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A Survey on Summability Methods in Metric Spaces

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Abstract

Summability theory has emerged as an important extension of ordinary convergence theory, particularly in situations where classical methods fail to capture the asymptotic behaviour of sequences. This survey systematically traces the development of statistical convergence and its associated summability methods in metric spaces, emphasizing the transition from ordinary statistical convergence to d-statistical convergence and its generalized forms. Related notions of boundedness and strong summability are also discussed, highlighting the interconnections among these structures within modern summability theory.
Keywords: summability theory, strong Cesàro summability, statistical convergence, metric spaces, lacunary convergence, deferred statistical convergence, modulus functions.

1. INTRODUCTION

Classical convergence plays a central role in mathematical analysis and summability theory. However, many sequences arising in analysis, approximation theory, Fourier analysis and applied mathematics fail to converge in the ordinary sense while still exhibiting a certain asymptotic regularity. This limitation motivated the development of generalized convergence methods capable of capturing asymptotic behaviour beyond ordinary convergence [14, 34, 37], [2].

Among such methods, statistical convergence has become one of the most influential generalizations of classical convergence. The concept originated from ideas related to summability theory and density methods. Early forms appeared in Zygmund's work on strong density for trigonometric series [37], with formal development carried out independently by Fast [14] and Buck [6]. Statistical convergence replaces the ordinary notion of eventual behaviour by natural density-based behaviour [30], allowing a sequence to deviate from its limit on a sparse subset of indices.

Subsequent investigations by Šalát [32], Fridy [15], Tripathy [36] and others established statistical convergence and statistical boundedness as active areas of research within summability theory. Over time, the theory was extended in several directions, including lacunary statistical convergence [16], λ-statistical convergence [28], deferred statistical convergence [2, 26], modulus-based convergence [27, 3], and statistical convergence of order α [9, 8] together with their corresponding notions of statistical boundedness.

A particularly fruitful direction concerns statistical convergence in metric spaces. Here, ordinary absolute difference gives way to the metric structure of the underlying space, yielding d-statistical convergence and related summability methods [24, 25, 5, 20, 22, 29, 35, 13, 11, 21, 12]. These extensions offer a unified framework for studying generalized convergence in abstract spaces.

As outlined in the abstract, this survey traces the evolution from ordinary statistical convergence to its generalized forms in metric spaces, emphasizing the conceptual transitions connecting these structures and their close relationship with modern summability theory.

2. FROM DENSITY TO SUMMABILITY METHODS

The preceding discussion outlined the historical evolution of statistical convergence and its gradual extension within summability theory. We now turn to the fundamental concepts that connect density methods with generalized summability techniques.

At the core of this connection lie three closely related ideas:

- *Density methods*, which measure the asymptotic frequency of certain properties,
- *Summability methods*, which transform or average sequences in order to detect generalized limiting behaviour,
- *Metric structures*, which extend these notions beyond real-valued sequences by replacing ordinary absolute difference with abstract distance functions.

The notion of natural density [30] provides the basic framework for statistical convergence. For a subset $A \subseteq \mathbb{N}$, the natural density $\delta(A)$ measures the proportion of elements of A among the first n natural numbers as $n \to \infty$. In this way, statistical convergence replaces the classical requirement that a property holds eventually with the condition that it holds for "almost all" indices in the density sense.

The interaction between density [2, 30] and summability methods [14, 34, 37, 32, 15] subsequently led to several important generalizations of statistical convergence and summability theory [16, 28, 9, 27, 17, 36, 8, 3, 26, 7]. Classical Cesàro methods [7] introduced averaging processes through arithmetic means of partial sums, whereas deferred methods [2, 26] studied convergence over moving intervals of the form $(p(r), q(r)]$. Similarly, λ-methods [28, 9] generalized summability constructions by replacing ordinary averaging intervals with variable windows determined by sequences $\lambda = (\lambda_r)$ tending to infinity.

Statistical convergence naturally emerges from the combination of these density-based ideas with generalized summability techniques. The following section presents the core definitions systematically, beginning with real-valued sequences and then extending them to metric spaces, where the ordinary expression $|z_m - L|$ is replaced by the metric distance $d(z_m, L)$.

2.1. Core Definitions

Natural density forms the basic foundation of statistical convergence and density-based summability methods.

[[30]] The asymptotic density (or simply natural density) of a subset $M \subseteq \mathbb{N}$ is defined by

$$\delta(M) = \lim_{m\to\infty} \frac{1}{m}\big|\{k \le m : k \in M\}\big|,$$

provided the limit exists finitely.

