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Uniform boundedness principle in G-fuzzy normed linear spaces

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Abstract

In this paper two types of continuous operators namely G-fuzzy continuous operators and G-fuzzy sequentially continuous operators in G-fuzzy normed linear spaces are defined and relation between them is established. It is shown that G-fuzzy continuous linear operators are G-fuzzy bounded and vice-versa. Some basic results related to G-fuzzy continuous linear operators are studied. A beautiful characterization of complete G-fuzzy normed linear space is given. Finally Uniform boundedness principle in G-fuzzy normed linear space is established.

Keywords: G-fuzzy normed linear space, G-fuzzy continuous operator, G-fuzzy sequentially continuous operator, G-fuzzy bounded linear operator, G-fuzzy operator norm, Uniform boundedness principle

Mathematics Subject Classification: 46S40, 03E72, 47A30

1. INTRODUCTION

Fuzzy metrics and fuzzy norms play key role in the development of fuzzy functional analysis. The most popular definition of fuzzy metric space is due to Kramosil & Michalek[3] ten years later the appearance of the fuzzy set theory. Following the definition of fuzzy metric introduced by Kramosil & Michalek, several authors introduced the concept of fuzzy metrics in different approaches and studied various results in such spaces (for reference please see[5, 7, 17, 18, 20, 23, 27, 29, 31]).
On the other hand A. K. Katsaras[4] first introduced the concept of fuzzy norm in 1984. After that, following this definition, many authors gave the definition of

fuzzy norm in different approaches and developed many results (for reference please see[6, 8, 10, 12, 13, 15, 16, 22, 24, 25, 26, 28, 30, 32, 33, 34]).
The concept of G-metric is introduced by Z. Mustafa & B. Sims[14] in 2006 and definition of Q-fuzzy metric is given by G. P. Sun & K. Yang[17] in 2010. Alternatively K. A. Khan[21] initiated the definition of G-norm in 2014. Following these concepts, S. Chatterejee, T. Bag & S. K. Samanta[28] brought up the idea of G-fuzzy norm in 2018 and studied some results. Same authors[30] also developed some fixed point theorems in such spaces in 2019. Later in 2023, concept of G-fuzzy operator norm is introduced by S. Chatterjee & T. Bag[33].
Uniform boundedness principle(UBP) is one of the four fundamental theorems of functional analysis. Many resarchers developed UBP in normed linear spaces (for fererence please see[1, 9, 11]). In later time UBP was studied in fuzzy settings by many authors e.g.[24, 26]. Recently S. Chatterjee, T. Bag and S. K. Samanta[32] has established UBP in fuzzy normed linear spaces. In this paper, following the concept of Chatterjee, Bag & Samanta we have investigated the UBP in G-fuzzy normed linear space and studied some related results.
This paper is organized in the following manner:
Section 2 consists of some preliminary results. In Section 3, closed ball, G-fuzzy continuous linear operators, G-fuzzy sequentially continuous linear operators in G-fuzzy normed linear spaces are defined and relations between G-fuzzy bounded linear operator, G-fuzzy continuous linear operators, G-fuzzy sequentially continuous linear operators are established. Then some basic theorems are proved in complete G-fuzzy normed linear spaces. In Section 4 uniform boundedness principle in G-fuzzy normed linear spaces is studied.

2. PRELIMINARIES

This section consists of some preliminary results which are used to study various results in this paper. [28] Let X be a linear space over the field of scalars $\mathbb{K}$ and $*$ be a t-norm. Let G_N be a fuzzy subset of $X^3 \times \mathbb{R}$ that satisfies the following conditions:

[$(G_N 1)$]

1. $G_N(x, y, z, t) = 0$ for $t \leq 0$;
2. $G_N(x, y, z, t) = 1\ \forall t > 0$ if and only if $x = y = z = \theta$;
3. $G_N(x, y, z, t) = G_N(p(x, y, z), t)$ where $p(x, y, z)$ is a permutation function of x, y, z;
4. $G_N(\alpha x, \alpha y, \alpha z, t) = G_N(x, y, z, \frac{t}{|\alpha|})$ for every $\alpha(\neq 0) \in \mathbb{K}$;

5. $G_N(x + x', y + y', z + z', s + t) \geq G_N(x, y, z, s) * G_N(x', y', z', t)$;

6. $\lim_{t\to\infty} G_N(x, y, z, t) = 1$;

7. $G_N(x + y, \theta, z, t) \geq G_N(x, y, z, t)$

for every $x, y, z, x', y', z' \in X$ and for every $s, t \in \mathbb{R}$. The triplet $(X, G_N, *)$ is said to be G-fuzzy normed linear space and G_N is said to be G-fuzzy norm on X. [30] Let us consider the linear space $X = \mathbb{R}$ over the field $\mathbb{R}$ and t-norm $*$ defined by $a * b = ab$. We define $G_N : X^3 \times \mathbb{R} \to [0, 1]$ as

$$G_N(x, y, z, t) = \begin{cases} \frac{t}{|x|+|y|+|z|+t}, & \text{if } t > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X, G_N, *)$ is a G-fuzzy normed linear space. [28] Let $(X, G_N, *)$ be a G-fuzzy normed linear space. An open ball $B(x, r, t)$ with center at x, radius $r \in (0, 1)$ and $t > 0$ is defined by the following:

$$B_{G_N}(x, r, t) = \{y \in X : G_N(x - y, y - x, \theta, t) > 1 - r\}.$$

[28] Let $(X, G_N, *)$ be a G-fuzzy normed linear space. Then

$$\tau = \{A \subset X : x \in A \text{ iff } \exists r \in (0, 1) \text{ and } \exists t > 0 \text{ such that } B_{G_N}(x, r, t) \subset A\}$$

