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New Congruences for 2-Regular Partitions with Designated Summands

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Abstract

The study of partitions with designated summands originated with the work of Andrews, Lewis, and Lovejoy in 2002, who introduced the partition function $PD(n)$. Subsequent investigations have focused on the congruence properties of $PD_k(n)$, the number of k-regular partitions with designated summands. Building upon these developments, we establish several new Ramanuja-type congruence results for $PD_2(n)$ and derive several new infinite families of congruences that generalize and complement previously known results for $PD_2(n)$

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1. INTRODUCTION

Throughout the paper, we assume $|q| < 1$ and for positive integers n, we use the standard notation

$$(a;q)_\circ := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

A partition of a nonnegative integer n is a non-increasing sequence of positive integers known as its parts, whose sum is n. The number of partitions of n is represented by $p(n)$. For example $p(5) = 7$ given as

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)$$

The study of the arithmetic properties of the number $p(n)$ was introduced by Ramanujan[20, 21]. He proved the following three congruences for all $n \geq 0$

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)} = \frac{1}{(q;q)_\infty} = \frac{1}{f_1}$$

where $f_k := (q^k; q^k)_\infty$, for integers $k \geq 0$

Andrews, Lewis, and Lovejoy [1] introduced the partition function $PD(n)$ in 2002 to enumerate partitions with designated summands. Given an ordinary partition of n, a partition with designated summands is formed by distinguishing exactly one occurrence of each distinct part appearing in the partition. The function $PD(n)$ counts the total number of such designated partitions of n. For instance, there are 15 partitions with designated summands corresponding to $n = 5$, namely:

$$5',\ 4'+1',\ 3'+2',\ 3'+1'+1,\ 3'+1+1',\ 2'+2+1',\ 2+2'+1',$$
$$2'+1'+1+1,\ 2'+1+1'+1,\ 2'+1+1+1',\ 1'+1+1+1+1,$$
$$1+1'+1+1+1,\ 1+1+1'+1+1,\ 1+1+1+1'+1,\ 1+1+1+1+1'.$$

A k-regular partition is a partition whose parts are all relatively prime to k, or equivalently, none of its parts is a multiple of k. Let $PD_k(n)$ denote the number of k-regular partitions of n with designated summands. As an illustration, one finds that $PD_2(5) = 8$, given as:

$$\overline{5},\ \overline{3}+\overline{1}+1,\ \overline{3}+1+\overline{1},\ \overline{1}+1+1+1+1,\ 1+\overline{1}+1+1+1,$$
$$1+1+\overline{1}+1+1,\ 1+1+1+\overline{1}+1,\ 1+1+1+1+\overline{1}$$

The study of congruence properties of $PD(n)$ and $PD_k(n)$ has developed considerably following the foundational work of Andrews, Lewis, and Lovejoy [1]. Inspired by Ramanujan's celebrated partition congruences, many authors have established analogous congruence relations for these functions. A substantial body of literature now exists on this topic; see [8, 9, 10, 11, 12, 13, 14, 15, 16] for representative results and further discussion.

Andrews, Lewis and Lovejoy [1, Corollary 3], obtained the generating function of $PD_k(n)$ as follows

$$\sum_{n=0}^{\infty} PD_k(n)\, q^n = \frac{(q^6;q^6)_\infty\ (q^k;q^k)_\infty\ (q^{2k};q^{2k})_\infty\ (q^{3k};q^{3k})_\infty}{(q;q)_\infty\ (q^2;q^2)_\infty\ (q^3;q^3)_\infty\ (q^{6k};q^{6k})_\infty}$$

$$= \frac{f_6 f_k f_{2k} f_{3k}}{f_1 f_2 f_3 f_{6k}} \tag{1}$$

Motivated by the aforementioned works and the rich arithmetic structure exhibited by partitions with designated summands, the present paper focuses on the function $PD_2(n)$. Several new Ramanujan-type congruences satisfied by $PD_2(n)$ are established. In addition, infinitely many families of congruences modulo powers of 2 are derived. The proofs make use of generating function manipulations, q-series identities, and coefficient extraction techniques.

2. PRELIMINARIES

We recall that Ramanujan's general theta function is given by

$$f(a,b) = \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

The functions $\varphi(q)$, $\psi(q)$, and $f(-q)$ are obtained as special cases of Ramanujan's general theta function and, for $|q| < 1$, are given by: [7, p.36, Entry 22],

$$\phi(q) = f(q,q) = \sum_{k=0}^{\infty} q^{k^2} = \frac{f_2^5}{f_1^2 f_4^2} \tag{2}$$

$$\psi(q) = f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{f_2^2}{f_1}, \tag{3}$$

$$f(-q) = f(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = f_1 \tag{4}$$

The classical Jacobi's Identity[5, Theorem 1.3.9]

$$f_1^3 = \sum_{n=0}^{\infty} (2n+1) q^{\frac{n(n+1)}{2}} \tag{5}$$

The Euler's Identity [6, Eq. 1.6.1],

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}. \tag{6}$$

The derivation of the main results relies on several 2- and 3-dissection formulas, which are presented in this section. Furthermore, p-dissection formulas for $\psi(q)$ and f_1 are established.