In particular,

$$\delta(\mathbb{N}) = 1, \qquad \delta(k\mathbb{N}) = \frac{1}{k},$$

while

$$\delta(\mathbb{N}^k) = 0, \qquad \delta(k^{\mathbb{N}}) = 0,$$

where $k \ge 2$ is fixed. Moreover,

$$\delta(\mathbb{N} \setminus M) = 1 - \delta(M).$$

Deferred density generalizes natural density by replacing initial segments with variable intervals.

[[2]] Let $p = (p(r))$ and $q = (q(r))$ be sequences of non-negative integers satisfying

$$0 \leq p(r) < q(r), \qquad q(r) \to \infty.$$

The *deferred density* of $M \subseteq \mathbb{N}$ is defined by

$$\delta_{p,q}(M) = \lim_{r\to\infty} \frac{\big|\{m : p(r) < m \leq q(r),\, m \in M\}\big|}{q(r) - p(r)},$$

provided the limit exists finitely.

If $p(r) = 0$ and $q(r) = r$, then deferred density reduces to ordinary natural density.

A further generalization is obtained through λ-density.

[[28]] Let

$$\lambda = (\lambda_r)$$

be a non-decreasing sequence of positive integers satisfying

$$\lambda_{r+1} \leq \lambda_r + 1, \qquad \lambda_1 = 1, \qquad \lambda_r \to \infty.$$

The class of all such sequences is denoted by Λ.

For $\lambda = (\lambda_r) \in \Lambda$, define

$$I_r = [\, r - \lambda_r + 1, r\,].$$

The λ-*density* of $M \subseteq \mathbb{N}$ is defined by

$$\delta_\lambda(M) = \lim_{r\to\infty} \frac{1}{\lambda_r}\big|\{m \in I_r : m \in M\}\big|,$$

provided the limit exists finitely.

If $\lambda_r = r$, then λ-density reduces to ordinary natural density.

Lacunary sequences provide another important density structure associated with sparse intervals.

[[16]] A sequence

$$\theta = (k_r)$$

is called a *lacunary sequence* if

$$k_0 = 0 < k_1 < k_2 < \cdots, \qquad h_r = k_r - k_{r-1} \to \infty.$$

The intervals

$$I_r = (k_{r-1}, k_r]$$

are called *lacunary intervals*, while

$$q_r = \frac{k_r}{k_{r-1}}, \qquad r \geq 2,$$

are called the *lacunary quotients*.

Generalized summability methods naturally arise from these density structures.

[[2]] For a sequence $z = (z_m)$, the *deferred Cesàro mean* is defined by

$$C_{p(r),q(r)}(z) = \frac{1}{q(r) - p(r)} \sum_{m=p(r)+1}^{q(r)} z_m.$$

[[28]] A sequence $z = (z_m)$ is said to be *strongly* (V, λ)-*summable* to L if

$$\lim_{r\to\infty} \frac{1}{\lambda_r} \sum_{m\in I_r} |z_m - L| = 0.$$

The class of all such sequences is denoted by $[V, \lambda]$. If $\lambda_r = r$, this reduces to strong Cesàro summability $[C, 1]$.

Modulus functions further generalize density and summability methods.

[[27, 3]] A function

$$f : [0, \infty) \to [0, \infty)$$

is called a *modulus function* if:

1. $f(x) = 0 \iff x = 0$,
2. $f(x + y) \leq f(x) + f(y)$,
3. f is increasing,
4. f is right-continuous at 0.

Examples include

$$f(x) = x^p \ (p > 0), \qquad f(x) = 1 - e^{-x}.$$

Statistical convergence forms the central notion connecting density methods with summability theory.

[[15]] A sequence $z = (z_m)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$,

$$\delta\big(\{m \in \mathbb{N} : |z_m - L| \geq \varepsilon\}\big) = 0.$$

In this case, we write

$$st\text{-}\lim z_m = L.$$

[[36]] A sequence $z = (z_m)$ is said to be *statistically bounded* if there exists $K > 0$ such that

$$\delta\big(\{m \in \mathbb{N} : |z_m| \geq K\}\big) = 0.$$

The above concepts collectively provide the fundamental framework underlying generalized statistical convergence and summability theory. These density-based structures and averaging methods form the basis for subsequent extensions to metric spaces, deferred convergence methods, and generalized d-statistical convergence techniques.

3. d-STATISTICAL CONVERGENCE IN METRIC SPACES

The extension of statistical convergence from real-valued sequences to metric-valued sequences represents a significant development in modern summability theory. By replacing the ordinary absolute difference with a metric distance function, statistical convergence can be studied within abstract metric spaces, thereby enlarging the scope of convergence analysis beyond classical settings. This approach led to the introduction of d-statistical convergence and d-statistical boundedness together with several related summability structures in metric spaces.

