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Product Order Divisor Graph of a Group

R. Ponraj[1] **and T. Sutharson**[2]

[1]*Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627 412, India. E-mail: ponrajmaths@gmail.com*

[2]*Research Scholor, Reg. No. 241112312016, Department of Mathematics, Sri Paramakalyani College, Alwarkurichi–627 412, India (Affiliated to Manonmaniam Sundaranar University, Tirunelveli-627 012, Tamilnadu, India).*
E-mail: sutharson94@gmail.com

Abstract

In this paper we introduce the concept of a product order divisor graph $\Gamma_{pod}(G)$ of a finite group G. The product order divisor graph $\Gamma_{pod}(G)$ is a graph with $V(\Gamma_{pod}(G)) = G$ and two vertices a and b are adjacent in $\Gamma_{pod}(G)$ if either $o(a) \mid o(ab)$ or $o(b) \mid o(ab)$. Also we obtain certain graph parameters such as clique number, chromatic number, independent number, covering number and domination number.

Keywords. product order divisor graph, complete graph, star graph, planar, bipartite graph.

2020 Mathematics Subject Classification. 05C38, 05C78.

1. INTRODUCTION

For standard terminology and notion in group theory and graph theory, we refer the reader to the text-books of Herstein [14] and Harary [11]. Throughout this paper, G denotes a group with identity e and $\Gamma_{pod}(G)$ denotes the product order divisor graph. Analytic number theory plays a fundamental role in understanding the distribution of prime numbers and arithmetic functions. A comprehensive treatment of the subject can be found in Apostol [2]. Akbari and Mohammadian [1] investigated structural properties of the zero-divisor graph of a commutative ring and established several important results concerning its graph-theoretic characteristics. In [3], construct the central graph of the identity graph of finite cyclic group and investigate some of its graph properties. P. Balakrishnan and R. Kala [4] introduced the order difference interval graph of a group and investigated its structural properties. Birch, Thibodeaux and Tucci [5] investigated the behavior of zero-divisor graphs under finite direct products and derived important structural characterizations. The concept of the power graph provides a natural graphical representation of the power structure of a finite group. Cameron and Ghosh [6] initiated the systematic study of this graph and established important foundational results. The structural analysis of central graphs arising from identity graphs of finite cyclic groups has been explored by Alib and Magpantay [7], who examined various graph-theoretic parameters in this context. Total domination, first systematically studied by Cockayne et al. [8], plays a central role in the theory of domination parameters in graphs.

The behavior of graph parameters under various graph operations has been examined in several works. Recently, Scaria et al. [9] discussed certain graph parameters and their interaction with graph operations. We recall some basic concepts from group theory and ring theory as presented in Chatterjee [10]. A comprehensive account of various domination parameters and their properties can be found in Haynes, Hedetniemi and Henning [12]. The concept of identity graphs associated with finite cyclic groups has been studied by Herawati and Henryanti [13], who analyzed their structural properties. Lucchini, Maróti and Roney-Dougal [15] investigated the generating graph of a finite simple group and established significant structural results. In [20], Sattanathan and Kala defined the order of prime graphs of finite groups and studied some properties of order prime graphs. Ma, Wei and Yang [16] introduced and investigated the coprime graph of a group and analyzed its structural properties. Extending the study of prime graphs, Rajendra and Siva Kota Reddy [17] analyzed the general order prime graph of finite groups and obtained several structural results. Order divisor graphs of finite groups were studied by Rehman et al. [18], who investigated their structural and combinatorial properties. The concept of the global domination number was introduced by Sampathkumar [19], and it has since become an important parameter in domination theory. Motivated by these developments, we investigate domination properties of the associated algebraic graph.

In our study, we focus on simple, connected, and undirected graphs, fundamental structures in graph theory. Basic terminologies from graph theory are essential for our discussions, as outlined in references [12]. The degree of a vertex v, denoted as $d(v)$, represents the number of vertices adjacent to v. The maximum degree of a graph Γ is denoted as Δ, while its minimum degree is denoted as δ. A vertex with degree one is termed a pendant vertex, and a vertex that is adjacent to every other vertex in Γ is known as a universal vertex. The neighbors of a vertex v, denoted as $N(v)$, comprise the set of all vertices adjacent to v. The complement of a graph Γ, denoted $\bar{\Gamma}$, shares the same vertex set as Γ, with edges between vertices that are not adjacent in Γ. Let a be an element of a group G. The order of a is the least positive integer r such that $a^r = e$ and is denoted by $o(a)$. An integer n is said to be divisible by an integer $m \neq 0$, in symbols $m \mid n$, if there exists some integer r such that $n = mr$. In this paper, we introduce the product order divisor graph of a finite group and investigate several graph-theoretic parameters associated with this construction.