Lemma 2.1. *[16, Lemma 2.1] The following 2-dissections hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8^8} \tag{7}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}} \tag{8}$$

Lemma 2.2. *The following 2-dissections hold:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3\, f_6^2}{f_2^2\, f_{12}} \ + \ q\,\frac{f_{12}^3}{f_4} \tag{9}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4\, f_6\, f_{12}^2}{f_2^5\, f_8\, f_{24}} \ + \ 2q\,\frac{f_4\, f_6^2\, f_8\, f_{24}}{f_2^4\, f_{12}} \tag{10}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^3\, f_6^3}{f_2^9\, f_{12}^2} \ + \ 3q\,\frac{f_4^2\, f_6\, f_{12}^2}{f_2^7} \tag{11}$$

$$\frac{1}{f_1\, f_3} = \frac{f_8^2\, f_{12}^5}{f_2^2\, f_4\, f_6^4\, f_{24}^2} \ + \ q\,\frac{f_4^5\, f_{24}^2}{f_2^4\, f_6^2\, f_8^2\, f_{12}} \tag{12}$$

Proof. (9) and (10) are equivalent to Eq. (22.7.5) and Eq. (30.9.9) of [6].(11) can be obtained by replacing q by $-q$ in Eq. (27.7.3) of [6]. Also, (12) is Eq. (30.12.3) of [6] □

Lemma 2.3. *[6, Equation (14.8.5)] The following 3-dissections holds:*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} + 4q^3\frac{f_3^2 f_{18}^6}{f_6^2 f_9^3} - 3q f_9^3 \tag{13}$$

Lemma 2.4. *[25, Equation 9] The following 3-dissection holds:*

$$\frac{f_2}{f_1^2} = \frac{f_6^4\, f_9^6}{f_3^8\, f_{18}^3} \ + \ 2q\,\frac{f_6^3\, f_9^3}{f_3^7} \ + \ 4q^2\,\frac{f_6^2\, f_{18}^3}{f_3^6} \tag{14}$$

Lemma 2.5. *[2, Theorem 3.1], and [8, Lemma 2.6], We have*

$$\frac{f_4}{f_1} = \frac{f_{12}\, f_{18}^4}{f_3^3\, f_{36}^2} \ + \ q\,\frac{f_6^2\, f_9^3\, f_{36}}{f_3^4\, f_{18}^2} \ + \ 2q^2\,\frac{f_6\, f_{18}\, f_{36}}{f_3^3} \tag{15}$$

Lemma 2.6. *[17, Theorem 2.1] For any odd prime p, we have*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}},\, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}).$$

Moreover, for $0 \le k \le \frac{p-3}{2}$,

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod 8.$$

Lemma 2.7. *[17, Theorem 2.2] For any prime $p \geq 5$, we have*

$$f_1 = \sum_{\substack{k=-(p-1)/2 \\ k\neq\pm(p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right) + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f_{p^2}.$$

where

$$\frac{\pm p - 1}{6} = \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \frac{-(p-1)}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

Lemma 2.8. *For all primes p and all $k, m, j \geq 1$, we have*

$$f_{pm}^{p^{k-1}j} \equiv f_m^{p^k j} \pmod{p^k} \tag{16}$$

In particular, the following congruences will be used frequently, so we may omit referring to this lemma on many occasions.

$$\begin{aligned} f_m^2 &\equiv f_{2m} \pmod 2 \\ f_m^4 &\equiv f_{2m}^2 \pmod 4 \\ f_m^8 &\equiv f_{2m}^4 \pmod 8 \\ f_m^{p^k} &\equiv f_{pm}^{p^{k-1}} \pmod{p^k} \end{aligned}$$

3. DISSECTION IDENTITIES FOR $PD_2(n)$

The paper begins by establishing several elementary dissection formulas for the generating function of $PD_2(n)$, which serve as the foundation for the subsequent analysis.

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty} PD_2(2n)q^n = \frac{f_4^2 f_6^4}{f_1^2 f_3^2 f_{12}^2} \tag{17}$$

and

$$\sum_{n=0}^{\infty} PD_2(2n+1)q^n = \frac{f_2^6 f_{12}^2}{f_1^4 f_4^2 f_6^2} \tag{18}$$

Proof. Setting $k = 2$ in (1), we have

$$\sum_{n=0}^{\infty} PD_2(n)q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}} \tag{19}$$

Proof of (17) and (18) can be found in [1, Theorem 21], □

Theorem 3.2. *We have*

$$\sum_{n=0}^{\infty} PD_2(3n)q^n = \frac{f_2^2 f_6^4}{f_1^4 f_{12}^2} \tag{20}$$

$$\sum_{n=0}^{\infty} PD_2(3n+1)q^n = \frac{f_2^4 f_3^3 f_{12}}{f_1^5 f_4 f_6^2} \tag{21}$$

$$\sum_{n=0}^{\infty} PD_2(3n+2)q^n = 2\,\frac{f_2^3 f_6 f_{12}}{f_1^4 f_4}. \tag{22}$$

Proof. Using (15) in (19), we have

$$\sum_{n=0}^{\infty} PD_2(n)q^n = \frac{f_6^2}{f_3 f_{12}}\left(\frac{f_{12}\, f_{18}^4}{f_3^3\, f_{36}^2} + q\,\frac{f_6^2\, f_9^3\, f_{36}}{f_3^4\, f_{18}^2} + 2q^2\,\frac{f_6\, f_{18}\, f_{36}}{f_3^3}\right) \tag{23}$$

Extracting the terms that involves q^{3n}, q^{3n+1} and q^{3n+2} from both sides and then dividing by q^0, q and q^2 respectively and replacing q^3 by q we obtain 20, 21 and 22. □

Remark: Above theorem yields the same conclusion as [8, Theorem1.1]. However, our proof relies on different identities and consequently provides an independent derivation of the results.

Theorem 3.3. *We have*

$$\sum_{n=0}^{\infty} PD_2(4n)\,q^n = \frac{f_4^4 f_6^8}{f_1^4 f_3^4 f_{12}^4} + q\frac{f_2^{12} f_{12}^4}{f_1^8 f_4^4 f_6^4} \tag{24}$$

$$\sum_{n=0}^{\infty} PD_2(4n+1)\,q^n = \frac{f_2^{12} f_6^2}{f_1^8 f_3^2 f_4^4} \tag{25}$$

$$\sum_{n=0}^{\infty} PD_2(4n+2)\,q^n = 2\frac{f_2^6 f_6^2}{f_1^6 f_3^2} \tag{26}$$

$$\sum_{n=0}^{\infty} PD_2(4n+3)\,q^n = 4\frac{f_4^4 f_6^2}{f_1^4 f_3^2} \tag{27}$$

Proof. The proof is analogous to that of [4, Theorem 4] and is therefore omitted. □

