\left[\frac{C_4(\phi)-1}{24} - \frac{C_2(\phi)-1}{3} - \frac{2}{3}\big(C_1(\phi) + C_0(\phi)\big)\right], & \text{if } \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}; \\ \rho_2\, \frac{2^{-\frac{(k-1)}{2}}}{-3}\left(\frac{1}{2^k} + x - \frac{n}{2^{k-1}}\right), & \text{if } \frac{n}{2^{k-1}} \le x \le 1, \end{cases} \quad (2.13)$$

$$R_{nm}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{n-1}{2^{k-1}}; \\ \rho_m\, 2^{-\frac{3(k-1)}{2}-3}\left[\begin{array}{c} \frac{C_{m+2}(\phi)-(-1)^{m+2}}{2(m+1)(m+2)} - \frac{C_m(\phi)-(-1)^m}{2(m+1)(m)} \\ -\frac{C_m(\phi)-(-1)^m}{2(m-1)(m)} + \frac{C_{m-2}(\phi)-(-1)^{m-2}}{2(m-1)(m-2)} \\ +\left(1 + C_1(\phi)\right)\lambda_m \end{array}\right], & \text{if } \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}; \\ \rho_m\, 2^{-\frac{3(k-1)}{2}-3}\left[\begin{array}{c} \frac{1-(-1)^{m+2}}{2(m+1)(m+2)} - \frac{1-(-1)^m}{2(m+1)(m)} \\ -\frac{1-(-1)^m}{2(m-1)(m)} + \frac{1-(-1)^{m-2}}{2(m-1)(m-2)} \\ +2\mu_m + 2^k\left(x - \frac{n}{2^{k-1}}\right)\gamma_m \end{array}\right], & \text{if } \frac{n}{2^{k-1}} \le x \le 1, \end{cases} \quad (2.14)$$

where $\phi = 2^k x - 2n + 1$, $\gamma_m = \frac{1-(-1)^{m+1}}{m+1} - \frac{1-(-1)^{m-1}}{m-1}$, and $\lambda_m = \frac{(-1)^{m-1}}{m-1} - \frac{(-1)^{m+1}}{m+1}$. Moreover, $C_m(x)$ represents for the Chebyshev polynomials of the first kind of degree

m. For the sake of simplicity and understandability the use of unnecessary notations has been avoided. Thus, for any continuous $f(x)$ throughout this section we have the following relations:

$$\begin{aligned} f(x) &\simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} s_{nm} P_{nm}(x) = \boldsymbol{SP}(x), \\ \int_0^x f(\tau) d\tau &\simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} s_{nm} Q_{nm}(x) = \boldsymbol{SQ}(x), \\ \int_0^x \int_0^\xi f(\tau) d\tau \, d\xi &\simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} s_{nm} R_{nm}(x) = \boldsymbol{SR}(x). \end{aligned}$$

By definiing the Kronecker product that is denoted by $\otimes$ notation. For any $x = [x_1, x_2, ...x_{2^{k-1}(M-1)}]$, we have

$$\boldsymbol{x} \otimes \boldsymbol{R}(1) = \begin{bmatrix} x_1 R_{10}(1) & \cdots & x_1 R_{20}(1) & \cdots & x_1 R_{2^{k-1}(M-1)}(1) \\ x_2 R_{10}(1) & \cdots & x_2 R_{20}(1) & \cdots & x_2 R_{2^{k-1}(M-1)}(1) \\ \vdots & \ddots & \cdots & \ddots & \vdots \\ x_{2^{k-1}(M-1)-1} R_{10}(1) & \cdots & x_{2^{k-1}(M-1)-1} R_{20}(1) & \cdots & x_{2^{k-1}(M-1)-1} R_{2^{k-1}(M-1)}(1) \\ x_{2^{k-1}(M-1)} R_{10}(1) & \cdots & x_{2^{k-1}(M-1)} R_{20}(1) & \cdots & x_{2^{k-1}(M-1)} R_{2^{k-1}(M-1)}(1) \end{bmatrix}, \quad (2.15)$$

where

$$\boldsymbol{R}(1) = \left[R_{10}(1), \ldots, R_{1(M-1)}(1), R_{20}(1), \ldots, R_{2(M-1)}(1), \ldots, R_{2^{k-1}0}(1), \ldots, R_{2^{k-1}(M-1)}(1)\right]. \quad (2.16)$$

3. IMPLEMENTATION OF CHEBYSHEV WAVELET METHOD FOR HYPERBOLIC INVERSE PROBLEM

Let N_t is the number of divisions of the time interval such that $\Delta t = \frac{T}{N_t}$.Where $t \in [t_l, t_{l+1})$, the approximate solution of Equation (1.1) is as:

$$\frac{\partial^4 y}{\partial t^2 \partial x^2}(x,t) \simeq \frac{\partial^4 Y}{\partial t^2 \partial x^2}(x,t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} s_{nm} P_{nm}(x) = \boldsymbol{SP}(x), t \in [t_l, t_{l+1}). \quad (3.1)$$

