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# Generalized $*$ -Projective Curvature Tensor of Kenmotsu Manifold

**S. K. Pandey and Prachi Mishra**

*Department of Mathematical Sciences, A.P.S. University, Rewa (M.P.) 486003, India.*
*E.mail: shravan.math@gmail.com; prachimishramath@gmail.com*

**Abstract**

In this paper,we extend the concept of the generalize $*$-projective curvature tensor of Kenmotsu manifolds by utilizing a newly generalized $(0, 2)$ symmetric tensor $Z^*$ introduced by Mantica and Suh $(2012)$. The study investigates several geometric characteristics associated with the generalized $*$-projective curvature tensor in the context of Kenmotsu manifolds. Furthermore,it is established that a generalized $*$-projective $\phi$-Symmetric Kenmotsu manifold necessarily satisfies the condition of being an $\eta$-Einstein manifold.

**Keywords:** Kenmotsu manifold, $*$-Ricci tensor, $*$-Z tensor, Generalized $*$-projective curvature tensor, Einstein manifold.

**2010 Mathematics Subject Classification:** 53C15, 53C25.

## 1. INTRODUCTION

The study of a recent branch of contact geometry was initiated in $1972$, when Kenmotsu [9] introduced a new class of almost contact Riemannian manifolds, known as Kenmotsu manifolds. The fundamental properties of the local structure of such manifolds have been extensively studied by many geometers such as Pitis [13] and Jun et al. [8]. Yildiz and De [20] investigates special classes of Kenmotsu manifolds under certain curvature conditions. These curvature constraints influence geometric properties such as Einstein conditions and tensor behaviour. The work contributes to the structural understanding of almost contact metric geometry within Kenmotsu manifolds. Chaubey et al.[4] examines the geometric properties of Kenmotsu manifolds when equipped with a semi-symmetric metric connection and derive results on curvature and Ricci tensor behaviour.

Mantica et al.[10] introduced a new generalized $(0,2)$-type symmetric tensor $Z$ and investigated its different geometric characteristics on Riemannian manifolds. A new tensor $Z$ defined as

$$Z(X_1, X_2) = S(X_1, X_2) + \psi g(X_1, X_2), \tag{1.1}$$

is called $Z$-tensor, where $\psi$ is an arbitrary scalar function. Now replacing $S$ by $S^*$ then equation (1.1) becomes generalized $Z^*$ symmetric $(0,2)$ tensor as

$$Z^*(X_1, X_2) = S^*(X_1, X_2) + \psi g(X_1, X_2), \tag{1.2}$$

where $S^*$ is a $*$-Ricci tensor.

In a $(2n+1)$-dimensional Kenmotsu manifold, the $*$-Ricci tensor $S^*$ is given by [18]

$$S^*(X_1, X_2) = S(X_1, X_2) + (2n-1)g(X_1, X_2) + \eta(X_1)\eta(X_2). \tag{1.3}$$

Using equation (1.3) in equation (1.2), we get

$$Z^*(X_1, X_2) = S(X_1, X_2) + (2n-1)g(X_1, X_2) + \eta(X_1)\eta(X_2) + \psi g(X_1, X_2). \tag{1.4}$$

Pandey et al.[15] studied Kenmotsu manifolds under various curvature conditions, particularly involving generalized curvature tensor and recurrence concepts. It establish several results showing when such manifolds satisfy special structures (like Einstein conditions). De[5] investigates the geometric properties of Kenmotsu manifolds under special types of connections and curvature conditions. It also derives conditions under which a vector field becomes a killing vector field, contributing to structural understanding of Kenmotsu geometry. Pandey et al.[17] studied some curvature properties of a semi-symmetric metric connection in a Kenmotsu manifold.Various geometric properties of kenmotsu manifolds have been explored by several researchers such as Pathak et al.[7],Pandey et al.[16]. Pokhariyal et al.[14] defined a tensor field $M$ on a Riemannian manifold as

$$\begin{aligned} M(U,V,X_1,X_2) = R(U,V,X_1,X_2) - \frac{1}{2(n-1)}[S(V,X_1)g(U,X_2) - S(U,X_1) \\ g(V,X_2) + S(U,X_2)g(V,X_1) - S(V,X_2)g(U,X_1)]. \end{aligned} \tag{1.5}$$

This tensor field $M$ is referred to as the $M$-projective curvature tensor and $R(U,V,X_1,X_2)$ is referred to the Riemannian curvature tensor and $S$ is the Ricci tensor of the manifold. De et al.[6] defined that the $M$-projective curvature tensor