Theorem 3.4. *We have,*

$$\sum_{n=0}^{\infty} PD_2(6n)q^n = \frac{f_3^4 f_2^{14}}{f_1^{12} f_6^2 f_4^4} \tag{28}$$

$$\sum_{n=0}^{\infty} PD_2(6n+1)q^n = \frac{f_2^3 f_3^2 f_6}{f_1^{10} f_4 f_{12}} + 6\frac{f_2^2 f_3 f_4 f_6^2 f_{12}}{f_1^7} \tag{29}$$

$$\sum_{n=0}^{\infty} PD_2(6n+2)q^n = 2\frac{f_2^{13} f_3 f_6}{f_1^{11} f_4^4} \tag{30}$$

$$\sum_{n=0}^{\infty} PD_2(6n+3)q^n = 4\frac{f_3^4 f_2^2 f_4^4}{f_1^8 f_6^2} \tag{31}$$

$$\sum_{n=0}^{\infty} PD_2(6n+4)q^n = 3\frac{f_2^5 f_6^5}{f_1^8 f_4 f_{12}} + 2\frac{f_2^3 f_3^3 f_4 f_{12}}{f_1^9 f_6^2} \tag{32}$$

$$\sum_{n=0}^{\infty} PD_2(6n+5)q^n = 8\frac{f_2 f_3 f_4^4 f_6}{f_1^7} \tag{33}$$

Proof. Utilizing (8) in (20), we obtain

$$\sum_{n=0}^{\infty} PD_2(3n)q^n = \frac{f_2^2 f_6^4}{f_{12}^2}\Big(\frac{1}{f_1^4}\Big) \tag{34}$$

$$= \frac{f_2^2 f_6^4}{f_{12}^2}\Big(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\Big) \tag{35}$$

Collecting the even and odd power of q from both sides of (35) we arrive at (28) and (31) Invoking (10) and (11) in (21), we obtain

$$\sum_{n=0}^{\infty} PD_2(3n+1)q^n = \frac{f_2^4\, f_{12}}{f_4\, f_6^2}\Big(\frac{f_3}{f_1^3}\Big)\Big(\frac{f_3^2}{f_1^2}\Big)$$

$$= \frac{f_2^4\, f_{12}}{f_4\, f_6^2}\Big(\frac{f_4^3\, f_6^3}{f_2^9\, f_{12}^2} + 3q\,\frac{f_4^2\, f_6\, f_{12}^2}{f_2^7}\Big)\Big(\frac{f_4^4\, f_6\, f_{12}^2}{f_2^5\, f_8\, f_{24}} + 2q\,\frac{f_4\, f_6^2\, f_8\, f_{24}}{f_2^4\, f_{12}}\Big)$$

Collecting the even and odd terms, we arrive at (29) and (32) Utilizing (8) in (22), we have

$$\sum_{n=0}^{\infty} PD_2(3n+2)\, q^n = 2\,\frac{f_2^3 f_6 f_{12}}{f_4}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\,\frac{f_4^2 f_8^4}{f_2^{10}}\right) \tag{36}$$

Finally, collecting the even and odd terms, we arrive at (30) and (33). □

Theorem 3.5. *We have*

$$\sum_{n=0}^{\infty} PD_2(8n+1)q^n = \frac{f_4^{19} f_6^{10}}{f_1^7 f_3^6 f_2^6 f_8^6 f_{12}^4} + \frac{f_2^6 f_4^{11} f_{12}^4}{f_1^{11} f_3^2 f_8^6 f_6^2}\, q + \frac{12\, f_2^2 f_4^2 f_6^4}{f_1^9 f_3^4 f_8^2}\, q + \frac{12\, f_2^{10} f_8^2 f_{12}^4}{f_1^{11} f_3^2 f_4^{15} f_6^2}\, q^2 + \frac{16\, f_2^6 f_8^6 f_6^4}{f_1^9 f_3^4 f_4^{24}}\, q^2 \tag{37}$$

$$\sum_{n=0}^{\infty} PD_2(8n+2)\, q^n = 2\left[\frac{f_2^{12} f_6^{10}}{f_1^{12} f_3^6 f_{12}^4} + q\,\frac{f_2^{24} f_{12}^4}{f_1^{16} f_3^2 f_4^8 f_6^2} + 8q\,\frac{f_2^6 f_4^4 f_6^4}{f_1^{10} f_3^4}\right] \tag{38}$$

$$\sum_{n=0}^{\infty} PD_2(8n+3)\, q^n = 4\left[\frac{f_2^2 f_4^9 f_6^{10}}{f_1^9 f_3^6 f_{12}^4 f_8^2} + q\,\frac{f_2^{14} f_4 f_{12}^4}{f_1^{13} f_3^2 f_6^2 f_8^2} + 4q\,\frac{f_2^{10} f_6^4 f_8^2}{f_1^{11} f_3^4 f_4^8}\right] \tag{39}$$

$$\sum_{n=0}^{\infty} PD_2(8n+5)q^n = 2\Bigg[\frac{f_4^{15} f_6^4}{f_1^9 f_3^4 f_8^6} + \frac{3\, f_4^6 f_6^{10}}{f_1^7 f_3^6 f_2^4 f_8^2 f_{12}^4} + \frac{3\, f_2^8 f_{12}^4}{f_1^{11} f_3^2 f_4^2 f_8^2 f_6^2}\, q + \frac{12\, f_2^4 f_8^2 f_6^4}{f_1^9 f_3^4 f_4^{11}}\, q + \frac{4\, f_8^6 f_6^{10}}{f_1^7 f_3^6 f_4^{20} f_{12}^4}\, q + \frac{4\, f_2^{12} f_8^6 f_{12}^4}{f_1^{11} f_3^2 f_4^{28} f_6^2}\, q^2\Bigg] \tag{40}$$

$$\sum_{n=0}^{\infty} PD_2(8n+6)\, q^n = 4\left[\frac{f_2^{18} f_6^4}{f_1^{14} f_3^4 f_4^4} + 2\,\frac{f_4^8 f_6^{10}}{f_1^8 f_3^6 f_{12}^4} + 2q\,\frac{f_2^{12} f_{12}^4}{f_1^{12} f_3^2 f_6^2}\right] \tag{41}$$

$$\sum_{n=0}^{\infty} PD_2(8n+7)\, q^n = 8\left[\frac{f_2^8 f_4^5 f_6^4}{f_1^{11} f_3^4 f_8^2} + \frac{f_2^4 f_4^3 f_6^{10} f_8^2}{f_1^9 f_3^6 f_{12}^4} + \frac{f_2^{16} f_8^2 f_{12}^4}{f_1^{13} f_3^2 f_4^{12} f_6^2}\right] \tag{42}$$

Proof. Using (7) and (12) in (25) and then extracting the even and odd terms, we arrive at (37) and (40).