Küçükaslan and Değer [24, 25] systematically investigated d-statistical convergence and d-statistical boundedness in metric spaces, emphasizing their relationship with ordinary convergence and boundedness. Their work established several important connections between statistical convergence methods and classical metric-space theory. In particular, they studied the interplay between ordinary boundedness and d-statistical boundedness, showing that neither notion necessarily implies the other in general metric spaces. They also investigated conditions under which the collection of all bounded sequences coincides with the class of all d-statistically bounded sequences.

Further, the authors examined the relationship between ordinary convergence and d-statistical convergence, including the behaviour of subsequences and the role of equivalent metrics inducing the same topology. Their results show that statistical convergence in metric spaces preserves many structural properties of ordinary convergence while also introducing new asymptotic phenomena associated with density-based methods.

The following definitions present the fundamental notions associated with statistical convergence in metric spaces. Let (Z, d) be a metric space, and let $\omega(Z)$ denote the set of all sequences of points of Z.

A sequence $z = (z_m) \in \omega(Z)$ is said to be *convergent* to a point $a \in Z$ if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(z_m, a) < \varepsilon \qquad \text{for all } m > n_0.$$

A sequence $z = (z_m) \in \omega(Z)$ is said to be d-*statistically convergent* to a point $a \in Z$ if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m} \big|\{k \leq m : d(z_k, a) \geq \varepsilon\}\big| = 0.$$

The class of all d-*statistically convergent* sequences in (Z, d) is denoted by S^d.

A sequence $z = (z_m) \in \omega(Z)$ is said to be bounded if there exist $a \in Z$ and $M > 0$ such that

$$d(z_m, a) < M \qquad \text{for all } m \in \mathbb{N}.$$

A sequence $z = (z_m) \in \omega(Z)$ is said to be d-*statistically bounded* if there exist a point $a \in Z$ and a real number $M > 0$ such that

$$\lim_{m\to\infty} \frac{1}{m} \big|\{k \leq m : d(z_k, a) \geq M\}\big| = 0.$$

The class of all d-*statistically bounded* sequences in (Z, d) is denoted by BS^d.

The above notions provide the basic framework for studying convergence and boundedness in metric spaces through statistical methods. These concepts serve as the basis for several generalized forms of statistical convergence obtained by modifying the underlying density structure or averaging process.

One important direction of generalization is obtained by replacing ordinary natural density with λ-density, leading to the concept of λ-statistical convergence.

3.1. λ-Statistical Convergence in Metric Spaces

The notion of λ-statistical convergence extends ordinary statistical convergence by replacing natural density with λ-density [28]. This generalization provides a broader framework for studying convergence behaviour through variable averaging intervals.

Throughout this subsection, $\lambda = (\lambda_r) \in \Lambda$ and $I_r = [\, r - \lambda_r + 1, r\,]$.

A sequence $z = (z_m)$ of real numbers is said to be λ-*statistically convergent* to a number L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{\lambda_r} \big|\{m \in I_r : |z_m - L| \geq \varepsilon\}\big| = 0.$$

In particular, if $\lambda_r = r$, then λ-statistical convergence reduces to ordinary statistical convergence.

Kayan and Çolak [20] extended these ideas to metric spaces through the introduction of λ_d-statistical convergence, λ_d-statistical boundedness and strong $(V, \lambda)_d$-summability. Their work established several inclusion relations between the corresponding classes of convergent, bounded and summable sequences associated with different choices of $\lambda = (\lambda_r)$. The following definition formalizes the notion of λ_d-statistical convergence in metric spaces.

Let (Z, d) be a metric space. A sequence $z = (z_m) \in \omega(Z)$ is said to be λ_d-*statistically convergent* to a point $a \in Z$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{\lambda_r} \big|\{m \in I_r : d(z_m, a) \geq \varepsilon\}\big| = 0.$$

The class of all λ_d-statistically convergent sequences in (Z, d) is denoted by S^d_λ.

If $z = (z_m)$ is λ_d-statistically convergent to $a \in Z$, then we write $z_m \to a\ [S^d_\lambda]$.

In particular, if $\lambda_r = r$, then S^d_λ reduces to the class S^d of d-statistically convergent sequences.

Let (Z, d) be a metric space. A sequence $z = (z_m) \in \omega(Z)$ is said to be λ_d-*statistically bounded* if there exist a point $a \in Z$ and a real number $M > 0$ such that

$$\lim_{r\to\infty} \frac{1}{\lambda_r} \big|\{m \in I_r : d(z_m, a) \geq M\}\big| = 0.$$

The class of all λ_d-statistically bounded sequences in (Z, d) is denoted by BS^d_λ.