$M(U, V, X_1, X_2)$ is skew-symmetric in both the first and last pairs of variables and remains invariant when the two pairs are interchanged. From equation(1.5), we get

$$\begin{aligned} M(U,V)X_1 = R(U,V)X_1 - \frac{1}{2(n-1)}[S(V,X_1)U - S(U,X_1)V \\ + g(V,X_1)QU - g(U,X_1)QV]. \end{aligned} \tag{1.6}$$

The covariant derivative of the $M$-projective curvature tensor is expressed as

$$\begin{aligned} (\nabla_W M)(U,V)X_1 = (\nabla_W R)(U,V)X_1 - \frac{1}{2(n-1)}[(\nabla_W S)(V,X_1)U - (\nabla_W S) \\ (U,X_1)V + (\nabla_W Q)(U)g(V,X_1) - (\nabla_W Q)(V)g(U,X_1)]. \end{aligned} \tag{1.7}$$

The divergence of the $M$-projective curvature tensor is expressed as

$$\begin{aligned} (divM)(U,V)X_1 = \frac{2n-3}{2(n-1)}[(\nabla_U S)(V,X_1) - (\nabla_V S)(U,X_1)] - \frac{1}{4(n-1)} \\ [g(V,X_1)dr(U) - g(U,X_1)dr(V)]. \end{aligned} \tag{1.8}$$

Ojha[11] defined and examined different geometric properties of the $M$-projective curvature tensor on Sasakian and Kählerian manifolds and he showed that $M$-projective curvature tensor establishes a connection between the conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor on one hand and the $M$-projective curvature tensor on the other. Pandey et al.[12] studied generalized $M$-projective curvature tensor of Kenmotsu manifold.

## 2. PRELIMINARIES

Consider an odd-dimensional almost contact Riemannian manifold $M^n$ endowed with the structure $(\phi, \xi, \eta, g)$, where $\phi$ denotes a $(1, 1)$ tensor field,$\eta$ is a 1-form and $g$ is the Riemannian metric. It is well known that Blair [3] and Yano et al.[19]

$$\eta(\xi) = 1, \tag{2.1}$$

$$\phi(\xi) = 0, \eta(\phi X_1) = 0, \tag{2.2}$$

$$\phi^2 X_1 = -X_1 + \eta(X_1)\xi, \tag{2.3}$$

$$g(X_1, \xi) = \eta(X_1), \tag{2.4}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \tag{2.5}$$

for any vector fields $X_1$,$X_2$ on $M^n$. If moreover,

$$(\nabla_{X_1}\phi)X_2 = -g(X_1, \phi X_2)\xi - \eta(X_2)\phi X_1, \tag{2.6}$$

$$\nabla_{X_1}\xi = X_1 - \eta(X_1)\xi, \tag{2.7}$$

where $\nabla$ denotes the Riemannian connection of $g$, then $(M^n, \phi, \xi, \eta, g)$ is called an almost Kenmotsu manifold [9].

In Kenmotsu manifolds, the following relation hold [9]:

$$(\nabla_{X_1}\eta)X_2 = g(\phi X_1, \phi X_2), \tag{2.8}$$

$$g(R(X_1, X_2)U, \xi) = \eta((R(X_1, X_2, U)) = g(X_1, U)\eta(X_2) - g(X_2, U)\eta(X_1), \tag{2.9}$$

$$R(X_1, X_2)\xi = \eta(X_1)X_2 - \eta(X_2)X_1, \tag{2.10}$$

$$R(\xi, X_1)X_2 = \eta(X_2)X_1 - g(X_1, X_2)\xi, \tag{2.11}$$

$$R(\xi, X_1)\xi = X_1 - \eta(X_1)\xi, \tag{2.12}$$

$$S(X_1, \xi) = -(n-1)\eta(X_1), \tag{2.13}$$

$$(\nabla_{X_3}R)(X_1, X_2)\xi = g(X_3, X_1)X_2 - g(X_2, X_3)X_1 - R(X_1, X_2)X_3, \tag{2.14}$$

where $R$ is the Riemannian curvature tensor and $S$ is the Ricci tensor.In a Riemannian manifold we also have

$$g(R(W, X_1)X_2, X_3) + g(R(W, X_1)X_3, X_2) = 0, \tag{2.15}$$

for every vector fields $X_1$, $X_2$, $X_3$.