2. PRELIMINARIES

Definition 2.1. *The group G is said to be a self-inverse group if $a^2 = e$ for all $a \in G$.*

Definition 2.2. *[12]Let $\Gamma = (V, E)$ be a graph. Then the subset D of V is said to be a dominating set if all vertices in Γ are either in D or adjacent to at least one vertex in D. The minimum cardinality of such a set is the domination number, denoted by $\gamma(\Gamma)$.*

Definition 2.3. *[19]Let $\Gamma = (V, E)$ be a graph. Then the subset D of V is said to be a global dominating set if D is also a dominating set of $\bar{\Gamma}$. The minimum cardinality of such a set is the global domination number denoted by $\gamma_g(\Gamma)$.*

Definition 2.4. *[12]Let $\Gamma = (V, E)$ be a graph. Then the subset D of V is said to be a total dominating set if D has no isolated vertex. The minimum cardinality of such a set is the total domination number, denoted as $\gamma_t(\Gamma)$.*

Definition 2.5. *[11]The vertex connectivity of Γ, denoted by $\kappa(\Gamma)$, is the minimum number of vertices whose removal disconnects Γ or reduces it to a trivial graph.*

Definition 2.6. *[11]The edge connectivity of Γ, denoted by $\lambda(\Gamma)$, is the minimum number of edges whose removal disconnects Γ.*

Definition 2.7. *[11]The chromatic number of Γ, denoted by $\chi(\Gamma)$, is the minimum number of colors required to color the vertices of Γ such that no two adjacent vertices receive the same color.*

Definition 2.8. *[11]The chromatic index of Γ, denoted by $\chi'(\Gamma)$, is the minimum number of colors required for a proper edge coloring of Γ.*

Definition 2.9. *[11]A clique in Γ is a complete subgraph of Γ. The clique number of Γ, denoted by $\omega(\Gamma)$, is the maximum cardinality of a clique in G.*

Definition 2.10. *[11]An independent set in Γ is a set of pairwise non-adjacent vertices. The independence number of Γ, denoted by $\alpha(\Gamma)$, is the maximum cardinality of an independent set in Γ.*

Definition 2.11. *[11]The girth of Γ, denoted by $g(\Gamma)$, is the length of a shortest cycle in Γ.*

Definition 2.12. *[11]The diameter of a connected graph Γ, denoted by $diam(\Gamma)$, is defined as $diam(\Gamma) = \max_{v \in V(\Gamma)} e_\Gamma(v)$.*

Definition 2.13. *[11]The radius of a connected graph Γ, denoted by $rad(\Gamma)$, is defined as $rad(\Gamma) = \min_{v \in V(\Gamma)} e_\Gamma(v)$.*

3. PRODUCT ORDER DIVISOR GRAPH

Definition 3.1. *Let G be a finite group. The product order divisor graph $\Gamma_{pod}(G)$ is a graph with $V(\Gamma_{pod}(G)) = G$ and two vertices a and b are adjacent in $\Gamma_{pod}(G)$ if either $o(a) \mid o(ab)$ or $o(b) \mid o(ab)$.*

Example 3.1. *Let $G = S_3$. The product order divisor graph of S_3, $\Gamma_{pod}(S_3)$ is given in Figure 1.*

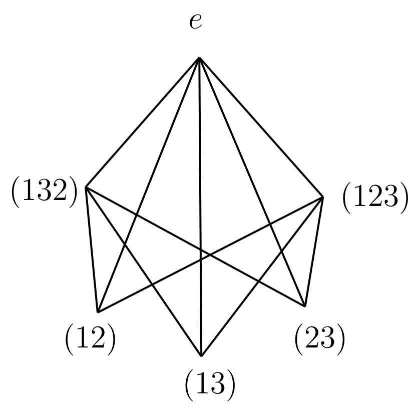

Figure 1.

Theorem 3.1. *Let G_1 and G_2 be two groups such that $G_1 \cong G_2$. Then $\Gamma_{pod}(G_1) \cong \Gamma_{pod}(G_2)$.*

Proof. Let f be an isomorphism of G_1 onto G_2. Let $a, b \in \Gamma_{pod}(G_1)$. This implies a and b are adjacent in $\Gamma_{pod}(G_1)$ if and only if $o(a) \mid o(ab)$ or $o(a) \mid o(ab)$ if and only if $o(f(a)) \mid o(f(ab))$ or $o(f(b)) \mid o(f(ab))$ if and only if $o(f(a)) \mid o(f(a)f(b))$ or $o(f(b)) \mid o(f(a)f(b))$ if and only if $f(a)$ and $f(b)$ are adjacent in $\Gamma_{pod}(G_2)$.