Here $y(x,t)$ and $Y(x,t)$ represent the exact and numerical solution.
Integrating Equation (3.1) with respect to t from t_l to t, we get

$$\left(\frac{\partial^3 Y}{\partial t \partial x^2}\right)(x,t) = (t - t_l)\boldsymbol{SP}(x) + \left(\frac{\partial^3 Y}{\partial t \partial x^2}\right)(x, t_l) \quad (3.2)$$

Integrating Equation (3.2) with respect to t from t_l to t, we get

$$\left(\frac{\partial^2 Y}{\partial x^2}\right)(x,t) = \frac{(t-t_l)^2}{2}\boldsymbol{SP}(x) + (t - t_l)\left(\frac{\partial^3 Y}{\partial t \partial x^2}\right)(x, t_l) + \left(\frac{\partial^2 Y}{\partial x^2}\right)(x, t_l) \quad (3.3)$$

Equation (3.3) can be written as

$$\left(\frac{\partial^2 Y}{\partial x^2}\right)(x,t) = \frac{(t-t_l)^2}{2}\boldsymbol{SP}(x) + (t-t_l)\left(\frac{\partial^2}{\partial x^2}\frac{\partial Y}{\partial t}\right)(x,t_l) + \left(\frac{\partial^2 Y}{\partial x^2}\right)(x,t_l) \quad (3.4)$$

Putting the initial condition in Equation (3.4), we have

$$\left(\frac{\partial^2 Y}{\partial x^2}\right)(x,t) = \frac{(t-t_l)^2}{2}\boldsymbol{SP}(x) + (t-t_l)f_2''(x) + f_1''(x) \quad (3.5)$$

Integrating Equation (3.1) with respect to x from 0 to x, we have

$$\left(\frac{\partial^3 Y}{\partial x \partial t^2}\right)(x,t) = \boldsymbol{SQ}(x) + \left(\frac{\partial^3 Y}{\partial x \partial t^2}\right)(0,t) \quad (3.6)$$

Integrating Equation (3.6) with respect to x from 0 to x, we have

$$\left(\frac{\partial^2 Y}{\partial t^2}\right)(x,t) = \boldsymbol{SR}(x) + x\left(\frac{\partial^3 Y}{\partial t^2 \partial x}\right)(0,t) + \left(\frac{\partial^2 Y}{\partial t^2}\right)(0,t) \quad (3.7)$$

Putting $x = 1$ in Equation (3.7) and apply boundary condition

$$\left(\frac{\partial^2 g_2}{\partial t^2}\right)(t) - \boldsymbol{SR}(1) - \left(\frac{\partial^2 g_1}{\partial t^2}\right)(t) = \left(\frac{\partial^3 Y}{\partial t^2 \partial x}\right)(0,t) \quad (3.8)$$

From Eq.(3.7) and Eq.(3.8), we have

$$\begin{aligned}\left(\frac{\partial^2 Y}{\partial t^2}\right)(x,t) &= \boldsymbol{S}\Big(\boldsymbol{R}(x) - x\otimes\boldsymbol{R}(1)\Big) + x\left(\frac{\partial^2 g_2}{\partial t^2}(t) - \frac{\partial^2 g_1}{\partial t^2}(t)\right) \\ &\quad + \left(\frac{\partial^2 g_1}{\partial t^2}\right)(t) \quad (3.9)\end{aligned}$$

Integrate Eq.(3.7) with respect to t from t_l to t, we have

$$\begin{aligned}\left(\frac{\partial Y}{\partial t}\right)(x,t) &= (t-t_l)\boldsymbol{SR}(x) + x\left(\frac{\partial^2 Y}{\partial t \partial x}(0,t) - \frac{\partial^2 Y}{\partial t \partial x}(0,t_l)\right) \\ &\quad + \left(\frac{\partial Y}{\partial t}\right)(0,t) - \left(\frac{\partial t}{\partial t}\right)(0,t_l) + \left(\frac{\partial Y}{\partial t}\right)(x,t_l) \quad (3.10)\end{aligned}$$

Putting $x = 1$ in Eq.(3.10) and using given boundary condition, we have

$$\begin{aligned}\left(\frac{\partial^2 Y}{\partial x \partial t}(0,t) - \frac{\partial^2 Y}{\partial x \partial t}(0,t_l)\right) &= \left(\frac{\partial g_2}{\partial t}(t) - (t-t_l)\boldsymbol{SR}(1) - \frac{\partial g_1}{\partial t}(t)\right. \\ &\quad \left. + \frac{\partial g_1}{\partial t}(t_l) - \frac{\partial g_2}{\partial t}(t_l)\right) \quad (3.11)\end{aligned}$$