Substituting (8) and (12) in (26), and extracting the even and odd terms, we obtain (38) and (41).

In similar fashion using (7) and (12) in (27), we obtain (39) and (42). □

Theorem 3.6. *We have*

$$\sum_{n=0}^{\infty} PD_2(9n)\, q^n = \frac{f_2^{12} f_3^{12}}{f_1^{16} f_4^2 f_6^6} + 16\, q\,\frac{f_2^9 f_3^3 f_6^3}{f_1^{13} f_4^2} \tag{43}$$

$$\sum_{n=0}^{\infty} PD_2(9n+3)\, q^n = 16\, q\,\frac{f_2^8 f_6^6}{f_1^{12} f_4^2} + 4\,\frac{f_2^{11} f_3^9}{f_1^{15} f_4^2 f_6^3} \tag{44}$$

$$\sum_{n=0}^{\infty} PD_2(9n+6)\, q^n = 12\,\frac{f_2^{10} f_3^6}{f_1^{14} f_4^2} \tag{45}$$

Proof. Using (14) in (20), we have

$$\sum_{n=0}^{\infty} PD_2(3n)q^n = \frac{f_6^4}{f_{12}^2}\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\,\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\,\frac{f_6^2 f_{18}^3}{f_3^6}\right)^2 \tag{46}$$

Extracting the terms that involves q^{3n}, q^{3n+1} and q^{3n+2} from both sides, dividing both sides by q^0, q and q^2 and then replacing q^3 by q, we arrive at (43), (44) and (45) □

4. CONGRUENCE RESULTS FOR $PD_2(n)$

Theorem 4.1. *For all* $n \geq 0$, *we have*

$$PD_2(3n+2) \equiv 0 \pmod 2, \tag{47}$$
$$PD_2(4n+r) \equiv 0 \pmod 2, \quad r \in \{2,3\}, \tag{48}$$
$$PD_2(6n+r) \equiv 0 \pmod 2, \quad r \in \{2,3,5\}, \tag{49}$$
$$PD_2(8n+r) \equiv 0 \pmod 2, \quad r \in \{2,3,5,6,7\}, \tag{50}$$
$$PD_2(9n+r) \equiv 0 \pmod 2, \quad r \in \{3,6\}, \tag{51}$$
$$PD_2(4n+3) \equiv 0 \pmod 4, \tag{52}$$
$$PD_2(6n+r) \equiv 0 \pmod 4, \quad r \in \{3,5\}, \tag{53}$$
$$PD_2(8n+r) \equiv 0 \pmod 4, \quad r \in \{3,6,7\}, \tag{54}$$
$$PD_2(9n+r) \equiv 0 \pmod 4, \quad r \in \{3,6\}. \tag{55}$$
$$PD_2(6n+5) \equiv 0 \pmod 8, \tag{56}$$
$$PD_2(8n+7) \equiv 0 \pmod 8. \tag{57}$$

Proof. We omit the details of the proof of the above results as they are straightforward and can be found in Theorem 3.2 -Theorem 3.6. □

Theorem 4.2. *For all* $n \geq 0$, *we have*

$$PD_2(3n) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = \frac{k(k+1)}{2} \text{ for some integer } k, \\ 0 \pmod 2, & \text{otherwise}. \end{cases} \tag{58}$$

$$PD_2(3n+1) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = k(3k+2) \text{ for some integer } k \\ 0 \pmod 2, & \text{otherwise} \end{cases} \tag{59}$$

Proof. From (20) and using Lemma 2.8 we obtain

$$\sum_{n=0}^{\infty} PD_2(3n)q^n \equiv f_1^3 \pmod 2 \tag{60}$$

Employing (5), we obtain

$$\sum_{n=0}^{\infty} PD_2(3n)q^n \equiv \sum_{n=0}^{\infty} q^{k(k+1)/2} \tag{61}$$

which yields (58)
We use the identity [[6], p. 273]

$$\Omega(q) = \sum_{k=-\infty}^{\infty} q^{k(3k+2)} = \frac{f_2^2\, f_3\, f_{12}}{f_1\, f_4\, f_6} \equiv \frac{f_3^3}{f_1} \pmod 2$$

From (21) and Lemma 2.8, we have

$$\sum_{n=0}^{\infty} PD_2(3n+1)q^n = \frac{f_2^4 f_3^3 f_{12}}{f_1^5 f_4 f_6^2} \equiv \frac{f_3^3}{f_1} \pmod 2 \tag{62}$$

$$\equiv \sum_{k=-\infty}^{k=\infty} q^{k(3k+2)} \pmod 2 \tag{63}$$

which yields (59) □

For all $n \geq 0$, we have

$$PD_2(12n+7) \equiv 0 \pmod 2 \tag{64}$$

$$PD_2(12n+10) \equiv 0 \pmod 2 \tag{65}$$

Proof. Since $12n + 7 = 3(4n + 2) + 1$, so if $4n + 2 = k(3k + 2)$, for some integer k, then

$$12n + 7 = 3k(4k + 2) + 1 = (3k + 1)^2 \tag{66}$$

which is not possible as $12n + 7 \equiv 3 \pmod 4$, and there are no squares congruent to 3 $\pmod 4$. Hence (64).
Again,

$$12n + 10 = 3(4n + 3) + 1 \quad \text{so, if} \quad 4n + 3 = k(3k + 2) \quad \text{for some integer } k \tag{67}$$