□

Theorem 3.2. *Let G be a cyclic group of order n. Then the following are hold:*

1. $\Gamma_{pod}(G)$ *is complete if and only if* $n \leq 2$;
2. $\Gamma_{pod}(G) \cong K_{1,n}$ *if and only if* $n \leq 3$.

Proof. Clearly $G \cong Z_n$.

(1) **Case (i)**: $n > 2$
Consider the elements $\frac{n-1}{2}$ and $\frac{n+1}{2}$. Now $o(\frac{n-1}{2}) = o(\frac{n+1}{2}) = n$. Also $o(\frac{n-1}{2} + \frac{n+1}{2}) = o(n) = 1$. Therefore the vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$ are not adjacent.
Case (ii): $n \leq 2$
Obviously $\Gamma_{pod}(G) \cong K_1$ for $n = 1$ and $\Gamma_{pod}(G) \cong K_2$ for $n = 2$.

(2) For $n = 1, 2, 3$, Clearly $\Gamma_{pod}(G) \cong K_1$, $\Gamma_{pod}(G) \cong K_{1,1}$, $\Gamma_{pod}(G) \cong K_{1,2}$.
Suppose $n > 3$, then there exists two elements $a, b \in \mathbb{Z}_n$, $a \neq 0, b \neq 0$ such that $o(a \oplus b) = o(\mathbb{Z}_n)$. Therefore a and b are adjacent in $\Gamma_{pod}(G)$.

□

Theorem 3.3. $\Gamma_{pod}(G)$ *is a complete graph if and only if G is a self-inverse group.*

Proof. Let $a \in G$, $a \neq e$. Then $a^2 = e$. Clearly $o(a) = 2$. Let $a, b \in G$. This implies $ab \in G$ and $o(ab) = 2$. Therefore a and b are adjacent in $\Gamma_{pod}(G)$. Hence $\Gamma_{pod}(G)$ is complete.
Conversely, let $\Gamma_{pod}(G)$ be complete. Suppose there exist $a, b \in G$ such that $a \neq e$, $b \neq e$, $a \neq b$ and $a^{-1} = b$. This implies $ab = e$. Then $o(ab) = 1$ but $o(a) > 1$ and $o(b) > 1$ a contradiction to a and b are adjacent.

□

4. CYCLIC GROUP OF ORDER n (n IS PRIME)

In this section, we consider the cyclic group with order n where $n > 2$ is prime. Clearly $G \cong (Z_n, \oplus)$.

Theorem 4.1. $\Gamma_{pod}(G)$ *is a* $(n, \frac{(n-1)^2}{2})$ *graph.*

Proof. Let $S = \{(x, y) : x \oplus y = 0\}$. Consider two elements $a, b \neq e$ in G such that $(a, b) \in S$. This implies $a \oplus b = 0$. Thus $o(a \oplus b) = 1$. Therefore a and b are not adjacent in $\Gamma_{pod}(G)$. Consider two elements a, b in G such that $(a \oplus b)$ not in S. This implies $a \oplus b = o(G)$. Hence $o(a) \mid o(a \oplus b)$. Therefore a and b are adjacent in $\Gamma_{pod}(G)$. Therefore the size of $\Gamma_{pod}(G) = \binom{n}{2} - (\frac{n-1}{2}) = \frac{(n-1)^2}{2}$.

□

Theorem 4.2. *The degree sequence of* $\Gamma_{pod}(G)$ *is* $(n-1, \underbrace{n-2, n-2, \cdots, n-2}_{(n-1)times})$.

Proof. Let $a \in G$. In the case of $a = e$. By the definition, the identity vertex is adjacent to every other vertex. Hence $d(e) = n - 1$. In the case of $a \neq e$. Since the graph is simple, a is not adjacent to itself. By the definition, a is not adjacent to $n - a$. Then the remaining vertex are adjacent to a. Therefore $d(a \neq e) = n - 2$. Hence the degree sequence is $(n-1, \underbrace{n-2, n-2, \cdots, n-2}_{(n-1)\text{times}})$.

□

Theorem 4.3. *The independent number of the product order divisor graph* $\Gamma_{pod}(G)$*,* $\alpha(\Gamma_{pod}(G)) = 2$ *and the covering number of* $\Gamma_{pod}(G)$*,* $\beta(\Gamma_{pod}(G)) = n - 2$.

Proof. Since G is group of prime order, $G \cong \mathbb{Z}_n$. Now $o(e) = 1$ and $o(a) = n$ for every $a \neq e$. Therefore the maximum independent in $\{n, n - a\}$. Clearly $S = \{a, n - a\}$ is an independent set. For any $x \neq a$, $x \neq n - a$ is adjacent to either a or $n - a$. Therefore the set S is a maximum independent set. Hence $\alpha(\Gamma_{pod}(G)) = 2$.
From 1, $\alpha(\Gamma_{pod}(G)) = 2$. AS $\alpha(\Gamma_{pod}(G)) + \beta(\Gamma_{pod}(G)) = n$, $\beta(\Gamma_{pod}(G)) = n - 2$.