From Eq.(3.10) and (3.11), we have

$$\begin{aligned}\left(\frac{\partial Y}{\partial t}\right)(x,t) &= (t-t_l)\boldsymbol{S}\Big(\boldsymbol{R}(x)-x\otimes\boldsymbol{R}(1)\Big)+x\Big(g_2'(t)-g_1'(t)+g_1'(t_l)-g_2'(t_l)\Big)\\ &\quad+\left(\frac{\partial Y}{\partial t}\right)(0,t)-\left(\frac{\partial Y}{\partial t}\right)(0,t_l)+\left(\frac{\partial Y}{\partial t}\right)(x,t_l) \qquad (3.12)\end{aligned}$$

Integrating Eq.(3.10) with respect to t from t_l to t, we have

$$\begin{aligned}Y(x,t) &= \frac{(t-t_l)^2}{2}\boldsymbol{S}\boldsymbol{R}(x)+x\Big(\frac{\partial Y}{\partial x}(0,t)-\frac{\partial Y}{\partial x}(0,t_l)-(t-t_l)\frac{\partial^2 Y}{\partial x\partial t}(0,t_l)\Big)\\ &\quad+Y(0,t)-Y(0,t_l)-(t-t_l)\left(\frac{\partial Y}{\partial t}\right)(0,t_l)\\ &\quad+(t-t_l)\left(\frac{\partial y}{\partial t}\right)(x,t_l)+Y(x,t_l) \qquad (3.13)\end{aligned}$$

Putting $x=1$ in Eq.(3.13) and apply boundary condition, we have

$$\begin{aligned}&\Big(\frac{\partial Y}{\partial x}(0,t)-\frac{\partial Y}{\partial x}(0,t_l)-(t-t_l)\frac{\partial^2 Y}{\partial x\partial t}(0,t_l)\Big)\\ &\qquad= g_2(t)-\frac{(t-t_l)^2}{2}\boldsymbol{S}\boldsymbol{R}(1)-g_1(t) \qquad (3.14)\\ &\qquad\quad+g_1(t_l)+(t-t_l)g_1'(t)\quad-(t-t_l)g_2'(t_l)-g_2(t_l)\end{aligned}$$

From Eq.(3.13) and Eq.(3.14), we have

$$\begin{aligned}Y(x,t) &= \frac{(t-t_l)^2}{2}\boldsymbol{S}\Big(\boldsymbol{R}(x)-x\otimes\boldsymbol{R}(1)\Big)+x\Big(g_2(t)-g_1(t)+g_1(t_l)\\ &\quad+(t-t_l)g_1'(t)-(t-t_l)g_2'(t_l)-g_2(t_l)\Big)+Y(0,t)-Y(0,t_l)\\ &\quad-(t-t_l)\left(\frac{\partial Y}{\partial t}\right)(0,t_l)+(t-t_l)\left(\frac{\partial Y}{\partial t}\right)(x,t_l)+Y(x,t_l) \qquad (3.15)\end{aligned}$$

The required coefficient to obtain the approximate solution at $t=t_{l+1}$. Substituting the expressions for $\left(\frac{\partial^2 Y}{\partial t^2}\right)(x,t_{l+1})$, $\left(\frac{\partial^2 Y}{\partial x^2}\right)(x,t_{l+1})$ and $Y(x,t_{l+1})$ in (1.1), we get the following

$$\begin{aligned}
&\boldsymbol{S}\Bigg(\boldsymbol{R}(x) - x\otimes \boldsymbol{R}(1)\Bigg) + x\Bigg(\frac{\partial^2 g_2}{\partial t^2}(t_{l+1}) - \frac{\partial^2 g_1}{\partial t^2}(t_{l+1})\Bigg) + \Bigg(\frac{\partial^2 g_1}{\partial t^2}\Bigg)(t_{l+1}) \\
&- \Bigg[\frac{(t_{l+1}-t_l)^2}{2}\boldsymbol{S}\boldsymbol{P}(x) + (t_{l+1}-t_l)f_2''(x) + f_1''(x)\Bigg] \\
&- X(t_{l+1})\Bigg[\frac{(t_{l+1}-t_l)^2}{2}\boldsymbol{S}\Bigg(\boldsymbol{R}(x) - x\otimes \boldsymbol{R}(1)\Bigg) \\
&+ x\Bigg(g_2(t_{l+1}) - g_1(t_{l+1}) + g_1(t_l) + (t_{l+1}-t_l)g_1'(t_{l+1}) - (t_{l+1}-t_l)g_2'(t_l) - g_2(t_l)\Bigg) \\
&+ Y(0,t_{l+1}) - Y(0,t_l) \qquad (3.16) \\
&- (t_{l+1}-t_l)\frac{\partial Y}{\partial t}(0,t_l) + (t_{l+1}-t_l)\frac{\partial Y}{\partial t}(x,t_l) + Y(x,t_l)\Bigg] = \xi(x,t_{l+1})
\end{aligned}$$