(68)

Then, we have

$$12n + 10 = (3k + 1)^2 \tag{69}$$

which is not possible as $12n + 10 \equiv 2 \pmod 4$, and there are no squares congruent to 2 $\pmod 4$ Hence, using (59), we have (65) □

For all prime $p > 3$ and all $n \geq 0$, we have

$$PD_2\Big(3(pn + r) + 1\Big) \equiv 0 \pmod 2 \tag{70}$$

if $3r + 1$ is a quadratic non-residue $\pmod p$

Proof. If $pn + r = k(3k+2)$ for some integer k,then $r \equiv k(3k+2) \pmod{p}$. Hence,

$$3r + 1 \equiv 3\big(k(3k+2)\big) + 1 = 9k^2 + 6k + 1 = (3k+1)^2 \pmod{p}. \tag{71}$$

which is a contradiction, as $3r + 1$ is assumed to be a quadratic non-residue modulo p, Hence by (59), (70) holds. Using the above corollary, we can find infinitely many Ramanujan-like congruences, for example, for all $n \geq 0$, the following congruences hold.

$$PD_2(15n + 3r + 1) \equiv 0 \pmod{2}, \quad r \in \{2, 4\}, \tag{72}$$
$$PD_2(21n + 3r + 1) \equiv 0 \pmod{2}, \quad r \in \{3, 4, 6\}, \tag{73}$$
$$PD_2(33n + 3r + 1) \equiv 0 \pmod{2}, \quad r \in \{2, 3, 4, 6, 9\}. \tag{74}$$

□

Theorem 4.3. *for all $n \geq 0$, we have*

$$PD_2(12n + 6) \equiv 0 \pmod{3}, \tag{75}$$
$$PD_2(12n + 10) \equiv 0 \pmod{3}. \tag{76}$$

Proof. Employing Lemma2.8 in (26),, we have

$$\sum_{n=0}^{\infty} PD_2(4n+2)\, q^n \equiv 2\, \frac{f_6^2\, f_6^2}{f_3^2\, f_3^2} \equiv 2\, \frac{f_6^4}{f_3^4} \pmod{3}. \tag{77}$$

We see that the R. H. S of (77) contains a power of q whose exponent is a multiple of 3. Thus extracting the terms that involve q^{3n+1} and q^{3n+2}, we get our results. □

Theorem 4.4. *For all integer $n > 0$, we have*

$$PD_2(3n) \equiv 0 \pmod{4} \tag{78}$$

Proof. Again using Lemma 2.8 in (20),we have

$$\sum_{n=0}^{\infty} PD_2(3n)\, q^n \equiv \frac{f_2^2 f_6^4}{f_1^4 f_{12}^2} \pmod{4} \equiv 1 \pmod{4} \tag{79}$$

Thus, the result follows. □

Theorem 4.5. *For all $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} PD_2\big(16n+2\big)q^n \equiv 2\,\frac{f_2^3}{f_3} \pmod 4 \tag{80}$$

$$\sum_{n=0}^{\infty} PD_2(48n+2)q^n \equiv 2f_1 \pmod 4 \tag{81}$$

$$\sum_{n=0}^{\infty} PD_2(48n+10)q^n \equiv 2f_1 f_4 \pmod 4 \tag{82}$$

$$\sum_{n=0}^{\infty} PD_2(48n+16i+10)q^n \equiv 0 \pmod 4 \quad \textit{for} \quad i \in \{1,2\} \tag{83}$$

Proof. Employing Lemma 2.8 in (38) and, we have

$$\begin{aligned}\sum_{n=0}^{\infty} PD_2(8n+2)\,q^n &\equiv 2\left(\frac{f_2^{12} f_6^{10}}{f_1^{12} f_3^6 f_{12}^4} + q\cdot\frac{f_2^{24} f_{12}^4}{f_1^{16} f_3^2 f_4^8 f_6^2}\right) \pmod 4\\ &\equiv 2\Big(\frac{f_4^3}{f_6} + q f_6^5\Big) \pmod 4 \end{aligned}\tag{84}$$

Upon extracting the even terms from (84), we deduce (80) and extracting the odd terms we deduce

$$\sum_{n=0}^{\infty} PD_2(16n+10)\,q^n \equiv 2f_3^5 \pmod 4 \tag{85}$$

Extracting the terms involving q^{3n}, from both sides of (85), we arrive at (82) Again extracting the terms that involves q^{3n+1} and q^{3n+2} from both sides of (85), we obtain (83).
Further, employing (13) into (80), we deduce

$$\sum_{n=0}^{\infty} PD_2(16n+2)\,q^n \equiv 2\frac{1}{f_3}\Big(f_3^2 - 3q^2 f_{18}^3\Big) \pmod 4 \tag{86}$$

From the q^{3n}-extraction of (86), we deduce (81). □

Theorem 4.6. *Let $p \geq 5$ be a prime with $\left(\frac{-4}{p}\right) = -1$ and $1 \leq u \leq p-1$. Then for all integer $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} PD_2\Big(48.p^{2\alpha}n + 10p^{2\alpha}\Big)q^n \equiv 2f_1 f_4 \pmod 4 \tag{87}$$

$$PD_2\Big(48\cdot p^{2\alpha+1}(pn+u) + 10p^{2\alpha+2}\Big) \equiv 0 \pmod 4 \tag{88}$$

Proof. We see that (82) is the $\alpha = 0$ case of the congruence (87). Assume that (87) holds for some integer $\alpha \geq 0$. Now using Lemma 2.7 in (87), we obtain

$$\sum_{n=0}^{\infty} PD_2\Big(48.p^{2\alpha}n + 10p^{2\alpha}\Big)q^n \equiv 2\Bigg[\sum_{\substack{x=-(p-1)/2 \\ x\neq\pm(p-1)/6}}^{(p-1)/2} q^{(3x^2+x)/2} f\left(-q^{(3p^2+(6x+1)p)/2}, -q^{(3p^2-(6x+1)p)/2}\right)$$
$$+ (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f_{p^2}\Bigg]$$
$$\times \Bigg[\sum_{\substack{y=-(p-1)/2 \\ y\neq\pm(p-1)/6}}^{(p-1)/2} q^{2(3y^2+y)} f\left(-q^{2(3p^2+(6y+1)p)}, -q^{2(3p^2-(6y+1)p)}\right)$$
$$+ (-1)^{(\pm p-1)/6} q^{(p^2-1)/6} f_{4p^2}\Bigg] \tag{89}$$