□

Theorem 4.4. *The clique number of* $\Gamma_{pod}(G)$*,* $\omega(\Gamma_{pod}(G)) = \frac{n+1}{2}$.

Proof. The order of identity is 1 and every non-identity element has order n. Let $a, b \neq e \in G$. Since $ab = e$, $o(ab) = o(e) = 1$. Since $p \nmid 1$, a and b are not adjacent. If $ab \neq e$, then $o(ab) = n$. Thus $n \mid n$. This implies a and b are adjacent. The $n - 1$ non identity elements split into $\frac{n-1}{2}$ pairs. Each pair is non adjacent but vertices from different pairs are all adjacent. From each pair we can select atmost one vertex and include identity. Thus $\omega(\Gamma_{pod}(G)) = \frac{n+1}{2}$.

□

Theorem 4.5. *1.* $\chi(\Gamma_{pod}(G)) = \frac{n+1}{2}$*;*

2. $\chi'(\Gamma_{pod}(G)) = n - 1$.

Proof. Since $d(e) = n - 1$, the identity element e is adjacent to all vertices in $\Gamma_{pod}(G)$. Let $S = \{a, n - a : 1 \leq a \leq \frac{n-1}{2}\}$. It is easy to verify that a and $n - a$ are not adjacent where $a, n - a \in S$. Assign the color $c_{\frac{n+1}{2}}$ to e. Next assign the color c_a to a and $n - a$, $1 \leq a \leq \frac{n-1}{2}$. Hence $\chi(\Gamma_{pod}(G)) = 1 + \frac{n-1}{2} = \frac{n+1}{2}$.
Since e is adjacent to all other vertices of $\Gamma_{pod}(G)$, $\Delta(\Gamma_{pod}(G)) = n - 1$. By vizing theorem, $\Delta(\Gamma_{pod}(G)) \leq \chi'(\Gamma_o(G)) \leq \Delta(\Gamma_{pod}(G)) + 1$. This implies $n - 1 \leq \chi'(\Gamma_o(G)) \leq n$. Thus $\chi'(\Gamma_o(G)) = n$ or $n-1$. Clearly $\chi'(\Gamma_o(G)) = n$ is not impossiple. Hence $\chi'(\Gamma_o(G)) = n - 1$.

□

Example 4.1. *Figure 2 illustrate the theorem* 4.5 *the chromatic number of* $\Gamma_{pod}(G)$.

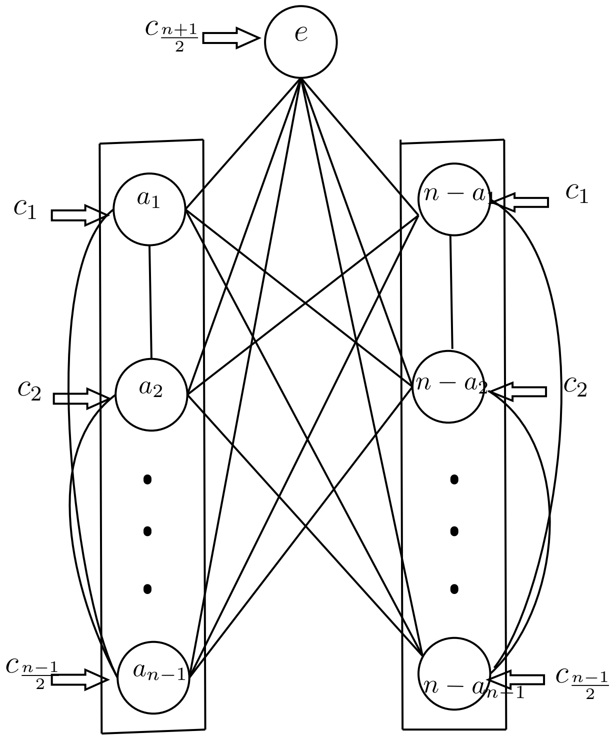

Figure 2.

Theorem 4.6. *In the product order divisor graph* $\Gamma_{pod}(G)$*, we have the following*

1. $diam(\Gamma_{pod}(G)) = 2$;
2. $grith(\Gamma_{pod}(G)) = 3$;
3. $rad(\Gamma_{pod}(G)) = 1$;

4. $C(\Gamma_{pod}(G)) = \begin{cases} \{e\}, & \text{if } n \geq 3 \\ V(\Gamma_{pod}(G)), & \text{if } n = 2. \end{cases}$

5. *$\Gamma_{pod}(G)$ has every non identity elements are peripheral.*

Proof.