## 3. GENERALIZED PROJECTIVE CURVATURE TENSOR

In this section, we define generalized $M$ -projective curvature tensor ,

$$M^*(U, V, X_1, X_2) = R(U, V, X_1, X_2) - \frac{1}{2(n-1)}[S(V, X_1)g(U, X_2) - S(U, X_1)$$
$$g(V, X_2) + S(U, X_2)g(V, X_1) - S(V, X_2)g(U, X_1)]. \tag{3.1}$$

From equation (3.1), we get

$$M^*(U, V)X_1 = R(U, V)X_1 - \frac{1}{2(n-1)}[S(V, X_1)U - S(U, X_1)V$$
$$+ g(V, X_1)QU - g(U, X_1)QV] \tag{3.2}$$

The covariant derivative of the $M^*$-projective curvature tensor is expressed as

$$(\nabla_W M^*)(U, V)X_1 = (\nabla_W R)(U, V)X_1 - \frac{1}{2(n-1)}[(\nabla_W S)(V, X_1)U - (\nabla_W S)$$
$$(U, X_1)V + (\nabla_W Q)(U)g(V, X_1) - (\nabla_W Q)(V)g(U, X_1)]. \tag{3.3}$$

The divergence of the $M^*$-projective curvature tensor is expressed as

$$(divM^*)(U,V)X_1 = \frac{2n-3}{2(n-1)}[(\nabla_U S)(V,X_1) - (\nabla_V S)(U,X_1)] - \frac{1}{4(n-1)} [g(V,X_1)dr(U) - g(U,X_1)dr(V)]. \tag{3.4}$$

## 4. GENERALIZED $*$ -PROJECTIVE CURVATURE TENSOR

In this section, we present a brief overview of the generalized $*$ -projective curvature tensor on Kenmotsu manifolds and investigate its various geometric properties.

**Theorem 4.1.** *If the scalar function $\psi$ is zero on the manifold $M^n$, then the $*$ -projective curvature tensor is identical to the generalized $*$ -projective curvature tensor.*

Proof: From equation (1.4), equation (3.1) can be expressed in the following form

$$\begin{aligned}\tilde{M}^*(U,V,X_1,X_2) &= R(U,V,X_1,X_2) - \frac{1}{2(n-1)}[Z^*(V,X_1)g(U,X_2)\\
&- Z^*(U,X_1)g(V,X_2) + Z^*(U,X_2)g(V,X_1) - Z^*(V,X_2)g(U,X_1)]\\
&+ \frac{(2n-1)}{2(n-1)}[g(V,X_1)g(U,X_2) - g(U,X_1)g(V,X_2) + g(U,X_2)g(V,X_1)\\
&- g(V,X_2)g(U,X_1)] + \frac{1}{2(n-1)}[\eta(V)\eta(X_1)g(U,X_2) - \eta(U)\eta(X_1)g(V,X_2)\\
&+ \eta(U)\eta(X_2)g(V,X_1) - \eta(V)\eta(X_2)g(U,X_1)] + \frac{\psi}{2(n-1)}[g(V,X_1)g(U,X_2)\\
&- g(U,X_1)g(V,X_2) + g(U,X_2)g(V,X_1) - g(V,X_2)g(U,X_1)].\end{aligned} \tag{4.1}$$

Define

$$\begin{aligned}\tilde{M}^*(U,V,X_1,X_2) &= R(U,V,X_1,X_2) - \frac{1}{2(n-1)}[Z^*(V,X_1)g(U,X_2)\\
&- Z^*(U,X_1)g(V,X_2) + Z^*(U,X_2)g(V,X_1) - Z^*(V,X_2)g(U,X_1)]\\
&+ \frac{(2n-1)}{2(n-1)}[g(V,X_1)g(U,X_2) - g(U,X_1)g(V,X_2) + g(U,X_2)g(V,X_1)\\
&- g(V,X_2)g(U,X_1)] + \frac{1}{2n-1}[\eta(V)\eta(X_1)g(U,X_2) - \eta(U)\eta(X_1)g(V,X_2)\\
&+ \eta(U)\eta(X_2)g(V,X_1) - \eta(V)\eta(X_2)g(U,X_1)].\end{aligned} \tag{4.2}$$

With the help of equation(4.2), equation(4.1) can be simplified as follows

$$\begin{aligned}M^*(U,V,X_1,X_2) &= \tilde{M}^*(U,V,X_1,X_2) + \frac{\psi}{2(n-1)}[g(V,X_1)g(U,X_2)\\
&- g(U,X_1)g(V,X_2) + g(U,X_2)g(V,X_1) - g(V,X_2)g(U,X_1)].\end{aligned} \tag{4.3}$$

The curvature tensor $\tilde{M}^*(U, V, X_1, X_2)$ introduced in equation (4.3) is referred to as the generalized $*$- projective curvature tensor of a Kenmotsu manifold. Which gives

$$\tilde{M}^*(U, V, X_1, X_2) = M^*(U, V, X_1, X_2) - \frac{\psi}{2(n-1)}[g(V, X_1)g(U, X_2) - g(U, X_1)g(V, X_2) + g(U, X_2)g(V, X_1) - g(V, X_2)g(U, X_1)]. \tag{4.4}$$

If $\psi = 0$, then from equation (4.4), we have

$$\tilde{M}^*(U, V, X_1, X_2) = M^*(U, V, X_1, X_2). \tag{4.5}$$

Which shows that the $*$-projective curvature tensor is identical to the generalized $*$-projective curvature tensor.