For prime $p \geq 5$ and $-\frac{p-1}{2} \leq x, y \leq \frac{p-1}{2}$, consider

$$\frac{(3x^2+x)}{2} + 4(3y^2+y) \equiv \frac{5(p^2-1)}{8} \tag{90}$$

This congruence is equivalent to

$$(6x+1)^2 + 4(6y+1)^2 \equiv 0 \pmod{p} \tag{91}$$

For $\left(\frac{-4}{p}\right) = -1$, (91) admits unique solution given by $x = y = \frac{\pm p-1}{6}$.
Extracting the terms that involves $q^{pn+\frac{5(p^2-1)}{8}}$ from both sides of the congruence (89), dividing both sides by $q^{\frac{5(p^2-1)}{8}}$ and then replacing q^p by q, we have

$$\sum_{n=0}^{\infty} PD_2\Big(48.p^{2\alpha}(pn + \frac{5(p^2-1)}{8}) + 10p^{2\alpha}\Big)q^n \equiv 2f_p f_{4p} \pmod{4} \tag{92}$$

which yields

$$\sum_{n=0}^{\infty} PD_2\Big(48.p^{2\alpha+1}n + 10p^{2\alpha+2}\Big)q^n \equiv 2f_p f_{4p} \pmod{4} \tag{93}$$

Again extracting the terms involving q^{pn} from both sides of the congruence (93) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} PD_2\Big(48.p^{2\alpha+2}n + 10p^{2\alpha+2}\Big)q^n \equiv 2f_1 f_4 \pmod{4} \tag{94}$$

This completes the inductive step by verifying the congruence (87) for $\alpha+1$. Accordingly, an application of mathematical induction shows that (87) remains valid for every positive integer α.
It follows by extracting the coefficients associated with q^{pn+u}, where $1 \leq u \leq p-1$, from both sides of (93) that

$$\sum_{n=0}^{\infty} PD_2\Big(16.p^{2\alpha+1}(pn+u)+10p^{2\alpha+2}\Big)q^n \equiv 0 \pmod 4 \tag{95}$$

This completes the proof of the theorem. □

Theorem 4.7. *Let $p \geq 5$ be a prime and $n,\ \beta \in \mathbb{Z}_{\geq 0}$ and $1 \leq u \leq p-1$, we have*

$$\sum_{n=0}^{\infty} PD_2\Big(48\cdot p^{2\alpha}n\ +\ 2p^{3\alpha}\Big)q^n \equiv 2f_1 \pmod 4 \tag{96}$$

$$PD_2\Big(48\cdot p^{2\beta+1}(pn+u)+2p^{3\beta}\Big) \equiv 0 \pmod 4 \tag{97}$$

Proof. An application of Lemma 2.7 to (81) shows that

$$\sum_{n=0}^{\infty} PD_2\Big(48\Big(pn+\frac{p^2-1}{24}\Big)+2\Big)q^n \equiv 2f_p \pmod 4 \tag{98}$$

thereby implying that

$$\sum_{n=0}^{\infty} PD_2\Big(48p^2n+2p^3\Big)q^n \equiv 2f_1 \pmod 4 \tag{99}$$

An inductive argument on α, based on the above relation, leads directly to (96).
Further, combining (96) and equation (98), we conclude that, for all $\alpha \geq 0$.

$$\sum_{n=0}^{\infty} PD_2\Big(48p^{2\alpha+1}n+2p^{3\alpha}\Big)q^n \equiv 2f_p \pmod 4 \tag{100}$$

Consequently,

$$\sum_{n=0}^{\infty} PD_2\Big(48p^{2\alpha+1}(pn+u)+2p^{3\alpha}\Big)q^n \equiv 0 \pmod 4 \tag{101}$$

□

Theorem 4.8. *For all integer $n \geq 0$ and $k \geq 0$, we have*

$$PD_2\Big(4^k(6n+2)\Big) \equiv PD_2\Big(6n+2\Big) \pmod 4. \tag{102}$$

Proof. From (30), we have on utilizing Lemma2.8

$$\sum_{n=0}^{\infty} PD_2(6n+2)q^n \equiv 2\frac{f_2^{13}f_3^3}{f_2^{13}f_1} \pmod 4 \equiv 2\frac{f_3^3}{f_1} \pmod 4 \tag{103}$$

Employing (9), we obtain

$$\sum_{n=0}^{\infty} PD_2(6n+2)q^n \equiv 2\Big(\frac{f_4^3\, f_6^2}{f_2^2\, f_{12}} \;+\; q\,\frac{f_{12}^3}{f_4}\Big) \pmod 4 \tag{104}$$

We extract the terms containing q^{2n} and q^{2n+1} from both sides of (104) dividing both sides by q^0 and q and replacing q^2 with q, we obtain

$$\sum_{n=0}^{\infty} PD_2(12n+2) \equiv 2\frac{f_2^3 f_3^2}{f_1^2 f_3^2} \equiv 2f_2^2 \equiv 2f_4 \pmod 4 \tag{105}$$

and

$$\sum_{n=0}^{\infty} PD_2(12n+8) \equiv 2\frac{f_6^3}{f_2} \pmod 4 \tag{106}$$

Extracting the terms containing q^{2n} and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} PD_2(24n+2) \equiv 2\frac{f_2^3}{f_1} \pmod 4 \tag{107}$$

which implies

$$PD_2\Big(4(6n+2)\Big) \equiv PD_2\Big(6n+2\Big) \tag{108}$$

This shows that the congruence (102) is true for $\alpha = 0$ and $\alpha = 1$.