is a topology on X. [28] Let $(X, G_N, *)$ be a G-fuzzy normed linear space. Further assume that:
$(G_N 8) : G_N(x, y, z, \cdot)$ is lower semi-continuous for each $(x, y, z) \neq (\theta, \theta, \theta)$;
$*$ is continuous at each point of $[0, 1] \times \{1\}$.
Then $\{B_{G_N}(x, r, t) : x \in X, r \in (0, 1), t > 0\}$ forms a base for τ(Definition 2.4). [28] Let X be a linear space over $\mathbb{K}$(the field of scalars) and θ be the origin of X. Suppose $Q = \{\|\cdot, \cdot, \cdot\|_\alpha : \alpha \in (0, 1)\}$ is a family of mappings from X^3 into $[0, \infty)$. Then Q is called a quasi G-norm family and (X, Q) is called generating space of quasi G-norm family if the following conditions are satisfied:

[(QN1)]

1. $\|x, y, z\|_\alpha = 0 \; \forall \alpha \in (0, 1) \iff x = y = z = \theta$;

2. $\|cx, cy, cz\|_\alpha = |c| \|x, y, z\|_\alpha \; \forall c \in \mathbb{K}$;

3. $\|x, y, z\|_\alpha = \|p(x, y, z)\|_\alpha$ for every $\alpha \in (0, 1)$ where $p(x, y, z)$ is a permutation function of x, y, z ;

4. For any $\alpha \in (0,1), \exists \beta \in (0,\alpha]$ such that $\|x+x', y+y', z+z'\|_\alpha \leq \|x,y,z\|_\beta + \|x',y',z'\|_\beta$;

5. $\|x,y,z\|_\alpha \geq \|x+y, \theta, z\|_\alpha \; \forall \alpha \in (0,1)$;

6. $\|x,y,z\|_\alpha$ is non-increasing for $\alpha \in (0,1)$.

[28] Let $(X, G_N, *)$ be a G-fuzzy normed linear space with $*$ is lower semi-continuous. Let us define $\|x,y,z\|_\alpha = \wedge\{t > 0 : G_N(x,y,z,t) \geq 1-\alpha\}$ and $Q = \{\|\cdot,\cdot,\cdot\|_\alpha : \alpha \in (0,1)\}$. Then (X,Q) is a generating space of quasi G-norm family. [30] Let $(X, G_N, *)$ be a G-fuzzy normed linear space. A sequence $\{x_n\}$ is said to be convergent in $(X, G_N, *)$ if $\exists x \in X$ such that

$$\lim_{l,m,n\to\infty} G_N(x_l - x, x_m - x, x_n - x, t) = 1 \; \forall t > 0.$$

Equivalently $\{x_n\}$ converges to $x \in X$ if for each $\alpha \in (0,1), t > 0, \; \exists n_0 = n_0(\alpha,t) \in \mathbb{N}$ such that

$$G_N(x_l - x, x_m - x, x_n - x, t) > 1 - \alpha \; \forall l,m,n \geq n_0.$$

[30] Let $(X, G_N, *)$ be a G-fuzzy normed linear space. A sequence $\{x_n\}$ is said to be Cauchy in $(X, G_N, *)$ if

$$\lim_{l,m,n\to\infty} G_N(x_l - x_m, x_m - x_n, x_n - x_l, t) = 1 \; \forall t > 0.$$

Equivalently $\{x_n\}$ is Cauchy in $(X, G_N, *)$ if for every $\alpha \in (0,1), t > 0, \exists n_0 = n_0(\alpha,t)$ such that

$$G_N(x_l - x_m, x_m - x_n, x_n - x_l, t) > 1 - \alpha \; \forall l,m,n \geq n_0.$$

[30] Let $(X, G_N, *)$ be a G-fuzzy normed linear space. If every Cauchy sequence in $(X, G_N, *)$ converges in X then X is said to be complete. [30] Let $(X, G_N, *)$ be G-fuzzy normed linear space with $*$ is continuous at $(1,1)$. Suppose $\{x_n\}$ is a sequence in X and $x \in X$. Then the followings are equivalent:

1. $\lim\limits_{n\to\infty} G_N(x_n - x, x_n - x, x_n - x, t) = 1 \; \forall t > 0$;

2. $\lim\limits_{n\to\infty} G_N(x_n - x, \theta, \theta, t) = 1 \; \forall t > 0$;

3. $\lim\limits_{n\to\infty} G_N(x_n - x, x - x_n, \theta, t) = 1 \; \forall t > 0$.

[30] Let $(X, G_N, *)$ be a G-fuzzy normed linear space and $A \subset X$. If $\lim_{n\to\infty} G_N(x_n - x, x_n - x, x_n - x, t) = 1\ \forall t > 0$ for some $x \in A$ and a sequence $\{x_n\}$ of points in A implies $x \in A$, then A is said to be closed in $(X, G_N, *)$. [30] Let $(X, G_N, *)$ be complete G-fuzzy normed linear space with $*$ is continuous at $(1, 1)$. Then every closed set in $(X, G_N, *)$ is complete. [33] Let $(X, G_{N_1}, *_1)$ and $(Y, G_{N_2}, *_2)$ be two G-fuzzy normed linear spaces. A linear operator $T : (X, G_{N_1}, *_1) \to (Y, G_{N_2}, *_2)$ is said to be G-fuzzy bounded if for each $\alpha \in (0, 1)$, $\exists M_\alpha > 0$ such that

$$G_{N_1}\left(x, y, z, \frac{t}{M_\alpha}\right) \geq 1 - \alpha \implies G_{N_2}(Tx, Ty, Tz, s) \geq \alpha\ \forall s > t,\ \forall t > 0.$$