**Theorem 4.2.** *The generalized $*$ -projective curvature tensor of a Kenmlotsu manifold $M^n$ is*

- *It is skew-symmetric in the first and the last pairs of variables.*
- *It remains underchanged under the interchange of these two pairs.*

Proof: From equation(4.4), we have

$$\tilde{M}^*(V, U, X_1, X_2) = M^*(V, U, X_1, X_2) - \frac{\psi}{2(n-1)}[g(U, X_1)g(V, X_2) - g(V, X_1)g(U, X_2) + g(V, X_2)g(U, X_1) - g(U, X_2)g(V, X_1)]. \tag{4.6}$$

Now by adding equation(4.4) and equation(4.6), with the fact that $M^*(U, V, X_1, X_2) + M^*(V, U, X_1, X_2) = 0$, which gives

$$\tilde{M}^*(U, V, X_1, X_2) + \tilde{M}^*(V, U, X_1, X_2 = 0, \tag{4.7}$$

which shows that the generalized $*$-projective curvature tensor $\tilde{M}^*$ is skew-symmetric in the first pairs of variables.

Again from equation(4.4), we get

$$\tilde{M}^*(U, V, X_2, X_1) = M^*(U, V, X_2, X_1) - \frac{\psi}{2(n-1)}[g(V, X_2)g(U, X_1) - g(U, X_2) g(V, X_1) + g(U, X_1)g(V, X_2) - g(V, X_1)g(U, X_2)]. \tag{4.8}$$

Now by adding equation(4.4) and equation(4.8), with the fact that

$M^*(U,V,X_1,X_2) + M^*(U,V,X_2,X_1) = 0,$

which gives

$$\tilde{M}^*(U,V,X_1,X_2) + \tilde{M}^*(U,V,X_2,X_1) = 0, \tag{4.9}$$

which shows that the generalized $*$-projective curvature tensor $\tilde{M}^*$ is skew-symmetric in the last pairs of variables.

Now again interchanging pairs of slots in equation (4.4), we get

$$\begin{aligned}\tilde{M}^*(X_1,X_2,U,V) &= M^*(X_1,X_2,U,V) - \frac{\psi}{2(n-1)}[g(X_2,U)g(X_1,V) \\ &- g(X_1,U)g(X_2,V) + g(X_1,V)g(X_2,U) - g(X_2,V)g(X_1,U)].\end{aligned} \tag{4.10}$$

Now subtracting equation(4.10) with equation(4.4), with the fact that

$M^*(U,V,X_1,X_2) - M^*(X_1,X_2,U,V) = 0$, which gives

$$\tilde{M}^*(U,V,X_1,X_2) - \tilde{M}^*(X_1,X_2,U,V) = 0, \tag{4.11}$$

which shows that the generalized $*$-projective curvature tensor remains underchanged under the interchange of these two pairs.

**Theorem 4.3.** *A generalized $*$-projective curvature tensor of Kenmotsu manifold $M^n$ satisfies Bianchi's first identity.*

Proof: By performing cyclic permutations of $U, V$ and $X_1$ in equation(4.4), two more equations are obtained, which are written as follows

$$\begin{aligned}\tilde{M}^*(V,X_1,U,X_2) &= M^*(V,X_1,U,X_2) - \frac{\psi}{2(n-1)}[g(X_1,U)g(V,X_2) \\ &- g(V,U)g(X_1,X_2) + g(V,X_2)g(X_1,U) - g(X_1,X_2)g(V,U)],\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\tilde{M}^*(X_1,U,V,X_2) &= M^*(X_1,U,V,X_2) - \frac{\psi}{2(n-1)}[g(U,V)g(X_1,X_2) \\ &- g(X_1,V)g(U,X_2) + g(X_1,X_2)g(U,V) - g(U,X_2)g(X_1,V)].\end{aligned} \tag{4.13}$$

Now adding equations (4.4),(4.12) and equation (4.13), with the fact that

$M^*(U,V,X_1,X_2) + M^*(V,X_1,U,X_2) + M^*(X_1,U,V,X_2) = 0$, we get

$$\tilde{M}^*(U,V,X_1,X_2) + \tilde{M}^*(V,X_1,U,X_2) + \tilde{M}^*(X_1,U,V,X_2) = 0, \tag{4.14}$$

which shows that the generalized $*$-projective curvature tensor satisfies Bianchi's first identity.