If $\lambda_r = r$, then λ_d-statistical boundedness reduces to ordinary d-statistical boundedness, and the corresponding class is denoted by BS^d.

The associated notion of strong $(V, \lambda)_d$-summability is defined through the generalized de la Vallée-Poussin mean in metric spaces. It is obtained by requiring

$$\lim_{r\to\infty} \frac{1}{\lambda_r} \sum_{m\in I_r} d(z_m, a) = 0.$$

The class of all strongly $(V,\lambda)_d$-summable sequences is denoted by $[V,\lambda]^d$.

When $\lambda_r = r$, this notion reduces to strong $(C,1)_d$-summability.

The above notions extend statistical convergence and summability methods through variable interval structures determined by the sequence λ. Consequently, these methods provide a flexible framework for studying generalized convergence and averaging processes in metric spaces.

Another important generalization of statistical convergence is obtained through the introduction of an additional parameter $\alpha \in (0,1]$, leading to the notion of statistical convergence of order α.

3.2. Statistical Convergence of order α in Metric Spaces

The concept of statistical convergence of order α extends ordinary statistical convergence by modifying the associated density growth through the parameter $\alpha \in (0,1]$.

Let $0 < \alpha \leq 1$. A sequence $z = (z_m)$ of real or complex numbers is said to be *statistically convergent of order* α to a number L if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m^\alpha}\big|\{k \leq m : |z_k - L| \geq \varepsilon\}\big| = 0.$$

For $\alpha = 1$, statistical convergence of order α reduces to ordinary statistical convergence.

Kayan et al. [22] extended these ideas to metric spaces through the notions of d-statistical convergence of order α, d-statistical boundedness of order α, and d-strong p-Cesàro summability of order α. They also investigated inclusion relations between the corresponding classes of convergent, bounded and summable sequences for various values of $\alpha \in (0,1]$.

Let (Z,d) be a metric space and let $0 < \alpha \leq 1$. A sequence $z = (z_m) \in \omega(Z)$ is said to be *d-statistically convergent of order* α to a point $a \in Z$ if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m^\alpha}\big|\{k \leq m : d(z_k, a) \geq \varepsilon\}\big| = 0.$$

If $z = (z_m)$ is d-statistically convergent of order α to $a \in Z$, then we write $S^d_\alpha\text{-}\lim z_m = a$.

The class of all d-statistically convergent sequences of order α in (Z,d) is denoted by S^d_α.

In particular, when $\alpha = 1$, the class S^d_α reduces to the class S^d of ordinary d-statistically convergent sequences.

The corresponding notion of d-statistical boundedness of order α is obtained by requiring the existence of a point $a \in Z$ and a real number $M > 0$ such that

$$\lim_{m\to\infty} \frac{1}{m^\alpha}\big|\{k \le m : d(z_k, a) \ge M\}\big| = 0.$$

The associated class is denoted by BS^d_α.

Similarly, a sequence $z = (z_m) \in \omega(Z)$ is said to be *d-strongly p-Cesàro summable of order α* to a point $a \in Z$ if

$$\lim_{m\to\infty} \frac{1}{m^\alpha} \sum_{k=1}^{m} (d(z_k, a))^p = 0,$$

where $p > 0$.

The class of all d-strongly p-Cesàro summable sequences of order α is denoted by $w^d_{p,\alpha}$.

In each of the above notions, taking $\alpha = 1$ reduces the corresponding concepts to their ordinary d-statistical and strong p-Cesàro summability counterparts.

The above notions generalize ordinary statistical convergence and strong Cesàro summability by incorporating variable rates of density growth through the parameter α. These extensions provide a more refined framework for analyzing convergence and summability behaviour in metric spaces.

3.3. Lacunary Statistical Convergence in Metric Spaces

Lacunary methods provide an important generalization of statistical convergence by replacing ordinary initial segments with lacunary intervals generated through sparse increasing sequences. This approach allows asymptotic behaviour to be analyzed relative to sparse variable interval structures and has led to several extensions of statistical convergence and boundedness in metric spaces.

Throughout this subsection, $\theta = (k_r)$ denotes a lacunary sequence with $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$.