(1) Since e is adjacent to all other vertices, $d(e) = n - 1$.
Case (i):$b \neq n - a$.
Consider two elements a and b where $b \neq n - a$ in Z_n. Then $o(a) \mid o(ab)$ or $o(b) \mid o(ab)$. Therefore a and b are adjacent. This implies $d(a, b) = 1$.
Case (ii):$b = n - a$.
In this case a and b are not adjacent. Choose c an element in Z_n such that $c \neq a$, $c \neq n - a$ then $a - c - b$ in $\Gamma_{pod}(G)$. Therefore $d(a, b) = 2$. Hence $diam(\Gamma_{pod}(G)) = 2$.

(2) Take three non identity elements a, b, c in G such that $ab \neq n, bc \neq n$ and $ac \neq n$. Clearly $a - b - c - a$ is a cycle in $\Gamma_{pod}(G)$. Therefore $grith(\Gamma_{pod}(G)) = 3$.

(3) Since e is adjacent to every non identity vertex, $d(e, a) = 1$. Let $a \neq e$. The $(a, n - a)$ pairs are not adjacent. Then $d(a, n - a) = 2$. Hence $rad(\Gamma_{pod}(G)) = 1$.

(4) When $n \geq 3$, by (3) $rad(\Gamma_{pod}(G)) = 1$ and $e(e) = 1$. Hence $C(\Gamma_{pod}(G)) = \{e\}$. Clearly when $n = 2$, $C(\Gamma_{pod}(G)) = V(\Gamma_{pod}(G))$.

(5) Let $a \neq e$. Clearly $d(a, n - a) = 2$. Since distance between any two elements is not exceed 2, $e(a) = 2$. By (1), $diam(\Gamma_o(G)) = 2$. Thus $e(a) = diam(\Gamma_o(G))$. Clearly every $a \neq e$ is peripheral. Hence $P(\Gamma_{pod}(G)) = V(\Gamma_{pod}(G)) - e$.

□

Theorem 4.7. *1. $\kappa(\Gamma_{pod}(G)) = n - 2$;*

2. $\lambda(\Gamma_{pod}(G)) = n - 2$.

Proof.

(1) Consider the set $S = \{0, a, n - a : a \in \mathbb{Z}_n, 1 \leq a \leq \frac{n-3}{2}\}$. It is easy to verify that the set S is the vertex connectivity. That is the removal of S to $V(\Gamma_{pod}(G))$ is a disconnected graph. Also, for any set S' such that $|S'| < |S|$, is not a vertex connectivity. Therefore $\kappa(\Gamma_{pod}(G)) = n - 2$.

(2) Consider a vertex $a \in V(\Gamma_{pod}(G))$, $a \neq e$. Removal of all edges incident with a, is a disconnected graph. There are $n-2$ edges adjacent to a. Therefore $\lambda(\Gamma_{pod}(G)) = n-2$. □

Theorem 4.8. *For the product order divisor graph $\Gamma_{pod}(G)$, we have the following*

1. $\gamma(\Gamma_{pod}(G)) = 1$;
2. $\gamma_t(\Gamma_{pod}(G)) = 2$;
3. $\gamma_c(\Gamma_{pod}(G)) = 1$;
4. $i(\Gamma_{pod}(G)) = 1$;
5. $\gamma_g(\Gamma_{pod}(G)) = \frac{n+1}{2}$;
6. $\gamma_p(\Gamma_{pod}(G)) = 2$;
7. $\gamma_U(\Gamma_{pod}(G)) = 2$;
8. $\nu(\Gamma_{pod}(G)) = \frac{n-1}{2}$.

Proof.

(1) Let $D = \{e\}$. Consider the element a in G. When $a = e$, $a \in D$. Since $d(e) = n-1$, a is adjacent to e for all $a \neq e$. Therefore D is a dominating set. Thus $\gamma(\Gamma_{pod}(G)) = 1$.

(2) Since every vertex in a total dominating set must be adjacent to another vertex in the set, no single vertex can totally dominate $\Gamma_{pod}(G)$. Hence $\gamma_t(\Gamma_{pod}(G)) \geq 2$. Consider $D = \{e, a : a \in \mathbb{Z}_p\}$. In this set e and a are mutually adjacent. Moreover every other vertex of $\Gamma_{pod}(G))$ is adjacent to e. Thus D is total dominate. Therefore $\gamma_t(\Gamma_{pod}(G)) \leq 2$. Hence $\gamma_t(\Gamma_{pod}(G)) = 2$.

(3) Since e is adjacent to all other vertices of $\Gamma_{pod}(G)$, $D = \{e\}$ dominates every vertex of $\Gamma_{pod}(G)$. The induced subgraph consists of a single vertex and is connected. Therefore $\gamma_c(\Gamma_{pod}(G)) = 1$.