Finally, we get the matrix equation (at time t_{l+1})

$$\boldsymbol{A}\boldsymbol{S} = \boldsymbol{b} \qquad (3.17)$$

Where

$$\begin{aligned}
\boldsymbol{A} &= (\boldsymbol{R}(x) - x\otimes \boldsymbol{R}(1)) - \Bigg(\frac{(t_{l+1}-t_l)^2}{2}\Bigg)\boldsymbol{P}(x) \\
&\quad -X(t_{l+1})\Bigg(\frac{(t_{l+1}-t_l)^2}{2}\Bigg)(\boldsymbol{R}(x) - x\otimes \boldsymbol{R}(1)) \qquad (3.18)
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{b} &= \xi(x,t_{l+1}) - x\Bigg(\frac{\partial^2 g_2}{\partial t^2}(t_{l+1}) \\
&\quad -\frac{\partial^2 g_1}{\partial t^2}(t_{l+1})\Bigg) - \Bigg(\frac{\partial^2 g_1}{\partial t^2}\Bigg)(t_{l+1}) + \Bigg((t_{l+1}-t_l)f_2''(x) + f_1''(x)\Bigg) \\
&\quad +X(t_{l+1})\Bigg(Y(0,t_{l+1}) - Y(0,t_l) - (t_{l+1}-t_l)\frac{\partial Y}{\partial t}(0,t_l) \\
&\quad +(t_{l+1}-t_l)\frac{\partial Y}{\partial t}(x,t_l) + Y(x,t_l) \\
&\quad +x\Bigg[g_2(t_{l+1}) - g_1(t_{l+1}) + g_1(t_l) + (t_{l+1}-t_l)g_1'(t_{l+1}) \qquad (3.19) \\
&\quad -(t_{l+1}-t_l)g_2'(t_l) - g_2(t_l)\Bigg]\Bigg)
\end{aligned}$$

Using matrix Equation (3.18), we obtained wavelet coefficients $\boldsymbol{S}$ for Chebyshev wavelet method via the `gmres` package in matlab.

3.1. Evaluation of the control parameter

From Equation (1.1), we have

$$\frac{\partial^2 y}{\partial t^2}(x^*,t) = \frac{\partial^2 y}{\partial x^2}(x^*,t) + X(t)y(x,t) + \xi(x^*,t) \tag{3.20}$$

Using the given overspecification condition(1.4), we have

$$\frac{\partial^2 Q}{\partial t^2}(t) = \frac{\partial^2 y}{\partial x^2}(x^*,t) + X(t)y(x^*,t) + \xi(x^*,t) \tag{3.21}$$

This implies

$$X(t) = \frac{\frac{\partial^2 Q}{\partial t^2}(t) - \frac{\partial^2 y}{\partial x^2}(x^*,t) - \xi(x^*,t)}{Q(t)} \tag{3.22}$$

The numerical solution for the parameter evaluated by the Taylor expansion, at $t = t_{l+1}$ as follows:

$$X(t_{l+1}) = \frac{\frac{\partial^2 Q}{\partial t^2}(t_{l+1}) - \frac{\partial^2 y}{\partial x^2}(x^*,t_l) - \frac{\partial^3 y}{\partial t \partial x^2}(x^*,t_l) - \xi(x^*,t_{l+1})}{Q(t_{l+1})} + \mathcal{O}(\Delta t^2) \tag{3.23}$$

4. CONVERGENCE ANALYSIS

4.1. The error estimations for the CWM

Theorem 4.1.1. *Let $f(x) \in \mathcal{C}^2[0,1]$ with second order derivative is bounded, that is $|f''(x)| \leq L$. $f(x)$ can be expanded as an infinite series of the Chebyshev wavelet, and the series converges uniformly to the function $f(x)$,*

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} s_{nm}P_{nm}(x).$$

Notice that $s_{nm} = \langle f(x), P_{nm}(x)\rangle$ where $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2_\alpha[0,1]$ given in Equation (2.4) and $P_{nm}(x)$ are the Chebyshev wavelet.

$$|s_{nm}| \leq \frac{\rho_m \pi L}{32 n^{5/2}(m-1)^2}, \quad m > 1. \tag{4.1}$$

where n denotes the resolutions of the interval for which $n = 1, 2, \ldots, 2^{k-1}$ for $k \in \mathbb{Z}^+$ and m stands for the degree of Chebyshev polynomials of the first kind.