A sequence $z = (z_m)$ of real numbers is said to be *lacunary statistically convergent* to a number $L \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\big|\{m \in I_r : |z_m - L| \ge \varepsilon\}\big| = 0.$$

Using the notion of lacunary sequences, Sengül et al. [35] introduced the concepts of lacunary d-statistical convergence and lacunary d-statistical boundedness in metric spaces. These notions extend d-statistical convergence by replacing ordinary density intervals with lacunary intervals associated with a lacunary sequence.

Let (Z, d) be a metric space. A sequence $z = (z_m) \in \omega(Z)$ is said to be *lacunary d-statistically convergent* (or S_θ^d-convergent) to a point $a \in Z$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \big|\{m \in I_r : d(z_m, a) \geq \varepsilon\}\big| = 0.$$

In this case, we write S_θ^d- $\lim z_m = a$.

The class of all lacunary d-statistically convergent sequences is denoted by S_θ^d.

In particular, if $\theta = (2^r)$, then lacunary d-statistical convergence reduces to ordinary d-statistical convergence.

The corresponding notion of lacunary d-statistical boundedness is obtained by requiring the existence of a point $a \in Z$ and a real number $M > 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \big|\{m \in I_r : d(z_m, a) \geq M\}\big| = 0.$$

The class of all lacunary d-statistically bounded sequences is denoted by BS_θ^d.

In particular, when $\theta = (2^r)$, lacunary d-statistical boundedness reduces to ordinary d-statistical boundedness.

The above notions extend statistical convergence and statistical boundedness through lacunary interval structures, thereby providing a flexible framework for studying asymptotic behaviour in metric spaces. These constructions provide a useful framework for studying asymptotic behaviour through sparse interval structures and their interaction with generalized convergence methods in metric spaces.

3.4. Deferred Statistical Convergence in Metric Spaces

Deferred statistical methods generalize ordinary statistical convergence by replacing natural density with deferred density determined through variable deferred intervals. This approach provides a flexible framework for studying asymptotic behaviour relative to two controlling sequences and has led to several extensions of statistical convergence and summability methods in metric spaces.

Throughout this subsection, $p = (p(r))$ and $q = (q(r))$ denote sequences of non-negative integers satisfying $p(r) < q(r)$ and $q(r) \to \infty \quad$ as $r \to \infty$.

A sequence $z = (z_m)$ of real numbers is said to be *deferred statistically convergent* to a number $L \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{q(r) - p(r)} \left|\{m : p(r) < m \leq q(r),\ |z_m - L| \geq \varepsilon\}\right| = 0.$$

In this case, we write $S_{p,q}\text{-}\lim z_m = L$.

If $p(r) = 0$ and $q(r) = r$, then deferred statistical convergence reduces to ordinary statistical convergence.

Et et al. [11] extended deferred statistical convergence to metric spaces through the notions of deferred d-statistical convergence and deferred strong d-Cesàro summability.

Let (Z, d) be a metric space. A sequence $z = (z_m) \in \omega(Z)$ is said to be *deferred d-statistically convergent* to a point $a \in Z$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{q(r) - p(r)} \left|\{m : p(r) < m \leq q(r),\ d(z_m, a) \geq \varepsilon\}\right| = 0.$$

The corresponding notion of deferred strong d-Cesàro summability is obtained by replacing the cardinality expression with the averaged metric sum

$$\frac{1}{q(r) - p(r)} \sum_{m=p(r)+1}^{q(r)} d(z_m, a).$$

The class of all deferred strongly d-Cesàro summable sequences is denoted by $Dw^d_{p,q}$.

In the above constructions, choosing $p(r) = 0$ and $q(r) = r$, reduces the deferred notions to their ordinary statistical counterparts.

Further refinements of deferred statistical convergence were obtained through the introduction of order-based density conditions involving a parameter $\alpha \in (0, 1]$. In this direction, Et et al. [12] extended deferred statistical convergence of order α and deferred strong Cesàro summability of order α to metric spaces, thereby providing a more flexible framework for analyzing asymptotic behaviour under varying rates of density growth.

Let (Z, d) be a metric space. A sequence $z = (z_m) \in \omega(Z)$ is said to be *deferred d-statistically convergent of order α* to a point $a \in Z$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{(q(r) - p(r))^\alpha} \left|\{m : p(r) < m \leq q(r),\ d(z_m, a) \geq \varepsilon\}\right| = 0.$$

The corresponding class is denoted by $S^{d,\alpha}_{p,q}$.

The corresponding notion of deferred strong d-Cesàro summability is obtained by replacing the cardinality expression with the corresponding averaged metric expression

$$\frac{1}{(q(r)-p(r))^{\alpha}}\sum_{m=p(r)+1}^{q(r)} d(z_m,a).$$

The class of all deferred strongly d-Cesàro summable sequences of order α is denoted by $w_{p,q}^{d,\alpha}$.