[33] Let $(X, G_{N_1}, *_1)$ and $(Y, G_{N_2}, *_2)$ be two G-fuzzy normed linear spaces. Suppose $T : (X, G_{N_1}, *_1) \to (Y, G_{N_2}, *_2)$ is a G-fuzzy bounded linear operator. Then the followings are equivalent:
(1): $G_{N_1}\left(x, y, z, \frac{t}{M_\alpha}\right) \geq 1 - \alpha \implies G_{N_2}(Tx, Ty, Tz, s) \geq \alpha\ \forall s > t,\ \forall t > 0,$
(2): $\|Tx, Ty, Tz\|^2_{1-\alpha} \leq M_\alpha \|x, y, z\|_\alpha\ \forall x, y, z \in X$
where $\|Tx, Ty, Tz\|_{1-\alpha} = \wedge\{t > 0 : G_{N_2}(Tx, Ty, Tz, t) \geq \alpha\}$ and $\|x, y, z\|_\alpha = \wedge\{s > 0 : G_{N_1}(x, y, z, s) \geq 1 - \alpha\}$ for every $\alpha \in (0, 1)$. [33] The set of all G-fuzzy bounded linear operators from a G-fuzzy normed linear space $(X, G_{N_1}, *_1)$ to a G-fuzzy normed linear space $(Y, G_{N_2}, *_2)$ is denoted by $BGF(X, Y)$. [33] Let $(X, G_{N_1}, *_1)$ and $(Y, G_{N_2}, *_2)$ be two G-fuzzy normed linear spaces. Then $BGF(X, Y)$ is a subspace of $L(X, Y)$ where $L(X, Y)$ is the linear space of all linear operators from X to Y. [33] Let $(X, G_{N_1}, *)$ and $(Y, G_{N_2}, \min)$ be two G-fuzzy normed linear spaces. Let us denote the set of all G-fuzzy bounded linear operators from $(X, G_{N_1}, *)$ to $(Y, G_{N_2}, \wedge)$ by $BGF(X, Y)$. Suppose $\{\|\cdot, \cdot, \cdot\|^1_\alpha : \alpha \in (0, 1)\}$ be a family of quasi-G-norms of G_{N_1} and $\{\|\cdot, \cdot, \cdot\|^2_\alpha : \alpha \in (0, 1)\}$ be a family of semi-G-norms of G_{N_2}. Further we assume that:
$(QN7) : \|x, y, z\|^1_\alpha > 0\ \forall x, y, z \in X$ with $(x, y, z) \neq (\theta, \theta, \theta), \forall \alpha \in (0, 1)$.
We define $\|T_1, T_2, T_3\|_\alpha = \bigvee_{\substack{x,y,z\in X \\ (x,y,z)\neq(\theta,\theta,\theta)}} \sum_{i=1}^{3} \dfrac{\|T_i x, T_i y, T_i z\|^2_\alpha}{\|x, y, z\|^1_{1-\alpha}}$.
Then the function $G_N : \{BGF(X, Y)^3 \times \mathbb{R} \to [0, 1]\}$ defined by

$$G_N(T_1, T_2, T_3, t) = \begin{cases} \vee\{\alpha \in (0, 1) : \|T_1, T_2, T_3\|_{1-\alpha} \leq t, & \text{if } (T_1, T_2, T_3, t) \neq (\bigcirc, \bigcirc, \bigcirc, 0)\} \\ 0, & \text{otherwise} \end{cases}$$

is a G-fuzzy normed linear space with respect to the t-norm "$\min$". Here G_N is called the G-fuzzy operator norm on $BGF(X, Y)$.

3. MAIN RESULT

In this section closed ball, G-fuzzy continuity, G-fuzzy sequential continuity of linear operators are defined and relation between them is established. Then some basic results are studied. Let $(X, G_N, *)$ be a G-fuzzy normed linear space. A closed ball with center at $x \in X$, radius $r \in (0,1)$ and for some $t > 0$ is denoted by $\overline{B_{G_N}(x,r,t)}$ and is defined by

$$\overline{B_{G_N}(x,r,t)} = \{y \in X : G_N(x-y, y-x, \theta, t) \geq 1-r\}.$$

Let $(X, G_{N_1}, *_1)$ and $(Y, G_{N_2}, *_2)$ be two G-fuzzy normed linear spaces. An operator $T : (X, G_{N_1}, *_1) \to (Y, G_{N_2}, *_2)$ is said to be G-fuzzy continuous at $x_0 \in X$ if for each $\alpha \in (0,1)$ and $\epsilon > 0$, $\exists \delta = \delta(\alpha, \epsilon) > 0$ such that

$$G_{N_1}(x-x_0, y-x_0, z-x_0, \delta) \geq \alpha \implies G_{N_2}(Tx-Tx_0, Ty-Tx_0, Tz-Tx_0, \epsilon) \geq 1-\alpha.$$

1.12 Let $(X, G_{N_1}, *_1)$ and $(Y, G_{N_2}, *_2)$ be two G-fuzzy normed linear spaces. Let $T : X \to Y$ be a linear operator. Then T is G-fuzzy continuous iff T is G-fuzzy bounded.

