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The L(2,1) Labeling of Triangular Book Families

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Abstract

An L(2,1) labeling (or distance-two labeling) of a graph $G$is a function$f: V(G) \rightarrow \{0,1,2,\ldots\}$ such that
$\mid f(x) - f(y) \mid \geq 2$ if $d(x,y) = 1$ and $\mid f(x) - f(y) \mid \geq 1$ if $d(x,y) = 2$. The L(2,1) labeling number $\lambda(G)$of G is the smallest integer k such that G admits an L(2,1) labeling with $\max\{f(v): v \in V(G)\} = k$.

In this paper, we determine the L(2,1) labeling number of the triangular book graph and the triangular book with bookmark graph.

Keywords: L(2,1) labeling, L(2,1) labeling number, triangular book graph, triangular book with bookmark graph.

AMS Subject Classification: 05C78.

Introduction

Graph labeling is the assignment of integers to the vertices, edges, or both, of a graph subject to certain conditions. The concept of graph labeling was introduced by Rosa in 1967. Since then, numerous graph labeling techniques have been studied extensively, resulting in over 2100 research papers. Labeled graphs provide useful models in various applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, FM radio channel assignment, and communication network addressing. Channel assignments for FM radio stations depend not only on transmission power and antenna height but also on geographical distances between stations. In wireless networks, radio spectrum management involves assigning frequencies to transmitters in such a way that interference is avoided. Interference may occur when geographically close transmitters are assigned frequencies that are too close.

Let G be a connected graph and k an integer. The distance between two vertices u and v in $G$is denoted by $d(u,v)$. The vertex set and edge set of G are denoted by

$V(G)$ and $E(G)$, respectively. Distance-two labeling was introduced by J. R. Griggs and R. K. Yeh[9], who proved that every graph with maximum degree Δ admits an L(2,1) labeling with span at most $\Delta^2 + 2\Delta$, and they verified the conjecture for 2-regular graphs. Later, Chang and Kuo improved the upper bound to $\Delta^2 + \Delta$ [6]. Chang et al. further generalized the result, obtaining $\Delta^2 + (d-1)\Delta$ as an upper bound for the minimum span of an L(d,1)-labeling [7]. K M Baby Smitha and K Thirusangu studied the distance two labeling of several graphs [1][2][3][4][5]. N B Rathod and K K Kanani studied the k-cordial Labeling of Triangular Book, Triangular Book with Book Mark & Jewel Graph[14].
Graph theoretic models for radio frequency assignment problems were first introduced in the early 1980s by Hale. Several studies on L(2,1) labeling of special classes of graphs have since been conducted. In particular, the L(2,1) labeling of various graph families has been investigated extensively.

Preliminaries

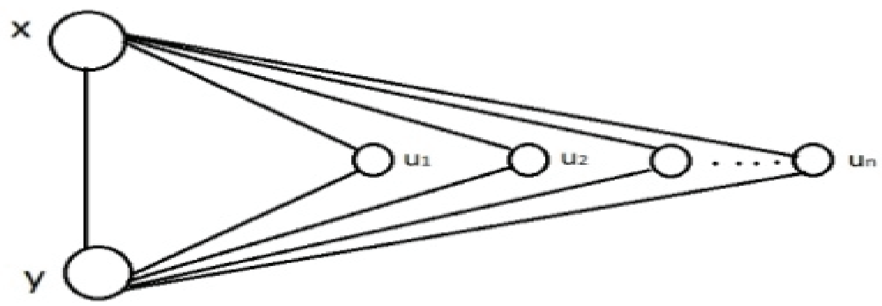
In this paper, all graphs considered are finite, undirected, and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

Definition 2.1

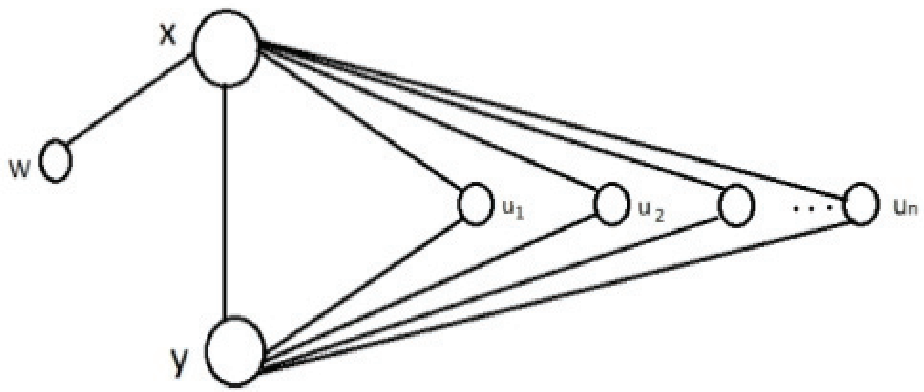
An L(2,1) labeling (or distance-two labeling) of a graph G is a function $f: V(G) \rightarrow \{0,1,2,\ldots\}$ such that
$| f(x) - f(y) | \geq 2$ whenever $d(x,y) = 1$ and $| f(x) - f(y) | \geq 1$ whenever $d(x,y) = 2$.
The L(2,1) labeling number $\lambda(G)$ of G is the smallest integer k such that $\max\{f(v): v \in V(G)\} = k$.