**Theorem 4.4.** *A generalized $*$-projective curvature tensor of Kenmotsu manifold satisfies the following:*

1. $\tilde{M}^*(\xi, V)X_1 = [\frac{-2\psi-n+1}{2(n-1)}]g(V, X_1)\xi + [\frac{2\psi+n-1}{2(n-1)}]\eta(X_1)V - \frac{1}{2(n-1)}S(V, X_1)\xi + \frac{1}{2(n-1)}Q(V)\eta(X_1)$
2. $\tilde{M}^*(U, V)\xi = \frac{-2\psi-n+1}{2(n-1)}[\eta(V)U - \eta(U)V] - \frac{1}{2(n-1)}[Q(U)\eta(V) - Q(V)\eta(U)]$,
3. $\eta(\tilde{M}^*(U, V)X_1) = \frac{-2\psi-n+1}{2(n-1)}[g(V, X_1)\eta(U) - g(U, X_1)\eta(V)] - \frac{1}{2(n-1)}[S(V, X_1)\eta(U) - S(U, X_1)\eta(V)]$

Proof: From equation(4.4), we get

$$\tilde{M}^*(U, V)X_1 = M^*(U, V)X_1 - \frac{\psi}{(n-1)}[g(V, X_1)U - g(U, X_1)V], \tag{4.15}$$

putting $U = \xi$ in equation(4.15) and using equation (2.4) and equation(3.1), we get

$$\tilde{M}^*(\xi, V)X_1 = R(\xi, V)X_1 - \frac{1}{2(n-1)}[S(V, X_1)\xi - S(\xi, X_1)V + Q(\xi)g(V, X_1) - Q(V)\eta(X_1)] - \frac{\psi}{(n-1)}[g(V, X_1)\xi - \eta(X_1)V]. \tag{4.16}$$

Now using equations(2.11) and (2.13) in equation (4.16), we have

$$\tilde{M}^*(\xi, V)X_1 = [\frac{-2\psi - n + 1}{2(n-1)}]g(V, X_1)\xi + [\frac{2\psi + n - 1}{2(n-1)}]\eta(X_1)V - \frac{1}{2(n-1)}S(V, X_1)\xi + \frac{1}{2(n-1)}Q(V)\eta(X_1). \tag{4.17}$$

Now putting $X = \xi$ in equation (4.15) and using equation (3.1), we get

$$\tilde{M}^*(U, V)\xi = R(U, V)\xi - \frac{1}{2(n-1)}[S(V, \xi)U - S(U, \xi)V + Q(U)g(V, \xi) - Q(V)g(U, \xi)] - \frac{\psi}{(n-1)}[g(V, \xi)U - g(U, \xi)V]. \tag{4.18}$$

Now in equation(4.18) using equations(2.10) and (2.13), we get

$$\tilde{M}^*(U, V)\xi = \frac{-2\psi - n + 1}{2(n-1)}[\eta(V)U - \eta(U)V] - \frac{1}{2(n-1)}[Q(U)\eta(V) - Q(V)\eta(U)]. \tag{4.19}$$

Computing the inner product of equation(4.15) with the vector field $\xi$, we get

$$\eta(\tilde{M}^*(U,V)X_1) = \eta(M^*(U,V)X_1) - \frac{\psi}{(n-1)}[g(V,X_1)\eta(U) - g(U,X_1)\eta(V)]. \tag{4.20}$$

Now using equation(3.2) and equation(2.10) in equation (4.20), we get

$$\begin{aligned}\eta(\tilde{M}^*(U,V)X_1) &= \frac{-2\psi - n + 1}{2(n-1)}[g(V,X_1)\eta(U) - g(U,X_1)\eta(V)] \\ &- \frac{1}{2(n-1)}[S(V,X_1)\eta(U) - S(U,X_1)\eta(V)].\end{aligned} \tag{4.21}$$

## 5. GENERALIZED ∗-PROJECTIVELY SEMI-SYMMETRIC KENMOTSU MANIFOLD

Here, we define the concept of a generalized ∗-projectively semi-symmetric Kenmotsu manifold and obtain certain important results.