Theorem 4.1.2. *Let $f(x) \in \mathcal{C}^2[0,1]$ with bounded second order derivative, that is $|f''(x)| \leq L$.Then, we have the following error estimations for the Chebyshev wavelet method (CWM) and its integral form*

$$\omega_{nm} \leq \frac{\sqrt{\pi}L}{8}\left(\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5}\frac{1}{(m-1)^4}\right)^{1/2}, \tag{4.2}$$

where ω_{nm} stands for the accuracy of the functional approximation of the CWM and its integral forms. Where, n represents the resolution of the interval for which $n = 1, 2, \ldots, 2^{k-1}$ for $k \in \mathbb{Z}^+$ and m denotes the degree of Chebyshev polynomial for the first kind.

Proof. The functional accuracy of CWM, that is, P_{nm}

$$\omega_{nm} \leq \frac{\sqrt{\pi}L}{8}\left(\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5}\frac{1}{(m-1)^4}\right)^{1/2},$$

For this study, a error analysis is required to investigate this problem that includes both Q_{nm} and R_{nm} which denotes the first and second integral of the CWM. Then, we start with the first integral form such that

$$\left\|\int_0^x f(\tau)d\tau - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} s_{nm}Q_{nm}(x)\right\|^2 = \int_0^1\left|\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} s_{nm}Q_{nm}(x) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} s_{nm}Q_{nm}(x)\right|^2 \alpha(x)dx.$$

This implies that

$$\begin{aligned}
\omega_{nm}^2 &\leq \int_0^1 \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|s_{nm}|^2\int_0^x P_{nm}^2(\tau)\,\alpha(\tau)d\tau\alpha(x)dx \\
&\leq \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|s_{nm}|^2\int_0^1\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\rho_m^2\,2^{k-1}\frac{C_m^2\left(2^k\tau-2n+1\right)}{\sqrt{1-\left(2^k\tau-2n+1\right)^2}}d\tau\alpha(x)dx \\
&\leq \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|s_{nm}|^2\int_0^1\int_{-1}^{1}\rho_m^2\frac{C_m^2(t)}{2\sqrt{1-t^2}}dt\alpha(x)dx,
\end{aligned}$$

where $t = (2^k\tau - 2n + 1)$.As we know that Chebyshev polynomials are orthogonal, one can obtain

$$\left\|\int_0^x f(\tau)d\tau - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} s_{nm}Q_{nm}(x)\right\|^2 \leq \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|s_{nm}|^2\rho_m^2\frac{\pi}{4}.$$

By virtue of $|\rho_m| \leq \frac{2}{\sqrt{\pi}}, \quad m = 0, 1, 2, \ldots$ and Eq (4.1), we have

□

$$\omega_{nm}^2 \leq \frac{\sqrt{\pi}L}{8}\left(\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5}\frac{1}{(m-1)^4}\right)^{1/2} \tag{4.3}$$

The second integral for the CWM also follows the line of the first integral . We us the bound of ρ_m and Equation (4.1), some integration property have been used

$$\left\|\int_0^x\int_0^u f(\tau)d\tau du - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} s_{nm}R_{nm}(x)\right\|^2 \leq \int_0^1 \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|s_{nm}|^2 R_{nm}(x)\alpha(x)dx,$$

$$\begin{aligned}\omega_{nm}^2 &\leq \int_0^1 \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|s_{nm}|^2\int_0^1\int_0^u P_{nm}^2(\tau)\,\alpha(\tau)d\tau\alpha(u)du\alpha(x)dx \\ &\leq \frac{\sqrt{\pi}L}{8}\left(\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5(m-1)^4}\right)^{1/2}.\end{aligned} \tag{4.4}$$

4.2. The final convergence result of the proposed method

Theorem 4.2.1. *Let $y \in L^2[0,1] \cap \mathcal{C}[0.t]$ be the exact solution of initial boundary problem stated in Equations (1.1)-(1.4). Let $\Delta t = \frac{T}{N_t}$ where N_t stands for the number of discretizations for the time interval. Suppose that Y(x,t) is the numerical solution obtained by the proposed method.*

$$\|y(x,t_l) - Y(x,t_l)\| \leq \|y(x,0) - Y(x,0)\| + l\delta\Delta t$$

. Notice that $\delta = \beta\omega_{nm},\ \beta \in \mathbb{R}$ where ω_{nm} denotes the error estimations of the present wavelet methods. Moreover, l denotes the time step for $l = 1, 2, ..., N_t$

Proof. we recall the numerical solution and the exact solution at $t = t_{l+1}$, respectively.

$$Y(x,t_{l+1}) = (t_{l+1}-t_l)\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} s_{nm}(R_{nm}(x) - x\otimes R_{nm}(1)) + Y(x,t_l) + bc(t_l^{l+1},x), \tag{4.5}$$

and

$$y(x,t_{l+1}) = (t_{l+1}-t_l)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} s_{nm}(R_{nm}(x) - x\otimes R_{nm}(1)) + y(x,t_l) + bc(t_l^{l+1},x) \tag{4.6}$$

where $bc(t_l^{l+1},x) = x(g_2(t_{l+1}) - g_1(t_{l+1}) - g_1(t_l) + (t_{l+1}-t_l)g_1'(t_{l+1}) - (t_{l+1}-t_l)g_2'(t_l) - g_2(t_l))$. □