(4) Clearly $D = \{e\}$ dominates all vertices of $\Gamma_{pod}(G)$. Since a single vertex forms an independent, D is independent dominate. We have $i(\Gamma_{pod}(G)) = 1$.

(5) Since e is isolated in $\overline{G}$, any dominating set must contain e. Remaining component $(a, n-a)$ pairs atleast one vertex to dominate the component. There are $\frac{n-1}{2}$ such components. Hence any dominating set must contain atleast $\frac{n+1}{2}$ vertices. Thus

$\gamma(\overline{\Gamma_{pod}(G)}) \geq \frac{n+1}{2}$. Any one vertex is eliminate the set is fails the dominate. Therefore $\gamma(\overline{\Gamma_{pod}(G)}) \leq \frac{n+1}{2}$. Hence $\gamma(\overline{\Gamma_{pod}(G)}) = \frac{n+1}{2}$. Also domination in $\Gamma_{pod}(G)$ already requires $\frac{n+1}{2}$ vertices. Thus $\gamma_g(\Gamma_{pod}(G)) \geq \frac{n+1}{2}$. But the dominating set already dominates $\Gamma_{pod}(G)$. Thus $\gamma_g(\Gamma_{pod}(G)) \leq \frac{n+1}{2}$. We have $\gamma_g(\Gamma_{pod}(G)) = \frac{n+1}{2}$.

(6) Clearly $\{e\}$ dominates all vertices of $\Gamma_{pod}(G)$. Since every paired dominating set must have cardinality even number. It follows that, $\gamma_p(\Gamma_{pod}(G)) \geq 2$. Let $D = \{0, a : a \in \mathbb{Z}_n\}$. The induced subgraph forms a perfect matching on D. This D is paired dominating set of cardinality two, and so $\gamma_p(\Gamma_{pod}(G)) \leq 2$. Hence $\gamma_p(\Gamma_{pod}(G)) = 2$.

(7) Since $\{e\}$ is adjacent to all other vertices of $\Gamma_{pod}(G)$, $\{e\}$ dominates every vertex of $\Gamma_{pod}(G)$.
Case (i): Sets containing e.
Let D be a set contain e. For any $a \neq e \in D$, $D - a$ is also a dominating set. Therefore the set contain $\{e\}$ only in the minimal dominating set.
Case (ii): Sets not containing e.
Let $D \subseteq \Gamma_{pod}(G)$. Clearly remove any one element in D the resulting set is not dominating set. Thus D is minimal dominating set. Hence $\gamma_U(\Gamma_{pod}(G)) = 2$.

(8) It is well known that the matching number of the complete graph is $\lfloor \frac{n}{2} \rfloor$. Since n is an odd prime, $\nu(\Gamma_{pod}(G)) = \frac{n-1}{2}$.

□

Theorem 4.9. *The $\Gamma_{pod}(G)$ is planar if and only if $n \leq 5$ (n is prime).*

Proof. Observe that $\{e\}$ is adjacent to all other vertices of $\Gamma_{pod}(G)$.
Case (i): $n = 2$
Since $\Gamma_{pod}(G) \cong K_2$, the graph is planar.
Case (ii): $n = 3$
Since $\Gamma_{pod}(G) \cong P_3$, the graph is planar.
Case (iii): $n = 5$
Since $\Gamma_{pod}(G) \cong K_5 - \{x, y\}$ where x and y are the edge of K_5, the graph is planar.
Case (iv): $n > 5$
Since the maximum clique size is $\frac{n+1}{2}$, the graph contains $K_{\frac{n+1}{2}}$. Since K_5 is non planar and $K_{\frac{n+1}{2}}$ contains K_5 as a subgraph. Thus $\Gamma_{pod}(G)$ is non planar.

□

Theorem 4.10. *1. $\Gamma_{pod}(G)$ is Hamiltonian;*

2. $\Gamma_{pod}(G)$ is not Eulerian.

Proof. By theorem 4.2, $d(e) = n - 1$ and $d(a \neq e) = n - 2$ for all $a \in \mathbb{Z}_n$.

(1) We observe that $\Delta(\Gamma_{pod}(G)) = n - 1$ and $\delta(\Gamma_o(G)) = n - 2$. By Dirac's theorem, $\Gamma_{pod}(G)$ is Hamiltonian.

(2) Since all the vertices have not even degree, $\Gamma_{pod}(G)$ is not Eulerian.

□

Theorem 4.11. *The following statements are hold:*

1. $\Gamma_{pod}(G)$ *is not bipartite;*
2. $\overline{\Gamma_{pod}(G)}$ *is bipartite.*

Proof.

(1) By (2) theorem 4.6, the vertices a, b, c are mutually adjacent. It forms a triangle. Which is a odd length of cycle. Thus $\Gamma_{pod}(G)$ is not bipartite.