Definition 2.2

The **triangular book graph** with n pages, denoted by $B(3,\mathrm{n})$ is defined as the graph obtained by taking n copies of the cycle C_3 and identifying a common edge, called the spine (or base) of the book. The two end vertices of the common edge are called the spine vertices. Each additional vertex from the n triangles is adjacent to both spine vertices. The number of vertices in the triangular book graph $B(3,\mathrm{n})$ is $n+2$.



Definition 2.3
The **triangular book with bookmark graph** is obtained from the triangular book graph $B(3,n)$by attaching a pendant edge to one of the spine vertices. Let x and y be the end vertices of the common edge (spine) of $B(3,n)$. If a new vertex w is introduced and joined to x by an edge xw, then the resulting graph is called the triangular book with bookmark graph. This graph is denoted by $TB_n(x,y)(x,w)$ where x and y are the spine vertices and $xw$is the pendant (bookmark) edge attached at x. The number of vertices in this graph is $n+3$.



Main Results

Theorem 3.1
For $n \geq 2$, the L(2,1) labeling number of the triangular book graph $B(3,n)$ is $\lambda(B(3,n)) = n+3$.

Proof
Let $G = B(3,n)$be the triangular book graph with vertex set $V(G) = \{x, y, u_1, u_2, \ldots, u_n\}$,
where $x$and $y$are the spine vertices. Thus, $|V(G)| = n+2$. Each vertex $u_i$is adjacent to both $x$and y, and the vertices $u_i$are mutually non-adjacent.

Labeling Construction
Define a function $f: V(G) \to \mathbb{N} \cup \{0\}$
by

$$f(u_i) = i-1, 1 \leq i \leq n,$$
$$f(y) = n+1,$$
$$f(x) = n+3.$$

Hence $f(V(G)) = \{0,1,2, \ldots, n-1, n+1, n+3\}$.

Therefore, $\max\{f(v): v \in V(G)\} = n+3$.

Verification of L(2,1) Conditions

Let $w_1, w_2 \in V(G)$.

Case 1: $w_1 = u_i, w_2 = u_j, i \neq j$

Since the vertices $u_i$are not adjacent,

$$d(u_i, u_j) = 2.$$

$$\mid f(u_i) - f(u_j) \mid = \mid (i-1) - (j-1) \mid = \mid i-j \mid \geq 1.$$

Condition satisfied.

Case 2: $w_1 = u_i, w_2 = y$

These vertices are adjacent:

$$d(u_i, y) = 1.$$

$$\mid f(y) - f(u_i) \mid = \mid (n+1) - (i-1) \mid = \mid n-i+2 \mid.$$

Since $1 \leq i \leq n$,

$$\mid f(y) - f(u_i) \mid \geq 2.$$

Condition satisfied.

Case 3: $w_1 = u_i, w_2 = x$

These vertices are adjacent:

$$d(u_i, x) = 1.$$

$$\mid f(x) - f(u_i) \mid = \mid (n+3) - (i-1) \mid = \mid n-i+4 \mid \geq 2.$$

Condition satisfied.

Case 4: $w_1 = x, w_2 = y$

$$d(x, y) = 1.$$

$$\mid f(x) - f(y) \mid = \mid (n+3) - (n+1) \mid = 2.$$

Condition satisfied.

Thus, in all possible cases, the L(2,1) labeling conditions hold. Therefore $\lambda(B(3,n)) = n+3$.

Example 3.1

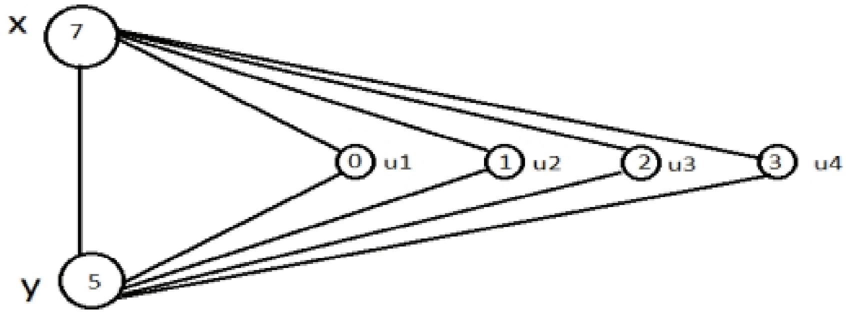


Figure 1 L(2,1) labeling of triangular book graph $\lambda(B(3,4) = 7$

Theorem 3.2
For $n \geq 2$, the L(2,1) labeling number of the triangular book with bookmark graph $TB_n(x,y)(x,w)$ is $\lambda(TB_n(x,y)(x,w)) = n + 3$.

Proof
Let $V(G) = \{x, y, w, u_1, u_2, \ldots, u_n\}$.

Here, $|V(G)| = n + 3$.

The vertex w is a pendant vertex adjacent only to x.