Subtracting Eq.(4.5) from Eq.(4.6), the local error has been written as follows:

$$\begin{aligned} |Y(x,t_{l+1}) - y(x,t_{l+1})| \;\leq\; & \Delta t \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |s_{nm}(R_{nm}(x) - x \otimes R_{nm}(1))| \\ & + |Y(x,t_l) - y(x,t_l)|\,, \end{aligned} \tag{4.7}$$

where $\Delta t = (t_{l+1} - t_l)$. Remark that the terms of $bc(t_l^{l+1}, x)$ are removed from the exact solution; hence, no effect of $bc(t_l^{l+1}, x)$ on the error estimation. Therefore, Equation (4.7) shows a strong connection between the convergence result of the proposed numerical method and error analysis of wavelet methods. By defining $d_r = |Y(x,t_r) - y(x,t_{r-1})|\,,\ r = 1, 2, ..., l+1$, we have

$$d_l \leq d_{l-1} + \delta \Delta t, \tag{4.8}$$

where

$$\delta = \begin{cases} \underbrace{\beta \left(\sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^5 (m-1)^4} \right)^{1/2}}_{\omega_{nm}}, & \text{for CWM.} \end{cases} \tag{4.9}$$

Here, β is a constant depends on L and $max_{1 \leq i \leq 2^{k-1}M} x_i$. From Eq.(4.8) and (4.9) gives that the proposed method is consistent. The error propagation can easily be seen by induction as

$$\begin{aligned} d_1 &\leq d_0 + \delta \Delta t, & (4.10) \\ d_2 &\leq d_1 + \delta \Delta t \leq d_0 + 2\delta \Delta t, & (4.11) \\ &\vdots & (4.12) \\ d_{N_t} &\leq d_0 + N_t \delta \Delta t. & (4.13) \end{aligned}$$

Moreover, it is importance that when at $t = 0$, the numerical solution is obtained by the help of the given initial condition of the equation, that is, $d_0 = 0$. From the definition of Δt, $N_t \Delta t = T$. The proposed method is stable. It is important to recall that $\delta = \beta \omega_{nm} \to 0$ as k and M increase. From the above, one can conclude that $d_{N_t} \to 0$ guarantees the convergence of the proposed method.

5. NUMERICAL EXAMPLES

In this section, we present numerical results based on the Chebyshev wavelet method applied to the one-dimensional hyperbolic inverse problem. In the numerical results, Chebyshev wavelet method is denoted by CWM.

Example 1.

$$\frac{\partial^2 y}{\partial t^2}(x,t) = \frac{\partial^2 y}{\partial x^2}(x,t) + X(t)y(x,t) - xcos(t) - txcos(t),\ 0 \le x \le 1, 0 < t \le T, \tag{5.1}$$

with initial condition

$$u(x,0) = x, \qquad \frac{\partial y}{\partial t}(x,0) = 0, \qquad 0 \le x \le 1, \tag{5.2}$$

and Dirichlet boundary conditions

$$y(0,t) = 0, \qquad 0 < t \le T, \tag{5.3}$$

$$y(1,t) = cos(t), \qquad 0 < t \le T, \tag{5.4}$$

subjet to the overspecified condition

$$y(0.5,t) = 0.5cos(t), \qquad 0 < t \le T. \tag{5.5}$$

The exact solution is

$$y(x,t) = xcos(t)$$

and

$$X(t) = t$$

x	Pointwise absolute error
	CWM
	$\Delta t = 10^{-3}$
	$k = 3, m = 4.$
0.0625	1.388×10^{-17}
0.1250	2.776×10^{-17}
0.1875	2.498×10^{-16}
0.2500	5.551×10^{-17}
0.3125	6.106×10^{-16}
0.3750	1.943×10^{-15}
0.4375	9.437×10^{-16}
0.5000	1.110×10^{-16}
0.5625	1.110×10^{-16}
0.6250	1.554×10^{-15}
0.6875	5.551×10^{-16}
0.7500	5.773×10^{-15}
0.8125	1.110×10^{-15}
0.8750	3.331×10^{-15}
0.9375	9.992×10^{-16}
1.000	2.220×10^{-16}

In the above table, we have shown pointwise absolute error in the numerical solutions obtained by the Chebyshev wavelet method (CWM). It is observed that at very less k and m, we are obtaining very good results. The errors are of order 10^{-15}.