**Definition 5.1.** *A Riemannian manifold is termed a semi-symmetric Kenmotsu manifold if it satisfies the condition [2]*

$$R(X_1, X_2).R = 0 \tag{5.1}$$

*where $R(X_1, X_2)$ is regarded as a derivation of the tensor algebra at each point of the manifold.*

**Definition 5.2.** *A Kenmotsu manifold $\tilde{M}^*$ is called generalized ∗- projectively semi-symmetric if it satisfies the condition*

$$R(X_1, X_2).\tilde{M}^* = 0, \tag{5.2}$$

*where $\tilde{M}^*$ is the generalized ∗-projective curvature tensor of Kenmotsu manifold and $R(X_1, X_2)$ is regarded as a derivation of the tensor algebra at each point of the Kenmotsu manifold.*

**Theorem 5.3.** *Every generalized ∗-projective semi-symmetric Kenmotsu manifold is an $\eta$-Einstein manifold.*

Proof:Assume that $(R(\xi, X_1).\tilde{M}^*)(U,V)X_2 = 0$,for arbitrary vector fields $X_1, X_2, U$ and $V$, where $\tilde{M}^*$ denotes the generalized ∗-projective curvature tensor of Kenmotsu manifold, then we get

$$\begin{aligned}&R(\xi, X_1).\tilde{M}^*(U,V)X_2 - \tilde{M}^*(R(\xi, X_1)U, V)X_2 - \tilde{M}^*(U, R(\xi, X_1)V)X_2 \\ &- \tilde{M}^*(U,V)R(\xi, X_1)X_2 = 0.\end{aligned} \tag{5.3}$$

Using equation(2.11), equation(5.3) becomes

$$\begin{aligned}&\eta(\tilde{M}^*(U,V)X_2)X_1 - \tilde{M}^*(U,V,X_2,X_1)\xi - \eta(U)\tilde{M}^*(X_1,V)X_2\\&+ g(X_1,U)\tilde{M}^*(\xi,V)X_2 - \eta(V)\tilde{M}^*(U,X_1)X_2 + g(X_1,V)\tilde{M}^*(U,\xi)X_2\\&- \eta(X_2)\tilde{M}^*(U,V)X_1 + g(X_1,X_2)\tilde{M}^*(U,V)\xi = 0.\end{aligned} \tag{5.4}$$

By computing the inner product of equation(5.4) with respect to $\xi$, we get

$$\begin{aligned}&\eta(\tilde{M}^*(U,V)X_2)\eta(X_1) - \tilde{M}^*(U,V,X_2,X_1)\eta(\xi) - \eta(U)\eta(\tilde{M}^*(X_1,V)X_2)\\&+ g(X_1,U)\eta(\tilde{M}^*(\xi,V)X_2) - \eta(V)\eta(\tilde{M}^*(U,X_1)X_2) + g(X_1,V)\eta(\tilde{M}^*(U,\xi)X_2)\\&- \eta(X_2)\eta(\tilde{M}^*(U,V)X_1) + g(X_1,X_2)\eta(\tilde{M}^*(U,V)\xi) = 0.\end{aligned} \tag{5.5}$$

Now by using equations (2.1),(4.4) and equation(4.21) in equation(4.5), we get

$$\begin{aligned}&2R(U,V,X_2,X_1) = g(V,X_2)g(X_1,U) - g(U,X_2)g(X_1,V)\\&+ \frac{1}{(n-1)}[S(U,X_1)g(V,X_2) - S(V,X_1)g(U,X_2)]\\&- [g(X_1,U)\eta(V) - g(X_1,V)\eta(U)]\eta(X_2)\\&+ \frac{1}{(n-1)}[S(U,X_1)\eta(V) - S(V,X_1)\eta(U)]\eta(X_2).\end{aligned} \tag{5.6}$$

Consider $\{e_i : 1,2,3...n\}$ be orthonormal basis vectors. Substituting $U = X_1 = e_i$ in equation(5.6) and summing over $i$, $1 \leq i \leq n$, we have

$$S(V,X_2) = \frac{r-(n+1)^2}{2n-1}g(V,X_2) - \frac{r+n-1+(n-1)^2}{2n-1}\eta(V)\eta(X_2), \tag{5.7}$$

which can be written as

$$S(V,X_2) = ag(V,X_2) + b\eta(V)\eta(X_2), \tag{5.8}$$

Hence, $M^n$ becomes an $\eta$- Einstein manifold,where

$a$=$\frac{r-(n+1)^2}{2n-1}$,

and

$b$=$\frac{r+n-1+(n-1)^2}{2n-1}$,

## 6. GENERALIZED $*$-PROJECTIVELY RICCI SEMI-SYMMETRIC KENMOTSU MANIFOLD

This section defines a generalized $*$-projectively Ricci semi-symmetric Kenmotsu manifold and establishes that it satisfies

$S^2(X_1,V)$=$-(n-1)(-2\psi+n-1)g(X_1,V) - (2\psi-2n+2)S(X_1,V)$.