Labeling Construction
Define $f(u_i) = i - 1, 1 \leq i \leq n,$

$f(w) = n,$

$f(y) = n + 1,$

$f(x) = n + 3.$

Thus,

$$f(V(G)) = \{0, 1, \ldots, n - 1, n, n + 1, n + 3\}.$$

Hence, $\max\{f(v)\} = n + 3.$

Verification
We verify all possible vertex pairs.

1. $Pair\ u_i, u_j$ $d = 2, \ |f(u_i) - f(u_j)| = |i - j| \geq 1.$

2. $Pair\ u_i, w$ $d = 2, \ |f(u_i) - f(w)| = |(i - 1) - n| = |n - i + 1| \geq 1.$

3. $Pair\ u_i, y$ $d = 1, \ |f(y) - f(u_i)| = |n + 1 - (i - 1)| = |n - i + 2| \geq 2.$

4. $\boldsymbol{Pair\ u_i, x}$ $d = 1,\ |\ f(x) - f(u_i)\ |=|\ n + 3 - (i - 1)\ |=|\ n - i + 4\ | \geq 2.$

5. $\boldsymbol{Pair\ w, y}$ $d = 2,\ |\ f(w) - f(y)\ |=|\ n - (n + 1)\ | = 1.$

6. $\boldsymbol{Pair\ w, x}$ $d = 1,\ |\ f(w) - f(x)\ |=|\ n - (n + 3)\ | = 3.$

7. $\boldsymbol{Pair\ x, y}$ $d = 1,\ |\ f(x) - f(y)\ | = 2.$

Thus, in all cases the L(2,1) conditions are satisfied.
Therefore $\lambda(TB_n(x, y)(x, w)) = n + 3.$

Example 3.2

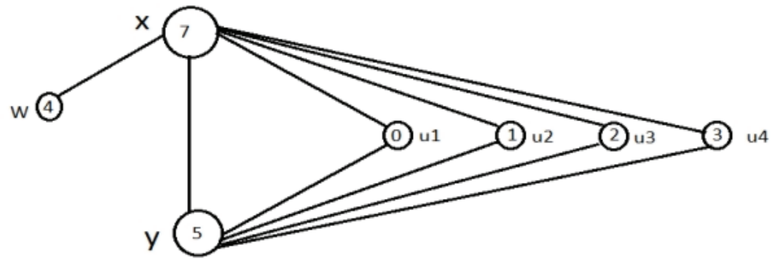


Figure 2 L(2,1) labeling of triangular book with bookmark graph $\lambda(TB_4(x, y)(x, w)) = 7.$

Conclusion
In this paper, we determined the L(2,1) labeling number of the triangular book graph and the triangular book with bookmark graph. For both graph families, the labeling number is shown to be $\lambda = n + 3, n \geq 2.$

References

[1] K. M. Baby Smitha and K. Thirusangu, Distance two labeling of Some Special Graphs, International Journal of Applied Engineering Research, 10(24) (2015), 45631–45636.
[2] K. M. Baby Smitha and K. Thirusangu, Distance two labeling of Certain Snake Graphs, International Mathematical Forum, 11(11) (2016), 503–512.
[3] K. M. Baby Smitha and K. Thirusangu, Distance two labeling of Quadrilateral Snake Families, International Journal of Pure and Applied Mathematical Sciences, 9(2) (2016), 283–298.

[4] K. M. Baby Smitha and K. Thirusangu, Distance two labeling of Cycle Related Graphs, International Journal of Pure and Applied Mathematical Sciences, 9(2) (2016), 299–312.
[5] K. M. Baby Smitha and K. Thirusangu, Distance two labeling of Triangular Snake Families, International Journal of Innovative Research in Science Engineering and Technology, 6(9) (2017), 18134–18146.
[6] G. J. Chang and D. Kuo, The L(2,1) labeling problem on graphs, SIAM Journal on Discrete Mathematics, 9 (1996), 309–316.
[7] G. J. Chang et al., On L(d,1)-labeling of graphs, Discrete Mathematics, 220 (2000), 57–66.
[8] J. A. Gallian, A Dynamic Survey of Graph Labeling, Electronic Journal of Combinatorics, (2015).
[9] J. R. Griggs and R. K. Yeh, Labeling with a condition of distance two, SIAM Journal on Discrete Mathematics, 5 (1992), 586–595.
[10] W. K. Hale, Frequency Assignment: Theory and Applications, Proceedings of the IEEE, 68 (1980), 1497–1514.
[11] F. Harary and R. Z. Norman, Some properties of line digraphs, Rendiconti del Circolo Matematico di Palermo, 9 (1960), 161–168.
[12] K. Jonas, Graph Coloring Analogues with a Condition at Distance Two: L(2,1)-Labeling and List Λ-Labelings, Ph.D. Thesis, University of South Carolina, (1993).
[13] K. Manimekalai and K. Thirusangu, Pair Sum labeling of some special graphs, International Journal of Computer Applications, 69 (2013), 34–38.
[14] N B Rathod and K K Kanani, k-cordial Labeling of Triangular Book, Triangular Book with Book Mark & Jewel Graph, Global Journal of Pure and Applied Mathematics,13(10), (2017), 6979-6989