Pointwise absolute error		
t	Exact X	CWM
0.1	0.1	9.933×10^{-4}
0.2	0.2	1.000×10^{-3}
0.3	0.3	1.000×10^{-3}
0.4	0.4	1.001×10^{-3}
0.5	0.5	1.002×10^{-3}
0.6	0.6	9.576×10^{-4}
0.7	0.7	1.006×10^{-3}
0.8	0.8	1.002×10^{-3}
0.9	0.9	1.003×10^{-3}
1.0	1.0	9.902×10^{-4}

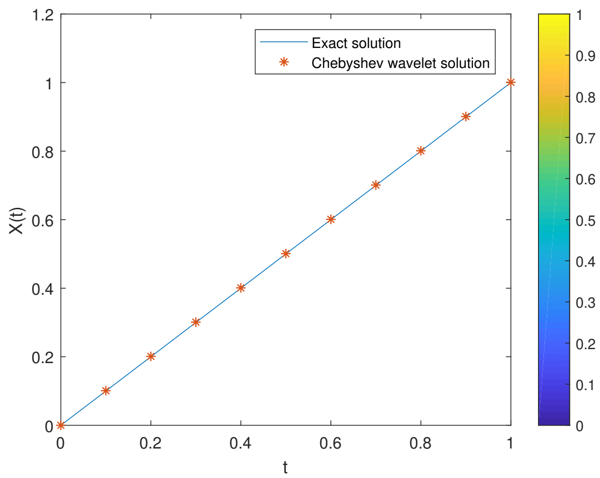

Figure 1: Exact and CWM solution of $X(t)$

In figure 1, we obtain exact and Chebyshev wavelet solution for X for $\Delta t = 10^{-3}$.

Example 2.

$$\frac{\partial^2 y}{\partial t^2}(x,t) = \frac{\partial^2 y}{\partial x^2}(x,t) + X(t)y(x,t) - xsin(t) - txsin(t),\ 0 \leq x \leq 1, 0 < t \leq T, \tag{5.6}$$

with initial condition

$$y(x,0) = 0, \qquad \frac{\partial y}{\partial t}(x,0) = x \qquad 0 \leq x \leq 1, \tag{5.7}$$

and Dirichlet boundary conditions

$$y(0,t) = 0, \qquad 0 < t \leq T, \tag{5.8}$$

$$y(1,t) = sin(t), \qquad 0 < t \leq T, \tag{5.9}$$

subject to the overspecified condition

$$y(0.5,t) = 0.5sin(t), \qquad 0 < t \leq T. \tag{5.10}$$

the exact solution is

$$y(x,t) = xsin(t),$$

and

$$X(t) = t.$$

x	pointwise absolute error
	CWM
	$\Delta t = 10^{-3}$
	$k = 3, m = 4.$
0.0625	2.429×10^{-17}
0.1250	4.857×10^{-17}
0.1875	3.469×10^{-16}
0.2500	9.714×10^{-17}
0.3125	1.665×10^{-16}
0.3750	6.106×10^{-16}
0.4375	6.384×10^{-16}
0.5000	1.943×10^{-16}
0.5625	1.665×10^{-16}
0.6250	2.220×10^{-16}
0.6875	2.220×10^{-16}
0.7500	1.776×10^{-15}
0.8125	4.441×10^{-16}
0.8750	1.221×10^{-15}
0.9375	1.110×10^{-16}
1.0000	3.886×10^{-16}

In the above table, we have shown pointwise absolute error in the numerical solutions obtaained by Chebyshev wavelet method(CWM). It is observed that at very less k and m, we are obtained very good results. The error are of order 10^{-15}.

Pointwise absolute error		
t	Exact X	CWM
0.1	0.1	9.982×10^{-4}
0.2	0.2	9.925×10^{-3}
0.3	0.3	0.013×10^{-3}
0.4	0.4	0.085×10^{-3}
0.5	0.5	0.006×10^{-3}
0.6	0.6	0.008×10^{-3}
0.7	0.7	0.012×10^{-3}
0.8	0.8	0.016×10^{-3}
0.9	0.9	9.996×10^{-4}
1.0	1.0	9.998×10^{-4}

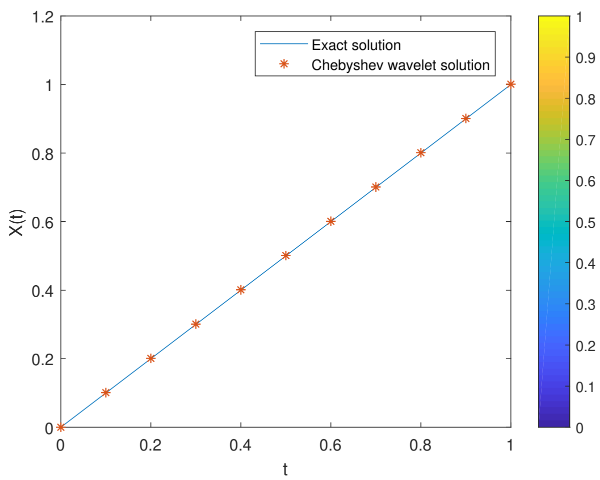

Figure 2: Exact and CWM solution of $X(t)$

In figure 2, we obtain exact and Chebyshev wavelet solution for X for $\Delta t = 10^{-3}$.