**Definition 6.1.** *A Riemannian manifold $M^n$ is termed Ricci semi-symmetric [1] if the condition*

$$R(X_1, X_2).S = 0, \tag{6.1}$$

*is satisfied for all vector fields $X_1$ and $X_2$.*

**Definition 6.2.** *A Kenmotsu manifold is called a generalized projectively Ricci semi-symmetric if it satisfies*

$$\tilde{M}^*(X_1, X_2).S = 0, \tag{6.2}$$

*is satisfied for all vector fields $X_1$ and $X_2$. $\tilde{M}^*$ is the generalized $*$-projective curvature tensor of the Kenmotsu manifold.*

**Theorem 6.3.** *A generalized $*$-projectively Ricci semi-symmetric Kenmotsu manifold satisfies*

$S^2(X_1, V)$=$-(n-1)(n-1+2\psi)g(X_1, V) - 2(2\psi + n - 1)S(X_1, V) + (n-1)^2\eta(X_1)\eta(V) - (n-1)^2\eta(X_1)$.

Proof: Suppose the manifold be a generalized $*$-projectively Ricci semi-symmetric manifold

$(\tilde{M}^*(\xi, X_1).S)(U, V) = 0$, which gives

$$S(\tilde{M}^*(\xi, X_1)U, V) + S(U, \tilde{M}^*(\xi, X_1)V) = 0. \tag{6.3}$$

Now by using equation(4.17) in equation(6.3), we get

$$\begin{aligned}
&\alpha(n-1)[g(X_1, V)\eta(U) + g(X_1, U)\eta(V)] + \alpha[S(X_1, V)\eta(U) + S(X_1, U)\eta(V)] \\
&+ \frac{1}{2}[S(X_1, V)\eta(U) + S(X_1, U)\eta(V)] + \frac{1}{2(n-1)}[S^2(X_1, V)\eta(U) \\
&+ S(U, QX_1)\eta(V)] = 0,
\end{aligned} \tag{6.4}$$

where

$\alpha$=$\frac{n-1-2\psi}{2(n-1)}$. Now putting $U = \xi$ in equation(6.4) and using equations(2.1),(2.4) and equation(2.13),we get

$$\begin{aligned}
&S^2(X_1, V) = -(n-1)(n-1+2\psi)g(X_1, V) - 2(2\psi + n - 1)S(X_1, V) \\
&+ (n-1)^2\eta(X_1)\eta(V) - (n-1)^2\eta(X_1).
\end{aligned} \tag{6.5}$$

## 7. GENERALIZED ∗-PROJECTIVELY $\phi$-SYMMETRIC KENMOTSU MANIFOLD

This section introduces a generalized ∗-projectively locally $\phi$-symmetric Kenmotsu manifold and a generalized ∗-projectively $\phi$-symmetric Kenmotsu manifold, and derives an interesting result.

**Definition 7.1.** *A Riemannian manifold $M^n$ is called locally $\phi$-symmetric [19], if*

$$\phi^2((\nabla_W R)(U,V)X_1) = 0, \tag{7.1}$$

*is satisfied for all vector fields $U$,$V$,$W$ and $X_1$ orthogonal to $\xi$.*

**Definition 7.2.** *A Riemannian manifold $M^n$ is called $\phi$-symmetric [19],if*

$$\phi^2((\nabla_W R)(U,V)X_1) = 0, \tag{7.2}$$

*is satisfied for all vector fields $U, V, W$ and $X_1$.*

**Definition 7.3.** *A Kenmotsu manifold $\tilde{M}^*$ is called generalized ∗-projectively locally $\phi$-symmetric if*

$$\phi^2(\nabla_W \tilde{M}^*(U,V)X_1) = 0, \tag{7.3}$$

*is satisfied for all vector fields $U, V, W$ and $X_1$ orthogonal to $\xi$.*

**Definition 7.4.** *A Kenmotsu manifold is called generalized ∗-projectively $\phi$-symmetric if*

$$\phi^2((\nabla_W \tilde{M}^*)(U,V)X_1) = 0, \tag{7.4}$$

*is satisfied for all vector fields $U, V, W$ and $X_1$.*

**Theorem 7.5.** *A generalized ∗-projectively $\phi$-symmetric Kenmotsu manifold is an $\eta$-Einstein manifold.*

Proof: From using equation(3.3) and equation(4.15), we get

$$\begin{aligned}(\nabla_W \tilde{M}^*)(U,V)X_1 &= (\nabla_W R)(U,V)X_1 - \frac{1}{2(n-1)}[\nabla_W S)(V,X_1)U \\ &- (\nabla_W S)(U,X_1)V) + (\nabla_W Q)Ug(V,X_1) - (\nabla_W Q)Vg(U,X_1)] \\ &- \frac{dr(W)}{(n-1)}[g(V,X_1)U - g(U,X_1)V].\end{aligned} \tag{7.5}$$