When $p(r) = 0$ and $q(r) = r$, these order-based deferred notions reduce to the corresponding ordinary statistical convergence and summability methods of order α. In particular, when $\alpha = 1$, they reduce to the ordinary d-statistical and strong d-Cesàro summability methods.

The above concepts illustrate how deferred averaging processes naturally extend statistical convergence and summability theory through variable interval structures and order-dependent density behaviour. Consequently, these methods provide refined tools for studying generalized asymptotic behaviour in metric spaces.

3.5. Modulus-Based Statistical Convergence in Metric Spaces

Modulus functions provide another important direction in the generalization of statistical convergence by replacing ordinary density measures with generalized growth conditions determined through modulus functions. These methods yield more flexible convergence structures and allow greater flexibility in the study of asymptotic behaviour and summability techniques.

Throughout this subsection, all results are established under the assumption that f is an unbounded modulus function, unless otherwise specified.

A sequence $z = (z_m)$ of real numbers is said to be *f-statistically convergent* to a number $L \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{f\big(|\{k \le m : |z_k - L| \ge \varepsilon\}|\big)}{f(m)} = 0.$$

If $f(x) = x$, then f-statistical convergence reduces to ordinary statistical convergence.

Kayan and Çolak [21] further extended these ideas to metric spaces through the notions of df-statistical convergence, df-statistical boundedness, and df-strong Cesàro summability associated with modulus functions. They also investigated the relationships between the corresponding classes of convergent and summable sequences.

Let (Z, d) be a metric space. A sequence $z = (z_m) \in \omega(Z)$ is said to be *df-statistically convergent* to a point $a \in Z$ if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{f\big(|\{k \le m : d(z_k, a) \ge \varepsilon\}|\big)}{f(m)} = 0.$$

If $z = (z_m)$ is df-statistically convergent to $a \in Z$, then we write $z_m \to a\ [S^{df}]$.

The class of all df-statistically convergent sequences in (Z, d) is denoted by $S^{df}(Z)$.

If $f(x) = x$, then df-statistical convergence reduces to ordinary d-statistical convergence, and the corresponding class is denoted by $S^d(Z)$.

The corresponding notion of df-statistical boundedness is obtained by requiring the existence of a point $a \in Z$ and a real number $M > 0$ such that

$$\lim_{m\to\infty} \frac{f\big(|\{k \le m : d(z_k, a) \ge M\}|\big)}{f(m)} = 0.$$

Similarly, a sequence $z = (z_m) \in \omega(Z)$ is said to be *df-strongly Cesàro summable* to a point $a \in Z$ if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} f\big(d(z_k, a)\big) = 0.$$

The class of all df-strongly Cesàro summable sequences in (Z, d) is denoted by $W^{df}(Z)$.

Here, the modulus function f need not be unbounded. In particular, when $f(x) = x$, the above notions reduce to the corresponding ordinary d-statistical and strong d-Cesàro summability methods.

The above concepts demonstrate how modulus functions naturally extend statistical convergence and summability methods through generalized density-type measures. Consequently, these methods provide refined tools for studying asymptotic behaviour and generalized summability structures in metric spaces.

Recent years have witnessed substantial further developments in the theory of statistical convergence in metric and generalized metric spaces. In particular, the notions of d-statistical convergence, λ_d-statistical convergence, lacunary statistical convergence, deferred statistical convergence, modulus-based convergence and statistical convergence of order α have been extended to several broader settings including fuzzy metric spaces, probabilistic normed spaces, cone metric spaces, partial metric spaces, g-metric spaces, neutrosophic normed spaces, intuitionistic fuzzy normed spaces and generalized difference sequence spaces. Contemporary research has also focused on hybrid structures obtained by combining lacunary methods, deferred averaging processes, modulus functions, generalized difference operators, ideal convergence techniques and order-based density methods within unified frameworks. Significant contributions in these directions may be found in the recent works of Abazari [1], Jan and Jalal [19], Kişi [23], Hossain and Mondal [18], Ersan [10], Nuray [31], Bayram et al. [4], and Sarikaya et al. [33]. These developments considerably broaden the scope of density-based convergence theory and strengthen its interaction with generalized summability methods and abstract metric structures. However, since the primary objective of the present work is to study the transition from statistical convergence to d-statistical convergence and its principal generalized forms in metric spaces, a detailed treatment of these recent hybrid and highly specialized extensions has not been included. Nevertheless, the foundational concepts discussed herein provide the essential framework upon which many of these modern developments are constructed.