$$\begin{aligned} PD_2(6n+2) &\equiv PD_2\Big(4(6n+2)\Big) \\ &\equiv PD_2\Big(6(4n+1)+2\Big) \\ &\equiv PD_2\Big(4^2.6n+4.6+4^0.6+2\Big) \end{aligned} \tag{109}$$

Successive iterations of (109) yields

$$\begin{aligned} PD_2(6n+2) &\equiv PD_2\big(4^k.6n+4^{k-1}.6+4^{k-2}.6+ \text{........} +4^0.6+2\big) \\ &\equiv PD_2\Big(4^k.6n+\frac{4^k-1}{3}.6+2\Big) \\ &\equiv PD_2\Big(4^k(6n+2)\Big) \end{aligned} \tag{110}$$

□

Theorem 4.9. *Let $p \geq 5$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. The for all integer $\alpha \geq o$ and $i \in \{1, 2, 3\}$ and $1 \leq u \leq p-1$ we have*

$$PD_2(24n + 6i + 3) \equiv \pmod 8 \tag{111}$$

$$\sum_{n=0}^{\infty} PD_2\Big(24.p^{2\alpha}n + 3.p^{2\alpha}\Big)q^n \equiv 4f_1f_2 \pmod 8 \tag{112}$$

$$PD_2\Big(24.p^{2\alpha+1}(pn+u) + 3 \cdot p^{2\alpha+2}\Big) \equiv 0 \pmod 8 \tag{113}$$

Proof. From (31) and using Lemma2.8, we have

$$\sum_{n=0}^{\infty} PD_2(6n+3)q^n \equiv 4\frac{f_6^2 f_2^{10}}{f_2^4 f_6^2} \pmod 4 \equiv 4f_4^3 \pmod 8 \tag{114}$$

which yields

$$PD_2(24n + 6i + 3)q^n \equiv 0 \pmod 8, \quad where \quad i \in \{1, 2, 3\} \tag{115}$$

By extraction of q^{4n} term from (114) and replacing q^4 by q then utilizing Lemma 2.8, we arrive at

$$\sum_{n=0}^{\infty} PD_2(24n+3)q^n \equiv 4f_1f_2 \pmod 8 \tag{116}$$

(116) is the $\alpha = 0$ case of (112). Assume that the congruence (112) is true for some integer $\alpha \geq o$. Thus, (112) can be expressed as

$$\sum_{n=0}^{\infty} PD_2\Big(24(p^{2\alpha}n + \frac{p^{2\alpha}-1}{8}) + 3\Big)q^n \equiv 4f_1f_2 \pmod 8 \tag{117}$$

Using Lemma 2.7, we obtain

$$\sum_{n=0}^{\infty} PD_2\Big(24(p^{2\alpha}n+\frac{p^{2\alpha}-1}{8}) + 3\Big)q^n \tag{118}$$

$$\equiv 4\Bigg[\sum_{\substack{x=-(p-1)/2 \\ x\neq \pm(p-1)/6}}^{(p-1)/2} q^{(3x^2+x)/2} f\left(-q^{(3p^2+(6x+1)p)/2}, -q^{(3p^2-(6x+1)p)/2}\right)$$
$$+ (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f_{p^2}\Bigg] \times$$
$$\Bigg[\sum_{\substack{y=-(p-1)/2 \\ y\neq \pm(p-1)/6}}^{(p-1)/2} q^{(3y^2+y)} f\left(-q^{(3p^2+(6y+1)p)}, -q^{(3p^2-(6y+1)p)/2}\right)$$
$$+ (-1)^{(\pm p-1)/6} q^{(p^2-1)/12} f_{2p^2}\Bigg] \pmod 8 \tag{119}$$

For a prime $p \geq 5$ and $-\frac{p-1}{2} \leq x,\ y \leq \frac{p-1}{2}$, consider

$$\frac{3x^2+x}{2} + (3y^2+y) \equiv \frac{p^2-1}{8} \tag{120}$$

which implies

$$(6x+1)^2 + 2(6y+1)^2 \equiv 0 \pmod{p} \tag{121}$$

Since it is given $\left(\frac{-2}{p}\right) = -1$, we have $x = y = \frac{\pm p-1}{6}$ is the unique solution of the congruence (119).
Extracting the terms that involves $q^{pn+\frac{(p^2-1)}{8}}$ from both sides of the congruence and then replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} PD_2\Big(24(p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{8}) + 3\Big)q^n \equiv 4f_pf_{2p} \pmod{8} \tag{122}$$

which yields

$$\sum_{n=0}^{\infty} PD_2\Big(24\cdot p^{2\alpha+1}n + 3p^{2\alpha+2}\Big)q^n \equiv 4f_pf_{2p} \pmod{8} \tag{123}$$

Finally collecting the terms containing q^{pn} and replacing q^p by q,we obtain

$$\sum_{n=0}^{\infty} PD_2\Big(24\cdot p^{2\alpha+2}n + 3p^{2\alpha+2}\Big)q^n \equiv 4f_1f_2 \pmod{8} \tag{124}$$

Thus, (112) is true for $\alpha+1$. This completes the proof of (112)
Similarly, extracting the terms that contains q^{pn+u}, where $1 \leq u \leq p-1$ from both sides of (123), we obtain (113) □

Theorem 4.10. *For any prime p with $\left(\frac{-6}{p}\right) = -1$ and $1 \leq u \leq p-1$.Then for all integer $n \geq 0$ and $\alpha \geq 0$, we have*

$$PD_2(24n+17) \equiv 0 \pmod{16} \tag{125}$$

$$\sum_{n=0}^{\infty} PD_2\Big(24\cdot p^{2\alpha}n + 5\cdot p^{2\alpha}\Big)q^n \equiv 8\psi(q)f_2 \pmod{16} \tag{126}$$

$$PD_2\Big(24\cdot p^{2\alpha+1}(pn+u) + 5\cdot p^{2\alpha+2}\Big) \equiv 0 \pmod{16} \tag{127}$$