Let the manifold be a generalized ∗-projectively $\phi$-symmetric, then by using equation (7.4) and equation(7.5), we get

$$\begin{aligned}&(\nabla_W R)(U,V)X_1 - \frac{1}{2(n-1)}[(\nabla_W S)(V,X_1)U - (\nabla_W S)(U,X_1)V\\&+ (\nabla_W Q)Ug(V,X_1) - (\nabla_W Q)Vg(U,X_1)] - \frac{dr(W)}{(n-1)}[g(V,X_1)U - g(U,X_1)V]\\&= \eta((\nabla_W R)(U,V)X_1 - \frac{1}{2(n-1)}[(\nabla_W S)(V,X_1)\eta(U) - (\nabla_W S)(U,X_1)\eta(V)\\&+ (\nabla_W S)(U,\xi)g(V,X_1) - (\nabla_W S)(V,\xi)g(U,X_1))]\xi - \frac{dr(W)}{(n-1)}[g(V,X_1)\eta(U)\\&- g(U,X_1)\eta(V))\xi]\end{aligned} \tag{7.6}$$

Now, taking the inner product of the equation(7.6) with $X_2$, we get

$$\begin{aligned}&g((\nabla_W R)(U,V)X_1,X_2) - \frac{1}{2(n-1)}[(\nabla_W S)(V,X_1)g(U,X_2)\\&- (\nabla_W S)(U,X_1)g(V,X_2) + (\nabla_W S)g(U,X_2)g(V,X_1) - (\nabla_W S)g(V,X_2)g(U,X_1)]\\&- \frac{dr(W)}{(n-1)}[g(V,X_1)g(U,X_2) - g(U,X_1)g(V,X_2)] = \eta((\nabla_W R)(U,V)X_1)\eta(X_2)\\&- \frac{1}{2(n-1)}[(\nabla_W R)(U,V)X_1)\eta(U) - (\nabla_W S)(U,X_1)\eta(V) + (\nabla_W S)(V,\xi)g(U,X_1))]\eta(X_2)\\&- \frac{dr(W)}{(n-1)}[g(V,X_1)\eta(U) - g(U,X_1)\eta(V)])\eta(X_2).\end{aligned} \tag{7.7}$$

Now, putting $U = X_2 = e_i$ and taking summation over $i$, $1 \leq i \leq n$, we get

$$\begin{aligned}&(\nabla_W S)(V,X_1) - \frac{1}{2(n-1)}[(n-2)(\nabla_W S)(V,X_1) + dr(W)g(V,X_1))]\\&- dr(W)g(V,X_1) = \eta((\nabla_W R)(e_i,V)X_1)\eta(e_i) - \frac{1}{2(n-1)}[(\nabla_W S)(V,X_1)\eta(e_i)\\&- (\nabla_W S)(e_i,X_1)\eta(V) + (\nabla_W S)(e_i,X_1))]\eta(e_i) - \frac{dr(W)}{(n-1)}[g(V,X_1)\eta(e_i)\\&- g(e_i,X_1)\eta(V)])\eta(e_i).\end{aligned} \tag{7.8}$$

Putting $X_1 = \xi$ in above equation and using equation (2.1) and equation (2.4), we get

$$\eta((\nabla_W R)(e_i,V)\xi)\eta(e_i) + \frac{2n-3}{2(n-1)}dr(W)\eta(V) - \frac{n}{2(n-1)}(\nabla_W S)(V,\xi) = 0. \tag{7.9}$$

Now

$$\eta((\nabla_W R)(e_i, V)\xi)\eta(e_i) = g((\nabla_W R)(e_i, V)\xi, \xi)g(e_i, \xi) \tag{7.10}$$

and

$$\begin{aligned} g((\nabla_W R)(e_i, V)\xi, \xi) &= g(\nabla_W R(e_i, V)\xi, \xi) - g(R(\nabla_W e_i, V)\xi, \xi) \\ &- g(R(E_i, \nabla_W V)\xi, \xi) - g(R(e_i, V)\nabla_W \xi, \xi \end{aligned} \tag{7.11}$$

As, $e_i$ is an orthonormal basis, so $\nabla_W e_i = 0$ and using equation(7.10), we get

$$(\nabla_W S)(V, \xi) = \frac{2n-3}{n} dr(W)\eta(V) \tag{7.12}$$

Now putting $V = \xi$ in equation(7.12), we get $dr(W) = 0$, which shows that $r$ is constant. From equation(7.12), we get

$(\nabla_W S)(V, \xi) = 0$, This gives

$$S(V, W) = ag(V, W) + b\eta(V)\eta(W) \tag{7.13}$$

where $a = \frac{r-(n+1)^2}{2n-1}$, and $b$=$\frac{r+n-1+(n-1)^2}{2n-1}$ which shows that $M^n$ is an $\eta$-Einstein Manifold.

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