Proof. First we suppose that T is G-fuzzy bounded. Let us take $x_0 \in X$ and $\epsilon > 0$. Then for $\alpha \in (0,1), \exists M_\alpha > 0$ such that

$$G_{N_1}(x-x_0, y-x_0, z-x_0, \frac{\epsilon}{2M_\alpha}) \geq \alpha \implies G_{N_2}(T(x-x_0), T(y-x_0), T(z-x_0), \epsilon) \geq 1-\alpha$$

for every $x, y, z \in X$. This implies that

$$G_{N_1}(x-x_0, y-x_0, z-x_0, \frac{\epsilon}{2M_\alpha}) \geq \alpha \implies G_{N_2}(Tx-Tx_0, Ty-Tx_0, Tz-Tx_0, \epsilon) \geq 1-\alpha.$$

Let us denote $\delta = \frac{\epsilon}{2M_\alpha}$. Then $\delta > 0$. So it follows that

$$G_{N_1}(x-x_0, y-x_0, z-x_0, \delta) \geq \alpha \implies G_{N_2}(Tx-Tx_0, Ty-Tx_0, Tz-Tx_0, \epsilon) \geq 1-\alpha.$$

Hence T is continuous at x_0. Since $x_0 \in X$ is arbitrary, so T is continuous on X. Conversely suppose that T is continuous on X. So T is continuous at $x = \theta$. Then for $\epsilon = 1, \exists \delta = \delta(\alpha, 1) > 0$ such that

$$G_{N_1}(u, v, w, \delta) \geq \alpha \implies G_{N_2}(Tu, Tv, Tw, 1) \geq 1-\alpha \; \forall u, v, w \in X.$$

Let us take $x, y, z \in X$ with $(x, y, z) \neq (\theta, \theta, \theta)$ and $t > 0$. Putting $x = ut, y = vt, z = wt$ we have

$$G_{N_1}(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, \delta) \geq \alpha \implies G_{N_2}(\frac{Tx}{t}, \frac{Ty}{t}, \frac{Tz}{t}, 1) \geq 1-\alpha$$

$$\text{i.e. } G_{N_1}(x, y, z, t\delta) \geq \alpha \implies G_{N_2}(Tx, Ty, Tz, t) \geq 1-\alpha.$$

We know that $G_{N_2}(Tx, Ty, Tz, s) \geq G_{N_2}(Tx, Ty, Tz, t)$ for $s > t$ so from previous we have $G_{N_1}(x, y, z, t\delta) \geq \alpha \implies G_{N_2}(Tx, Ty, Tz, s) \geq 1 - \alpha$ for every $s > t$.
Suppose $(x, y, z) = (\theta, \theta, \theta)$. Then for $s, t > 0$ we have $G_{N_1}(x, y, z, t) = 1 = G_{N_2}(Tx, Ty, Tz, s)$.
This implies that $G_{N_1}(x, y, z, t\delta) \geq \alpha \implies G_{N_2}(Tx, Ty, Tz, s) \geq 1 - \alpha$ for every $\alpha \in (0, 1)$ and for every $s > t > 0$.
Let us denote $M_\alpha = \frac{1}{\delta}$. So $M_\alpha > 0$. Hence for each $\alpha \in (0, 1), \exists M_\alpha$ such that

$$G_{N_1}(x, y, z, \frac{t}{M_\alpha}) \geq 1 - \alpha \implies G_{N_2}(Tx, Ty, Tz, s) \geq \alpha \; \forall s > t > 0.$$

Therefore it follows that T is G-fuzzy bounded. □

Let $(X, G_{N_1}, *_1)$ and $(Y, G_{N_2}, *_2)$ be two G-fuzzy normed linear spaces. An operator $T : X \to Y$ is said to be G-fuzzy sequentially continuous at $x_0 \in X$ if for sequences $\{x_n\}, \{y_n\}, \{z_n\}$ of X with $\lim_{n\to\infty} G_{N_1}(x_n - x_0, y_n - x_0, z_n - x_0, t) = 1 \; \forall t > 0$ implies $\lim_{n\to\infty} G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, t) = 1 \; \forall t > 0$. If T is G-fuzzy sequentially continuous at every $x \in X$ then T is said to be G-fuzzy sequentially continuous on X.
Let $(X, G_{N_1}, *_1)$ and $(Y, G_{N_2}, *_2)$ be two G-fuzzy normed linear spaces. If a linear operator $T : (X, G_{N_1}, *_1) \to (Y, G_{N_2}, *_2)$ is G-fuzzy continuous at $x_0 \in X$ then T is G-fuzzy sequentially continuous at $x_0 \in X$.

Proof. First suppose that T is G-fuzzy continuous at $x_0 \in X$. Let us consider three sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in X such that

$$\lim_{n\to\infty} G_{N_1}(x_n - x_0, y_n - x_0, z_n - x_0, t) = 1 \; \forall t > 0.$$

Let $\alpha \in (0, 1)$ be given. Then for $\epsilon > 0, \exists n_0 = n_0(\alpha, \epsilon) \in \mathbb{N}$ such that

$$G_{N_1}(x_n - x_0, y_n - x_0, z_n - x_0, t) > \alpha \; \forall n \geq n_0.$$

Since T is continuous at $x_0 \in X$, so $\exists \delta = \delta(\alpha, \epsilon) > 0$ such that if $G_{N_1}(x - x_0, y - x_0, z - x_0, \delta) \geq \alpha$ then $G_{N_2}(Tx - Tx_0, Ty - Tx_0, Tz - Tx_0, \epsilon) \geq 1 - \alpha$ for $x, y, z \in X$. So we have, $G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) \geq 1 - \alpha \; \forall n \geq n_0$. Thus it follows that

$$\underline{\lim}_{n\to\infty} G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) \geq 1 - \alpha.$$

Here $\alpha \in (0, 1)$ is arbitrary which implies that

$$\underline{\lim}_{n\to\infty} G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) \geq 1$$
$$\implies \lim_{n\to\infty} G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, \epsilon) = 1.$$

Since $\epsilon > 0$ is arbitrary, so we have

$$\lim_{n\to\infty} G_{N_2}(Tx_n - Tx_0, Ty_n - Tx_0, Tz_n - Tx_0, t) = 1 \,\forall t > 0.$$

Therefore T is G-fuzzy sequentially continuous at x_0. □

Let $(X, G_{N_1}, *)$ be a complete G-fuzzy normed linear space with $*$ is continuous at $(1,1)$. If $X = \bigcup_{i=1}^{\infty} A_i$ then there is at least one $k_0 \in \mathbb{N}$ such that A_{k_0} is not rare in X.