4. CONCLUSION

This chapter presented a systematic overview of the evolution of statistical convergence and generalized summability methods from classical density-based approaches to their extensions in metric spaces. Beginning with natural density and its associated summability structures, several generalized forms including deferred density, λ-density, lacunary density, modulus-based density and convergence of order α were discussed in a unified framework.

The chapter further examined the transition from ordinary statistical convergence to d-statistical convergence in metric spaces, together with corresponding notions of boundedness and strong summability. Various extensions such as lacunary d-statistical convergence, deferred d-statistical convergence, λ_d-statistical convergence, modulus-based df-statistical convergence and statistical convergence of order α were also reviewed. These concepts demonstrate how classical convergence theory can be generalized through density structures, variable interval methods, modulus functions and metric-space techniques. Based on these generalizations, several recent and diverse research directions in generalized metric and convergence spaces were also briefly discussed.

Overall, the results surveyed in this chapter highlight the rich interaction between summability theory, density methods and generalized convergence structures. The development of these notions not only broadens the scope of classical convergence analysis but also provides a flexible framework for further investigations in sequence spaces, functional analysis, approximation theory, and generalized summability methods in abstract spaces.

REFERENCES

[1] Abazari, R., 2022, "Statistical Convergence in g-Metric Spaces," *Filomat*, 36(5), pp. 1461–1468. doi:10.2298/FIL2205461A

[2] Agnew, R. P., 1932, "On Deferred Cesaro Means," *Ann. Math.*, 33(3), pp. 413–421. doi:10.2307/1968524

[3] Aizpuru, A., Lipton-Garela, M. C., and Rambla-Barreno, F., 2014, "Density by Moduli and Statistical Convergence," *Quaest. Math.*, 37(4), pp. 525–530. doi:10.2989/16073606.2014.981683

[4] Bayram, E., Bektaş, Ç. A., and Altın, Y., 2024, "On Statistical Convergence of Order α in Partial Metric Spaces," *Georgian Math. J.*, 31(4), pp. 557–565. doi:10.48550/arXiv.2304.02245

[5] Bilalov, B., and Nazarova, T., 2015, "On Statistical Convergence in Metric Spaces," *J. Math. Res.*, 7(1), pp. 37–43. doi:10.5539/jmr.v1n1p37

[6] Buck, R. C., 1953, "Generalized Asymptotic Density," *Amer. J. Math.*, 75, pp. 335–346. doi:10.2307/2372456

[7] Cesàro, E., 1890, "Sur la Multiplication des Séries," *Bull. Sci. Math.*, 14, pp. 114–120.

[8] Çolak, R., 2010, "Statistical Convergence of Order α," in *Modern Methods in Analysis and Its Applications*, Anamaya, New Delhi, India, pp. 121–129.

[9] Çolak, R., and Bektaş, Ç. A., 2011, "λ-Statistical Convergence of Order α," *Acta Math. Sci. Ser. B Engl. Ed.*, 31(3), pp. 953–959. doi:10.1016/S0252-9602(11)60288-9

[10] Ersan, S. A. F. İ. N. A. Z., 2025, "On Generalizations of Statistical Convergence and Lacunary Statistical Convergence in Neutrosophic Normed Spaces," *Math. Notes*, 117(3), pp. 562–573. doi:10.1134/S0001434625030228

[11] Et, M., Çınar, M., and Sengül, H., 2019, "Deferred Statistical Convergence in Metric Spaces," *Conf. Proc. Sci. Technol.*, 2(3), pp. 189–193.

[12] Et, M., Cinar, M., and Kandemir, H. S., 2020, "Deferred Statistical Convergence of Order α in Metric Spaces," *AIMS Math.*, 5(4), pp. 3731–3740. doi:10.3934/math.2020241

[13] Et, M., and Karataş, M., 2019, ”On Lacunary d-Statistical Convergence of Order α,” *AIP Conf. Proc.*, 2086, 030017. doi:10.1063/1.5095102

[14] Fast, H., 1951, ”Sur la Convergence Statistique,” *Colloq. Math.*, 2(3–4), pp. 241–244.