Proof. From (33) and using Lemma 2.7, we have

$$\begin{aligned}\sum_{n=0}^{\infty} PD_2(6n+5)q^n &\equiv 8\frac{f_2f_3f_2^8f_3^2}{f_1^7} \pmod{16}\\ &\equiv 8\frac{f_2^6f_3^3}{f_1} \pmod{16}\end{aligned} \tag{128}$$

Using (9) in (128), we obtain

$$\sum_{n=0}^{\infty} PD_2(6n+5)q^n \equiv 8f_2^6\Big(\frac{f_4^3\, f_6^2}{f_2^2\, f_{12}} \;+\; q\,\frac{f_{12}^3}{f_4}\Big) \pmod{16} \tag{129}$$

Extracting the terms that involves q^{2n} and q^{2n+1} from both sides of (129), replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} PD_2(12n+5)q^n \equiv 8f_2^5 \pmod{16} \tag{130}$$

$$\sum_{n=0}^{\infty} PD_2(12n+11)q^n \equiv 8f_2^2 f_6^3 \pmod{16} \tag{131}$$

Similarly, extracting the terms that involves q^{2n} and q^{2n+1} from both sides of (130), we have

$$\sum_{n=0}^{\infty} PD_2(24n+5)q^n \equiv 8f_1^5 \pmod{16} \tag{132}$$

$$\sum_{n=0}^{\infty} PD_2(24n+17)q^n \equiv 0 \pmod{16} \tag{133}$$

(132) would immediatly implies

$$\sum_{n=0}^{\infty} PD_2(24n+5)q^n \equiv 8f_2\psi(q) \pmod{16} \tag{134}$$

which is the $\alpha = 0$ case of (126). Assume that the congruence (126) holds for some integer $\alpha > 0$. Utilizing Lemma 2.6 and 2.7, we have

$$\begin{aligned}\sum_{n=0}^{\infty} PD_2\Big(24\cdot p^{2\alpha}n + 5\cdot p^{2\alpha}\Big)q^n \equiv 8\Bigg[\sum_{x=0}^{\frac{p-3}{2}} q^{\frac{x^2+x}{2}} f\left(q^{\frac{p^2+(2x+1)p}{2}},\, q^{\frac{p^2-(2x+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\Bigg]\\ \times\Bigg[\sum_{\substack{y=-(p-1)/2\\ y\neq\pm(p-1)/6}}^{(p-1)/2} q^{(3y^2+y)} f\left(-q^{(3p^2+(6y+1)p)}, -q^{(3p^2-(6y+1)p)}\right)\\ + (-1)^{(\pm p-1)/6} q^{(p^2-1)/12} f_{2p^2}\Bigg]\end{aligned} \tag{135}$$

Consider the congruence

$$\frac{x^2+x}{2} + (3y^2+y) = \frac{5(p^2-1)}{24}$$

which is equivalent to

$$6(2x+1)^2 + (12y+2)^2 \equiv 0 \pmod{p}$$

Since we have $\left(\frac{-6}{p}\right) = -1$, the above congruence has only the solution $x = \frac{p-1}{2}$ and $y = \frac{\pm p-1}{6}$
Therefore, extracting all terms that involves $q^{pn+\frac{5(p^2-1)}{24}}$ from both sides (135) and replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} PD_2\Big(24\cdot p^{2\alpha}(pn+\frac{5(p^2-1)}{24})+5\cdot p^{2\alpha}\Big)q^n \equiv 8f_p\psi(q^p) \pmod{16}$$

which implies

$$\sum_{n=0}^{\infty} PD_2\Big(24.p^{2\alpha+1}n+5.p^{2\alpha+2}\Big)q^n \equiv 8f_p\psi(q^p) \pmod{16} \tag{136}$$

Finally, extracting the terms that include q^{pn} and replacing q^p by q, we have

$$\sum_{n=0}^{\infty} PD_2\Big(24.p^{2\alpha+2}n+5.p^{2\alpha+2}\Big)q^n \equiv 8f_1\psi(q) \pmod{16} \tag{137}$$

Similarly, extracting the terms that involve q^{pn+u} where $1 \le u \le p-1$ from both sides of (136), we obtain the proof of (127) □

Theorem 4.11. *For any prime p with* $\left(\frac{-2}{p}\right) = -1$*. Then for all integers* $n \ge 0$*,* $\beta \ge 0$ *and* $1 \le v \le p-1$*, we have*

$$PD_2(24n+23) \equiv 0 \pmod{16} \tag{138}$$

$$\sum_{n=0}^{\infty} PD_2(24n+11)q^n \equiv 8f_2\psi(q^3) \pmod{16} \tag{139}$$

$$\sum_{n=0}^{\infty} PD_2\big(24.p^{2\beta}n+11p^{2\beta}\big)q^n \equiv 8f_2\psi(q^3) \pmod{16} \tag{140}$$

$$PD_2\Big(24.p^{2\beta+1}(pn+v)+11p^{2\beta}\Big) \equiv 0 \pmod{16} \tag{141}$$

Proof. Extracting the terms that involves q^{2n} and q^{2n+1} from both sides of (131) and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} PD_2(24n+11)q^n \equiv 8f_1^2f_3^3 \pmod{16} \equiv 8f_2\psi(q^3) \pmod{16} \tag{142}$$

$$\text{and} \quad PD_2(24n+23) \equiv 0 \pmod{16} \tag{143}$$

(142) is the $\beta = 0$ case of (140). Assume that the congruence is true for some $\beta > 0$. Utilizing Lemma 2.6 and 2.7 in (140) and then proceeding by similar steps as shown in the theorem.4.10, we obtain

$$\sum_{n=0}^{\infty} PD_2(24.p^{2\beta+1}n + 11p^{2\beta+2})q^n \equiv 8f_{2p}\psi(q^{3p}) \pmod{16} \tag{144}$$

Similarly, extracting the terms that involve q^{pn} and replacing q^p by q we obtain (140). Again, extracting the terms that involve q^{pn+v} from both sides of (144), we arrive at (141). This completes the proof of the Theorem. □

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