Proof. If possible suppose $X = \bigcup_{i=1}^{\infty} A_i$ where each A_i is rare in X. So $\bar{A}_1 \neq X$. Let us choose $p_1 \in X - \bar{A}_1$.
We consider an open ball $B_1 = B(p_1, r_1, t)$ with $r_1 < \frac{1}{2}$ such that $B_1 \subset \bar{A}_1{}^C$.
By assumption, A_2 is rare in X. So $\bar{A}_2{}^\circ = \phi$. Then $B_1 \subset \bar{A}_2$ does not hold. Hence $B_1 \cap \bar{A}_2{}^C \neq \phi$. Let us take $p_2 \in \bar{A}_2{}^C$ and choose an open ball $B_2 = B(p_2, r_2, t) \subset B(p_1, \frac{r_1}{2}, \frac{t}{2}) \subset B_1$ with $B_2 \subset \bar{A}_2{}^C$. Again A_3 is rare in X, so $B_2 \subset \bar{A}_3$ does not hold. Then $B_2 \cap \bar{A}_3{}^C \neq \phi$. Let us take $p_3 \in \bar{A}_3{}^C$ and choose an open ball $B_3 = B(p_3, r_3, t) \subset B(p_2, \frac{r_2}{2}, \frac{t}{2})$. Proceeding in this manner we obtain a sequence $\{p_k\}$ of points in X and a sequence of open balls $B_k = B(p_k, r_k, t)$ such that $B_k \subset \bar{A}_k{}^C$ and $B_k \subset B(p_{k-1}, \frac{r_{k-1}}{2}, \frac{t}{2}) \subset B_{k-1}$ for each $k \in \mathbb{N}$.
For $m, n \in \mathbb{N}$ and $m > n$ we have, $G_N(p_m - p_n, p_n - p_m, \theta, t) > 1 - r_n > 1 - \frac{1}{2^n}$. So $G_N(p_m - p_n, p_n - p_m, \theta, t) \to 1$ as $m, n \to \infty$. Thus $\{p_k\}$ is Cauchy. Since $(X, G_N, *)$ is complete, so $\exists p \in X$ such that

$$\lim_{n\to\infty} G_N(p_n - p, p - p_n, \theta, t) = 1 \,\forall t > 0.$$

For $m > n$ we have,

$$\begin{aligned} &G_N(p_n - p, p - p_n, \theta, t) \\ \geq &G_N(p_n - p_m, p_m - p_n, \theta, \frac{t}{2}) * G_N(p_m - p, p - p_m, \theta, \frac{t}{2}) \\ \geq &(1 - \frac{r_n}{2}) * G_N(p_m - p, p - p_m, \theta, \frac{t}{2}). \end{aligned}$$

Letting $m \to \infty$ it follows that

$$G_N(p_n - p, p - p_n, \theta, t) \geq 1 - \frac{r_n}{2} > 1 - r_n$$
$$\text{i.e. } G_N(p_n - p, p - p_n, \theta, t) > 1 - r_n.$$

Hence $p \in B(p_n, r_n, t)$ i.e. $p \in B_n$.
Since $B_n \subset \bar{A}_n{}^C$, so $p \in \bar{A}_n{}^C$. Thus $p \notin A_n$. This is true for every $n \in \mathbb{N}$. Consequently $p \notin \bigcup_{i=1}^{\infty} A_i$ i.e. $p \notin X$. Hence we arrive at a contradiction. Therefore our assumption is wrong. □

Let $(X, G_N, *)$ be a G-fuzzy normed linear space that satisfies the following: $G_N(x, y, z, \cdot)$ is continuous at $t = 0$ for $x, y, z \in X$ with $(x, y, z) \neq (\theta, \theta, \theta)$.
Let us define $\|x, y, z\|_\alpha = \wedge\{t > 0 : G_N(x, y, z, t) \geq 1 - \alpha\}$. Then $\|x, y, z\|_\alpha > 0$ for every $\alpha \in (0, 1)$ and for every $x, y, z \in X$ with $(x, y, z) \neq (\theta, \theta, \theta)$.