[15] Fridy, J. A., 1985, ”On Statistical Convergence,” *Anly.*, 5(4), pp. 301–313. doi:10.1524/anly.1985.5.4.301

[16] Fridy, J. A., and Orhan, C., 1993, ”Lacunary Statistical Convergence,” *Pac. J. Math.* , 160(1), pp. 43–51. doi:10.2140/pjm.1993.160.43

[17] Fridy, J. A., and Orhan, C., 1997, ”Statistical Limit Superior and Limit Inferior,” *Proc. Amer. Math. Soc.* , 125(12), pp. 3625–3631. doi:10.1090/S0002-9939-97-04000-8

[18] Hossain, N., and Mondal, R., 2024, ”Certain Aspects of Deferred Statistical Convergence of Sequences in Probabilistic Normed Spaces,” *arXiv Preprint*, arXiv:2408.06362. doi:10.48550/arXiv.2408.06362

[19] Jan, A. H., and Jalal, T., 2024, ”On Lacunary Δ^m-Statistical Convergence in g-Metric Space,” *Korean J. Math.*, 32(1), pp. 109–120.

[20] Kayan, E., and Çolak, R., 2017, ”λ_d-Statistical Convergence, λ_d-Statistical Boundedness and Strong $(V, \lambda)_d$-Summability in Metric Spaces,” in *Mathematics and Computing*, Springer, Singapore, 655, pp. 391–403.

[21] Kayan, E., and Çolak, R., 2019, ”df-Statistical Convergence in Connection with a Modulus in Metric Spaces,” *Commun. Stat.-Theory Methods*, 50(10), pp. 2270-2280. doi:10.1080/03610926.2019.1663212

[22] Kayan, E., Çolak, R., and Altın, Y., 2018, ”d-Statistical Convergence of Order α and d-Statistical Boundedness of Order α in Metric Spaces,” *UPB Sci. Bull. A: Appl. Math. Phys.*, 80(4), pp. 229–238.

[23] Kişi, Ö., 2021, ”Ideal Convergence of Sequences in Neutrosophic Normed Spaces,” *J. Intell. Fuzzy Syst.*, 41(2), pp. 2581–2590. doi:10.3233/JIFS-201568

[24] Küçükaslan, M., 2012, ”On Statistical Boundedness of Metric Valued Sequences,” *Eur. j. pure appl. math.*, 5(2), pp. 174–186

[25] Küçükaslan, M., Değer, U., and Dovgoshey, O., 2014, ”On the Statistical Convergence of Metric-Valued Sequences,” *Ukr. Math. J.*, 66(5), pp. 796–805. doi:10.1007/s11253-014-0974-z

[26] Küçükaslan, M., and Yılmaztürk, M., 2016, ”On Deferred Statistical Convergence of Sequences,” *Kyungpook Math. J.*, 56(2), pp. 357–366. doi:10.5666/KMJ.2016.56.2.357

[27] Maddox, I. J., 1986, ”Sequence Spaces Defined by a Modulus,” *Math. Proc. Camb. Philos. Soc.*, 100(1), pp. 161–166. doi:10.1017/S0305004100065968

[28] Mursaleen, M., 2000, ”λ-Statistical Convergence,” *Math. Slovaca*, 50(1), pp. 111–115.

[29] Nabiev, A. A., Savaş, E., and Gürdal, M., 2019, ”Statistically Localized Sequences in Metric Spaces,” *J. Appl. Anal. Comput.*, 9(2), pp. 739–746. doi:10.11948/2156-907X.20180157

[30] Niven, I., Zuckerman, H. S., and Montgomery, H. L., 1991, *An Introduction to the Theory of Numbers*, John Wiley and Sons Inc. ISBN-13:978-0471625469

[31] Nuray, F., 2022, ”Statistical Convergence in Partial Metric Spaces,” *Korean J. Math.*, 30(1), pp. 155–160. doi:10.11568/kjm.2022.30.1.155

[32] Salát, T., 1980, ”On Statistically Convergent Sequences of Real Numbers,” *Math. Slovaca*, 30(2), pp. 139–150.

[33] Sarikaya, S., Kasap, M., Altın, Y., and Altınok, H., 2026, ”On Statistical Convergence of Order α in Partial Cone Metric Spaces,” *Mathematics*, 14(7), 1168. doi:10.3390/math14071168

[34] Schoenberg, I. J., 1959, ”The Integrability of Certain Functions and Related Summability Methods,” *Am. Math. Mon.*, 66(5), pp. 361–375. doi:10.1080/00029890.1959.11989303

[35] Sengül, H., Et, M., and Çakalli, H., 2019, ”Lacunary d-Statistical Convergence and Lacunary d-Statistical Boundedness in Metric Spaces,” *AIP Conf. Proc.*, 2183(1), 050015. doi:10.1063/1.5136153

[36] Tripathy, B. C., 1997, ”On Statistically Convergence and Statistically Bounded Sequences,” *Bull. Malays. Math. Soc*, 20(1), pp. 31–33.

[37] Zygmund, A., 1959, *Trigonometric Series*, Cambridge University Press, Cambridge, UK.