(2) In $\overline{\Gamma_{pod}(G)}$, $V_1 = \{e, a : 1 \leq a \leq \frac{n-1}{2}\}$ and $V_2 = \{n - a : 1 \leq a \leq \frac{n-1}{2}\}$. Then (V_1, V_2) is a bipartition of $\overline{\Gamma_{pod}(G)}$.

□

5. CYCLIC GROUP OF ORDER mn (m AND n ARE DISTINCT PRIMES)

In this section, we consider the cyclic group with order mn where m and n are distinct primes, $m \geq 3$, $n \geq 3$. Clearly $G \cong (\mathbb{Z}_{mn}, \oplus)$.

Theorem 5.1. *The* $\Gamma_{pod}(G)$ *is a* $(mn, \frac{m^2n^2 - m^2n - mn^2 + m^2 + n^2 - m - n + 1}{2})$ *graph.*

Proof. The divisors of mn are $1, m, n, mn$. By Euler function, $\phi(mn) = |\{x : gcd(x, mn) = 1\}|$. Thus $\phi(m) = m - 1$, $\phi(n) = n - 1$, $\phi(mn) = (m - 1)(n - 1)$. The identity has order 1 and is adjacent to all vertices. Let $S_1 = \{a \in G : o(a) = m\}$. Then $|S_1| = \phi(m)$. Clearly there are $\frac{m-1}{2}$ elements in S_1 which are not adjacent. Let $S_2 = \{a \in G : o(a) = n\}$. Then$|S_2| = \phi(n)$. Clearly there are $\frac{n-1}{2}$ elements in S_2 which are not adjacent. Let $S_3 = \{a \in G : o(a) = mn\}$. Then $|S_3| = \phi(mn)$. Clearly there are $\frac{(m-1)(n-1)(m+n-3)}{2}$ elements in S_3 which are not adjacent. Similarly there are (m-1)(n-1)elements between S_1, S_3 and S_2, S_3 they are not adjacent. From this conclusion we have the size of $\Gamma_{pod}(G)$ is $\frac{m^2n^2 - m^2n - mn^2 + m^2 + n^2 - m - n + 1}{2}$.

□

Theorem 5.2. *The degree sequence of* $\Gamma_{pod}(G)$ *is*

$$(mn-1,\underbrace{mn-(m+n),mn-(m+n),\cdots,mn-(m+n)}_{(m-1)(n-1)times},$$
$$\underbrace{mn-(m+1),mn-(m+1),\cdots,mn-(m+1)}_{(n-1)times},$$
$$\underbrace{mn-(n+1),mn-(n+1),\cdots,mn-(n+1)}_{(m-1)times}).$$

Proof. Since m and n are distinct odd primes, every vertex x belongs to exactly one of the following cases.

Case 1: $gcd(x, mn) = 1$

Here neither m nor n divides x. $d(x) = 1 + (n-1) - 1 + (m-1) - 1 + (m-1)(n-1) - (m-1+n-1) = mn - (m+n)$.

Case 2: $gcd(x, mn) = m$

Then $m \mid x$ and $n \nmid x$. Thus $d(v) = 1 + (m-1)(n-1) + n - 3 = mn - (m+1)$.

Case 3: $gcd(x, mn) = n$

Then $n \mid x$ and $m \nmid x$. Thus $d(v) = 1 + (m-1)(n-1) + m - 3 = mn - (n+1)$.

□

Theorem 5.3. *The following are hold:*

1. $\kappa(\Gamma_{pod}(G)) = mn - m - n$;
2. $\lambda(\Gamma_{pod}(G)) = mn - m - n$.

Proof.

(1) Consider the set $S = \{a : d(a) = \delta\}$. Delete all the vertices which are adjacent to a vertex in S. Choose a vertex a such that $gcd(a, mn) = 1$. Therefore $\kappa(\Gamma_{pod}(G)) = mn - m - n$.

(2) Let $S = \{a : gcd(a, mn) = 1\}$. Removal of all edges which are incident with vertex in S. Therefore $\lambda(\Gamma_{pod}(G)) = mn - m - n$.

□

Theorem 5.4. *The independent number of product order divisor graph* $\Gamma_{pod}(G)$ *is* $\frac{(m-1)(n-1)}{2}$ *and the covering number of* $\Gamma_{pod}(G)$ *is* $\frac{mn+m+n-1}{2}$.

Proof. Now $o(e) = 1$ and $o(a) = m$ or n or mn for every $a \neq e$. Let $S = \{a : gcd(a, mn) = 1\}$. There are $\frac{(m-1)(n-1)}{2}$ vertices in S which are not mutually adjacent. This is a minimum possible vertices in $\Gamma_{pod}(G)$. Hence $\alpha(\Gamma_{pod}(G)) = \frac{(m-1)(n-1)}{2}$.