Proof. If possible suppose $\exists \alpha_0 \in (0, 1)$ and $\exists x_0, y_0, z_0 \in X$ such that $\|x_0, y_0, z_0\|_{\alpha_0} = 0$.

$$\begin{aligned}&\text{Now } \|x_0, y_0, z_0\|_{\alpha_0} = 0\\ &\implies \wedge\{t > 0 : G_N(x_0, y_0, z_0, t) \geq 1 - \alpha_0\} = 0\\ &\implies G_N(x_0, y_0, z_0, t) \geq 1 - \alpha_0 \text{ for any } t > 0.\end{aligned}$$

By definition, $G_N(x_0, y_0, z_0, 0) = 0$. Since $G_N(x_0, y_0, z_0, \cdot)$ is continuous at $t = 0$ so for any $\epsilon > 0,\ \exists \delta > 0$ such that

$$G_N(x_0, y_0, z_0, t) < \epsilon \text{ for every } 0 < t < \delta.$$

Let us take $\epsilon = \frac{1-\alpha_0}{2}$. Then $\exists \delta > 0$ such that $G_N(x_0, y_0, z_0, t) < \epsilon$ for every $0 < t < \delta$.
We choose $0 < t_0 < \delta$. Then $G_N(x_0, y_0, z_0, t_0) < \frac{1-\alpha_0}{2}$.
But already we have, $G_N(x_0, y_0, z_0, t_0) \geq 1 - \alpha_0$.
Thus we arrive at a contradiction. Hence the result holds. □

4. UNIFORM BOUNDEDNESS PRINCIPLE IN G-FUZZY NORMED LINEAR SPACES

In this section uniform boundedness principle in G-fuzzy normed linear spaces is stated and proved with the help of results discussed in Section 3. [**Uniform Boundedness Principle**] Let $(X, G_{N_1}, *_1)$ be a complete G-fuzzy normed linear space with $*_1$ is continuous at $(1, 1)$ and $(Y, G_{N_2}, \min)$ be a G-fuzzy normed linear space. Suppose θ, θ' are zero vectors of X, Y respectively. Further we assume that

- for $x, y, z \in X$ with $(x, y, z) \neq (\theta, \theta, \theta),\ G_{N_1}(x, y, z, \cdot)$ is continuous at $t = 0$.

Suppose $\mathcal{F}$ is a family of G-fuzzy bounded linear operators from $(X, G_{N_1}, *_1)$ to $(Y, G_{N_2}, \min)$. If for each $x \in X,\ \exists M_x > 0$ such that

$$\bigvee_{f \in \mathcal{F}} \bigvee_{\alpha \in (0,1)} [\wedge\{t > 0 : G_{N_2}(f(x), f(x), f(x), t) \geq 1 - \alpha\}] \leq M_x$$

then $\exists \alpha_0 \in (0, 1)$ and $\exists M > 0$ (independent of x) such that

$$\bigvee_{f \in \mathcal{F}} [\wedge\{t > 0 : G_N(f, f, f, t) \geq \alpha_0\}] \leq M$$

where G_N is the G-fuzzy operator norm on $\mathcal{F}$.

Proof. We define

$$A_k = \{x \in X : [\bigvee_{f\in\mathcal{F}} \bigvee_{\alpha\in(0,1)} [\wedge\{t > 0 : G_{N_2}(f(x), f(x), f(x), t) \geq 1 - \alpha\}]] \leq k\}$$

for every $k \in \mathbb{N}$. Let us consider a sequence $\{x_n\}$ of points in A_k such that

$$\lim_{j\to\infty} G_{N_1}(x_j - x, x_j - x, x_j - x, t) = 1 \; \forall t > 0.$$

Since each $f \in \mathcal{F}$ is G-fuzzy bounded, so each $f \in \mathcal{F}$ is G-fuzzy continuous. Then from Theorem 3.5 we have

$$\lim_{j\to\infty} G_{N_2}(f(x_j) - f(x), f(x_j) - f(x), f(x_j) - f(x), t) = 1 \; \forall t > 0 \tag{1}$$

for every $f \in \mathcal{F}$.
Here $x_j \in A_k$ for every j. So

$$\bigvee_{f\in\mathcal{F}} \bigvee_{\alpha\in(0,1)} [\wedge\{t > 0 : G_{N_2}(f(x_j), f(x_j), f(x_j), t) \geq 1 - \alpha\}] \leq k.$$

Thus we have

$$\wedge\{t > 0 : G_{N_2}(f(x_j), f(x_j), f(x_j), t) \geq 1 - \alpha)\} \leq k \; \forall \alpha \in (0, 1), \forall f \in \mathcal{F}. \tag{2}$$

Let $\beta \in (0, 1)$ be fixed. Choose $\gamma \in (0, 1)$ such that $1 - \gamma > 1 - \beta$.
Let us consider $\epsilon > 0$. From (1),

$$\lim_{j\to\infty} G_{N_2}(f(x_j) - f(x), f(x_j) - f(x), f(x_j) - f(x), \epsilon) = 1.$$

This implies that $\exists j_1 \in \mathbb{N}$ such that

$$G_{N_2}(f(x_j) - f(x), f(x_j) - f(x), f(x_j) - f(x), \frac{\epsilon}{2}) > 1 - \gamma \; \forall j \geq j_1, \forall f \in \mathcal{F}. \tag{3}$$

Again from (2),

$$\wedge\{t > 0 : G_{N_2}(f(x_j), f(x_j), f(x_j), t) \geq 1 - \gamma\} \leq k \; \forall f \in \mathcal{F}.$$

Then

$$G_{N_2}(f(x_j), f(x_j), f(x_j), k + \frac{\epsilon}{2}) \geq 1 - \gamma \; \forall f \in \mathcal{F}. \tag{4}$$

Thus from (3) and (4) we have

$$\begin{aligned}
&G_{N_2}(f(x_j), f(x_j), f(x_j), k + \epsilon) \\
\geq &\min\{G_{N_2}(f(x) - f(x_j), f(x) - f(x_j), f(x) - f(x_j), \frac{\epsilon}{2}), G_{N_2}(f(x), f(x), f(x), k + \frac{\epsilon}{2})\} \\
\geq &\min\{(1 - \gamma), (1 - \gamma)\} \\
= &1 - \gamma \\
> &1 - \beta.
\end{aligned}$$