Example 3.

$$\frac{\partial^2 y}{\partial t^2}(x,t) = \frac{\partial^2 y}{\partial x^2}(x,t) + X(t)y(x,t) - (2xsin(t) + 2xcos(t) + t^2xcos(t) + t^2xsin(t)) \; 0 \le x \le 1, 0 < t \le T, \tag{5.11}$$

with initial condition

$$y(0,t) = x, \qquad \frac{\partial y}{\partial t}(x,0) = x \qquad\qquad 0 \leq x \leq 1, \tag{5.12}$$

and Dirichlet boundary conditions

$$y(0,t) = 0, \qquad\qquad 0 < t \leq T, \tag{5.13}$$

$$y(1,t) = sin(t) + cos(t), \qquad\qquad 0 < t \leq T, \tag{5.14}$$

subject to the overspecified condition

$$y(0.5,t) = 0.5(sin(t) + cos(t)), \qquad\qquad 0 < t \leq T.$$

The exact solution is

$$y(x,t) = x(sin(t) + cos(t)),$$

and

$$X(t) = 1 + t^2$$

x	pointwise absolute error
	CWM
	$\Delta t = 10^{-3}$
	$k = 3, m = 4.$
0.0625	1.360×10^{-15}
0.1250	5.551×10^{-17}
0.1875	6.661×10^{-16}
0.2500	1.110×10^{-16}
0.3125	9.992×10^{-16}
0.3750	4.552×10^{-15}
0.4375	3.331×10^{-16}
0.5000	2.220×10^{-16}
0.5625	3.220×10^{-15}
0.6250	5.551×10^{-16}
0.6875	2.442×10^{-15}
0.7500	6.661×10^{-15}
0.8125	1.776×10^{-15}
0.8750	4.663×10^{-15}
0.9375	3.331×10^{-15}
1.000	4.441×10^{-16}

In the above table, we have shown pointwise absolute error in the numerical solutions obtained by Chebyshev wavelet method(CWM). It is observed that at very less k and m, we are obtaining very good results. The error are of order 10^{-15}.

Pointwise absolute error		
t	Exact X	CWM
0.1	1.01	2.010×10^{-4}
0.2	1.04	4.010×10^{-4}
0.3	1.09	6.009×10^{-4}
0.4	1.16	8.099×10^{-4}
0.5	1.25	0.001×10^{-3}
0.6	1.36	0.108×10^{-3}
0.7	1.49	0.228×10^{-3}
0.8	1.64	0.388×10^{-3}
0.9	1.81	0.501×10^{-3}
1.0	2.00	0.002×10^{-3}

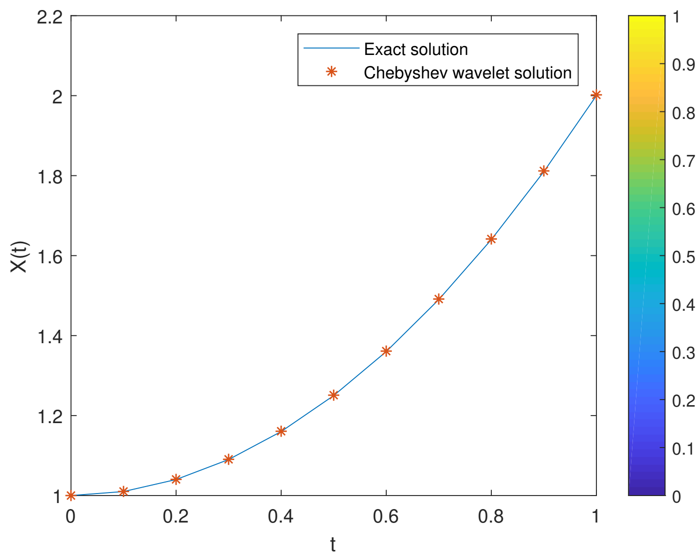

Figure 3: Exact and CWM solution of $X(t)$

In figure 3, we obtain exact and Chebyshev wavelet solution for X for $\Delta t = 10^{-3}$.

6. CONCLUSION

In this work, we developed highly efficient and accurate numerical method based on Chebyshev wavelet method for parameter identification in hyperbolic inverse problem. Error estimates have been derived in order to prove the convergence of the numerical method. The proposed method is tested on few 1D PDEs and it is found that numerical solution are in good agreement with exact solution. This method can be extended to 2D and 3D hyperbolic inverse problems easily.

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