From the independent number of $\Gamma_{pod}(G)$, we have $\alpha(\Gamma_{pod}(G)) = \frac{(m-1)(n-1)}{2}$. AS $\alpha(\Gamma_{pod}(G)) + \beta(\Gamma_{pod}(G)) = mn$, $\beta(\Gamma_{pod}(G)) = \frac{mn+m+n-1}{2}$.

□

6. CYCLIC GROUP OF ORDER $2m$ (m IS PRIME)

In this section, we consider the cyclic group with order $2m$ where m is odd prime.

Theorem 6.1. *The $\Gamma_{pod}(G)$ is a $(2m, \frac{3m^2-4m+3}{2})$ graph.*

Proof. The divisors of mn are $1, 2, m, 2m$. By Euler function, $\phi(2m) = |\{x : gcd(x, 2m) = 1\}|$. Thus $\phi(m) = m-1$, $\phi(2) = 1$, $\phi(2m) = (m-1)$. The identity has order 1 and is adjacent to all vertices. Let $S_1 = \{a \in G : o(a) = 2\}$. Then $|S_1| = \phi(2)$. Let $S_2 = \{a \in G : o(a) = m\}$. Then$|S_2| = \phi(m)$. Clearly there are $\frac{m-1}{2}$ pair of elements in S_2 which are not adjacent. Let $S_3 = \{a \in G : o(a) = 2m\}$. Then $|S_3| = \phi(2m)$. Clearly there are $(m-1)$ elements in S_3 which are not mutually adjacent. Similarly there are any two elements between S_1 and S_3 they are not mutually adjacent. From this conclusion we have the size of $\Gamma_{pod}(G)$ is $\frac{3m^2-4m+3}{2}$.

□

Theorem 6.2. *The degree sequence of $\Gamma_{pod}(G)$ is* $(2m-1, m, \underbrace{m-1, m-1, \cdots, m-1}_{(m-1)times}, \underbrace{2m-3, 2m-3, \cdots, 2m-3}_{(m-1)times})$.

Proof. First consider the identity element e. Since $o(e) = 1$ and $1 \mid o(ex)$ for every $x \in G$, the vertex e is adjacent to all other vertices of the graph. Hence $d(e) = 2m-1$. Next consider the elements which are coprime to $2m$. Such elements have order $2m$ and their number is $\varphi(2m) = m-1$. For each such element a, the adjacency condition implies that a is adjacent to exactly $m-1$ vertices. Therefore $d(a) = m-1$. Now consider the elements of order m. In $\mathbb{Z}_{2m}$ there are $\varphi(m) = m-1$ elements of order m. Each such vertex is adjacent to all vertices except two particular vertices, hence $d(a) = 2m-3$. Finally, there exists exactly one element of order 2 in $\mathbb{Z}_{2m}$. This vertex is adjacent to m vertices and therefore $d(a) = m$. From this conclusion we have the degree sequence $\Gamma_{pod}(G)$ is $(2m-1, m, \underbrace{2m-2, 2m-2, \cdots, 2m-2}_{(m-1)\text{times}}, \underbrace{2m-3, 2m-3, \cdots, 2m-3}_{(m-1)\text{times}})$.

□

7. CYCLIC GROUP OF ORDER n^α (n IS PRIME AND $\alpha \geq 2$)

In this section, we consider the cyclic group with order n^α where n is prime and $\alpha \geq 2$.

Theorem 7.1. *In the product order divisor graph $\Gamma_{pod}(G)$, the degree of the identity vertex $d(e) = n^\alpha - 1$ and degree of the vertex $a \neq e$ for all $a \in \Gamma_{pod}(G)$, $1 \leq i \leq \alpha$ is*
$$\begin{cases} n^\alpha - n^{i-1} - 1, & \text{if } n \geq 3 \\ 2^\alpha - 2^{i-1}, & \text{if } n = 2. \end{cases}$$

Proof. **Case (i)**: $n \geq 3$.
Let $S_i = \{kn^{\alpha-i} : 1 \leq i \leq \alpha, gcd(k, n^\alpha) = 1, 1 \leq k \leq 2^i - 1\}$. Clearly if $a \in S_i$ then $o(a) = n^i$. Note that if $a, b \in S_i$ then they are not adjacent also if $a \in S_i$ and $b \in S_j$, $i \neq j$ then a and b are adjacent. Therefore $d(e) = n^\alpha - 1$ and $d(a) = n^\alpha - n^{i-1} - 1$ for all a, $1 \leq i \leq \alpha$.
Case (ii): $n = 2$.
The proof follows as case (i). Therefore $d(e) = 2^\alpha - 1$ and $d(a) = 2^\alpha - 2^{i-1}$ for all a, $1 \leq i \leq \alpha$.

□

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