So $G_{N_2}(f(x), f(x), f(x), k+\epsilon) > 1-\beta \, \forall f \in \mathcal{F}$.
Then $\wedge\{t > 0 : G_{N_2}(f(x), f(x), f(x), t) \geq 1-\beta\} \leq k+\epsilon \, \forall f \in \mathcal{F}$.
Here $\beta \in (0,1)$ is arbitrary which implies that

$$\bigvee_{f\in\mathcal{F}} \bigvee_{\alpha\in(0,1)} [\wedge\{t > 0 : G_{N_2}(f(x), f(x), f(x), t) \geq 1-\alpha\}] \leq k+\epsilon.$$

Since $\epsilon > 0$ is taken arbitrarily so it follows that

$$\bigvee_{f\in\mathcal{F}} \bigvee_{\alpha\in(0,1)} [\wedge\{t > 0 : G_{N_2}(f(x), f(x), f(x), t) \geq 1-\alpha\}] \leq k.$$

Hence $x \in A_k$. Thus A_k is closed.
By assumption, each $f \in \mathcal{F}$ is pointwise bounded. As a consequence $X = \bigcup_{i=1}^{n} A_i$.
Again X is complete, so by Theorem 3.6, $\exists k_0 \in \mathbb{N}$ such that A_{k_0} is nowhere dense in X. Then $\exists t_0 > 0, \alpha_1 \in (0,1)$ such that $B(x_0, \alpha_1, t_0) \subset A_{k_0}$. Hence $\exists \alpha_0 \in (0,1)$ such that $B[x_0, \alpha_0, t_0] \subset A_{k_0}$.
Let us take $x, y, z \in X$ with $(x, y, z) \neq (\theta, \theta, \theta)$. We choose $t_1 > 0$ in such a way that

$$t_1 = 2 \wedge \{t > 0 : G_{N_1}(x, y, z, t) \geq 1-\alpha_0\}. \tag{5}$$

Since $*_1$ is continuous at $(1,1)$ so $\exists \beta_0 \in (0,1)$ such that $(1-\beta_0) *_1 (1-\beta_0) *_1 (1-\beta_0) \geq 1-\alpha_0$. Then it is clear that $t_1 \geq t_x + t_y + t_z$ where

$$\begin{aligned} t_x &= \wedge\{t > 0 : G_{N_1}(x, \theta, \theta, t) \geq 1-\beta_0\}, \\ t_y &= \wedge\{t > 0 : G_{N_1}(y, \theta, \theta, t) \geq 1-\beta_0\}, \\ t_z &= \wedge\{t > 0 : G_{N_1}(z, \theta, \theta, t) \geq 1-\beta_0\}. \end{aligned}$$

Set $r = \frac{t_0}{2t_1}$. Clearly $r > 0$. Then $x_0 + rx, x_0 + ry, x_0 + rz \in B[x_0, \alpha_0, t_0]$. We denote $x_1 = x_0 + rx, y_1 = y_0 + ry, z_1 = x_0 + rz$. So $x_1, y_1, z_1 \in B[x_0, \alpha_0, t_0] \subset A_{k_0}$. This implies that

$$\begin{aligned} &\bigvee_{f\in\mathcal{F}} \bigvee_{\alpha\in(0,1)} [\wedge\{t > 0 : G_{N_2}(f(x_1), f(x_1), f(x_1), t) \geq 1-\alpha\}] \leq k_0, \\ &\bigvee_{f\in\mathcal{F}} \bigvee_{\alpha\in(0,1)} [\wedge\{t > 0 : G_{N_2}(f(y_1), f(y_1), f(y_1), t) \geq 1-\alpha\}] \leq k_0, \\ &\bigvee_{f\in\mathcal{F}} \bigvee_{\alpha\in(0,1)} [\wedge\{t > 0 : G_{N_2}(f(z_1), f(z_1), f(z_1), t) \geq 1-\alpha\}] \leq k_0, \\ &\bigvee_{f\in\mathcal{F}} \bigvee_{\alpha\in(0,1)} [\wedge\{t > 0 : G_{N_2}(f(x_0), f(x_0), f(x_0), t) \geq 1-\alpha\}] \leq k_0. \end{aligned}$$

Let $\alpha' \in (0,1)$ and $\epsilon > 0$ be given. Then from above

$$G_{N_2}(f(x_1), f(x_1), f(x_1), k_0 + \frac{\epsilon}{2}) \geq 1 - \alpha', \tag{6}$$
$$G_{N_2}(f(y_1), f(y_1), f(y_1), k_0 + \frac{\epsilon}{2}) \geq 1 - \alpha', \tag{7}$$
$$G_{N_2}(f(z_1), f(z_1), f(z_1), k_0 + \frac{\epsilon}{2}) \geq 1 - \alpha', \tag{8}$$
$$G_{N_2}(f(x_0), f(x_0), f(x_0), k_0 + \frac{\epsilon}{2}) \geq 1 - \alpha'. \tag{9}$$

for every $f \in \mathcal{F}$.
So from (6),(7),(8),(9) we have

$$\begin{aligned}
&G_{N_2}(f(x), f(y), f(z), \frac{6k_0 + 3\epsilon}{3|r|}) \\
=&G_{N_2}(f(3rx), f(3ry), f(3rz), 6k_0 + 3\epsilon) \\
=&G_{N_2}(f(3x_1 - 3x_0), f(3y_1 - 3x_0), f(3z_1 - 3x_0), 6k_0 + 3\epsilon) \\
\geq& \min\{G_{N_2}(f(3x_1), f(3y_1), f(3z_1), 3k_0 + 3\frac{\epsilon}{2}), G_{N_2}(f(3x_0), f(3x_0), f(3x_0), 3k_0 + 3\frac{\epsilon}{2})\} \\
\geq& \min\{G_{N_2}(f(3x_1), \theta', \theta', k_0 + \frac{\epsilon}{2}), G_{N_2}(f(3y_1), \theta', \theta', k_0 + \frac{\epsilon}{2}), G_{N_2}(f(3z_1), \theta', \theta', k_0 + \frac{\epsilon}{2}), \\
&\quad G_{N_2}(f(x_0), f(x_0), f(x_0), k_0 + \frac{\epsilon}{2})\} \\
\geq& \min\{G_{N_2}(f(x_1), f(x_1), f(x_1), k_0 + \frac{\epsilon}{2}), G_{N_2}(f(y_1), f(y_1), f(y_1), k_0 + \frac{\epsilon}{2}), \\
&\quad G_{N_2}(f(z_1), f(z_1), f(z_1), k_0 + \frac{\epsilon}{2}), G_{N_2}(f(x_0), f(x_0), f(x_0), k_0 + \frac{\epsilon}{2})\} \\
\geq& \min\{1 - \alpha', 1 - \alpha', 1 - \alpha', 1 - \alpha'\} \\
=&1 - \alpha'.
\end{aligned}$$

i.e. $G_{N_2}(f(x), f(y), f(z), \frac{2k_0+\epsilon}{|r|}) \geq 1 - \alpha'$.
Therefore $\wedge\{t > 0 : G_{N_2}(f(x), f(y), f(z), t) \geq 1 - \alpha'\} \leq \frac{2k_0+\epsilon}{|r|}$ $\forall f \in \mathcal{F}$.
Here $\alpha' \in (0,1)$ is taken arbitrarily, which implies that

$$\bigvee_{f \in \mathcal{F}} [\wedge\{t > 0 : G_{N_2}(f(x), f(y), f(z), t) \geq \alpha_0\}] \leq \frac{2k_0 + \epsilon}{|r|}.$$

This is true for any $\epsilon > 0$. Hence the following holds:

$$
\begin{aligned}
& \bigvee_{f\in\mathcal{F}} [\wedge\{t>0 : G_{N_2}(f(x),f(y),f(z),t) \geq \alpha_0\}] \leq \frac{2k_0}{|r|} \\
\Longrightarrow\ & \bigvee_{f\in\mathcal{F}} [\wedge\{t>0 : G_{N_2}(f(x),f(y),f(z),t) \geq \alpha_0\}] \leq \frac{2k_0}{t_0}2t_1 \\
\Longrightarrow\ & \bigvee_{f\in\mathcal{F}} [\wedge\{t>0 : G_{N_2}(f(x),f(y),f(z),t) \geq \alpha_0\}] \leq \frac{4k_0}{t_0}t_1 \\
\Longrightarrow\ & \bigvee_{f\in\mathcal{F}} [\wedge\{t>0 : G_{N_2}(f(x),f(y),f(z),t \geq \alpha_0)\}] \leq \frac{4k_0}{t_0}2\|x,y,z\|^1_{\alpha_0} \\
\Longrightarrow\ & \bigvee_{f\in\mathcal{F}} \|f(x),f(y),f(z)\|^2_{1-\alpha_0} \leq \frac{8k_0}{t_0}\|x,y,z\|^1_{\alpha_0} \text{ for } (x,y,z) \neq (\theta,\theta,\theta) \\
\Longrightarrow\ & \bigvee_{f\in\mathcal{F}} \frac{\|f(x),f(y),f(z)\|^2_{1-\alpha_0}}{\|x,y,z\|^1_{\alpha_0}} \leq \frac{8k_0}{t_0} \text{ for } (x,y,z) \neq (\theta,\theta,\theta) \\
\Longrightarrow\ & \frac{\|f(x),f(y),f(z)\|^2_{1-\alpha_0}}{\|x,y,z\|^1_{\alpha_0}} \leq \frac{8k_0}{t_0} \text{ for } (x,y,z) \neq (\theta,\theta,\theta)
\end{aligned}
$$

for every $f \in \mathcal{F}$. Thus we obtain $M_0 > 0$ such that

$$
\begin{aligned}
& \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \frac{\|f(x),f(y),f(z)\|^2_{1-\alpha_0}}{\|x,y,z\|^1_{\alpha_0}} \leq M_0 \\
\Longrightarrow\ & 3 \bigvee_{\substack{x,y,z\in X\\(x,y,z)\neq(\theta,\theta,\theta)}} \frac{\|f(x),f(y),f(z)\|^2_{1-\alpha_0}}{\|x,y,z\|^1_{\alpha_0}} \leq M(=3M_0>0) \\
\Longrightarrow\ & \|f,f,f\|_{1-\alpha_0} \leq M.
\end{aligned}
$$

This is true for every $f \in \mathcal{F}$. As a consequence we obtain the following:

$$\bigvee_{f\in\mathcal{F}}[\wedge\{t>0 : G_N(f,f,f,t) \geq \alpha_0\}] \leq M.$$

□

CONCLUSION

In this paper we have established uniform boundedness principle in G-fuzzy normed linear spaces by assuming some conditions of the underlying t-norm and G-fuzzy norms. For our convenience two different kinds of G-fuzzy continuities of operators are defined and some results about them are studied. Further works like open mapping theorem, closed graph theorem and so on can be carried out in G-fuzzy normed linear spaces.

CONFLICT OF INTEREST

There is no conflict of interest.

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