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Orthogonality in bicomplex space

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Abstract

A definition of orthogonality in the bicomplex space $_2$ is proposed using the idempotent inner product (IIP), extending the classical concept from $_1$. This formulation is supported by examples, counter-examples, and theorems that highlight properties unique to the bicomplex setting. Orthogonal and orthonormal sets, as well as unit elements, are defined accompanied by examples that clarify the conditions; furthermore, the orthogonal complements of subspaces are investigated, and their geometric representations within the four-dimensional real model of $_2$ are presented. A injective map is established between a circle in $_1$ and a ball of equal radius in $_2$, thereby providing a novel connection between these two spaces. These results lay a foundation for further developments in bicomplex functional analysis and geometry.

Keywords: Bicomplex Numbers, Conjugation, Vector Space and Linear transformation.

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INTRODUCTION

The study of bicomplex numbers has witnessed a steady growth over recent decades, driven by their rich algebraic structure and applications in various branches of mathematics and physics see. The bicomplex space $_2$, generated by two commuting imaginary units, extends the classical complex field $_1$ and offers a broader framework for analysis. Within this setting, fundamental notions from complex analysis can be revisited and generalized, often revealing structural behaviors absent in the classical case. Orthogonality is a cornerstone concept in linear algebra, functional analysis, and geometry, serving as a foundation for inner product spaces, orthonormal bases, and projections. While its properties in $_1$ are well understood, the extension to $_2$ is not straightforward due to the presence of zero divisors, multiple conjugations, and the idempotent decomposition of bicomplex numbers. These features necessitate a re-examination of orthogonality in the bicomplex context. Previous attempts to define orthogonality in $_2$ have either relied on direct analogies with $_1$ or employed partial adaptations of inner products without fully accounting for the idempotent structure. Such approaches, although insightful, are often not sufficiently successful in capturing the full range of geometric and algebraic subtleties present in bicomplex spaces. This motivates the need for a formulation that is both structurally consistent with the bicomplex framework and mathematically robust.

In this paper, we propose a new definition of orthogonality in $_2$ based on the Idempo- tent Inner Product (IIP) [4], a construction that naturally incorporates the decomposition of bicomplex numbers into idempotent components [9]. This approach not only generalizes the classical complex case but also reveals distinct properties unique to $_2$ [6]. The validity and scope of the proposed definition are examined through illustrative examples, counterexamples, and theorems. Furthermore, we investigate and defines of orthogonal and orthonormal sets in $_2$, characterize the unit element within this framework, and explore the orthogonal complements of subspaces. Geometric interpretations are also provided in the four-dimensional real representation of $_2$, offering visual insight into the abstract theory. An additional contribution is the construction of a injective mapping between a circular ring in $_1$ and a ball in $_2$ of equal radius r, establishing a novel link between the two spaces.

This paper is divided into two sections: the first section contains the preliminaries and fundamental concepts, while the second section discusses presents our new research, which is primarily based on the idempotent inner product and the idempotent norm, and provides a systematic study of orthogonality in the classical sense. The results presented here lay a rigorous foundation for further developments in bicomplex functional analysis and geometry, with potential applications in operator theory, spectral analysis, and higher-dimensional complex systems.

1. PRELIMINARIES AND BASIC CONCEPTS

This part of the paper consolidates the fundamental material and notation essential for the theoretical developments that follow. The discussion is kept self-contained so that the necessary background on bicomplex numbers and their connection with the classical complex plane $\mathbb{C}_1$ is readily accessible. Emphasis is placed on those algebraic and structural characteristics that distinguish the bicomplex space $\mathbb{C}_2$ from $\mathbb{C}_1$, as such distinctions form the backbone of the framework established here.

Notation and Algebraic Structure of Bicomplex space: A bicomplex number ξ can be expressed as

$$\xi = z_1 + z_2 i_2, \quad z_1, z_2 \in \mathbb{C}_1,$$

where the imaginary units i_1 and i_2 satisfy $i_1^2 = i_2^2 = -1$ and commute with each other. Their product $i_1 i_2$ behaves as a hyperbolic unit, fulfilling $(i_1 i_2)^2 = 1$. The bicomplex space is defined by

$$\mathbb{C}_2 = \{z_1 + z_2 i_2 \mid z_1, z_2 \in \mathbb{C}_1\},$$

equipped with addition and multiplication carried out componentwise and term wise respectivly over $\mathbb{C}_1$. An another way to express a Bicomplex number is that, those numbers which are of the form :

$$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,$$

all $x_i^\prime s$ are real numbers, $1 \leqslant i \leqslant 4$, $i_p^2 = -1., p = 1, 2$ and $i_1 i_2 \ = \ i_2 i_1$. The symbols $_2, \mathbb{C}_1$, and $\mathbb{C}_0$, denotes to the set of all bicomplex numbers, the set of all complex numbers and the set of all real numbers respectively.The bicomplex space $_2$ is defined in the following forms:

$$\begin{array}{ll} \mathbb{C}_2 & \{x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 \ : \ x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}, \\ \mathbb{C}_2 & \{z_1 + i_2 z_2 \ : \ z_1, z_2 \in \mathbb{C}_1\}, \text{and} \\ \mathbb{C}_2 & \{\xi^-.e_1 + \xi^+.e_2 \ : \ \xi^-, \xi^+ \in_1\}, \text{where } e_1 = \dfrac{1 + i_1 i_2}{2}, e_2 = \dfrac{1 - i_1 i_2}{2}, \end{array}$$

these above three forms are known as real, complex, and idempotent forms of $_2$ respectively. $_2$ contains divisors of zero and therefore $_2$ not forms a field but it forms an algebra over $_1$ with modified consistency of ordinary norm. It is well know that $_2$ is an commutative modified Banach algebra over $_1$ with binary operations "+" which is coordinate wise,"$\times$" term by term and scalar multiplication "." also coordinate wise. so it is easily a vector space over $_1$ with dimension 2 and also over $_0$ with dimension 4 [8]. $_2$ contains exactly four idempotent elements namely $0, 1, e_1$, and e_2 where e_1, e_2 represent two nontrivial idempotent elements they forms a basis of $_2$ over $_1$ due to their linearly independent nature.

Remark 1.1. *[2, 9] The operations of addition and multiplication of bicomplex numbers can be expressed conveniently in their idempotent representation. For $\xi, \eta \in_2$, we have*

$$\begin{aligned} \xi \cdot \eta &= \left(\xi^- \eta^-\right) e_1 + \left(\xi^+ \eta^+\right) e_2, \\ \textit{and} \quad \xi + \eta &= \left(\xi^- + \eta^-\right) e_1 + \left(\xi^+ + \eta^+\right) e_2. \end{aligned}$$

A notable observation is that the multiplication of two bicomplex numbers in the complex form is given by

$$\xi \cdot \eta = (z_1 + i_2 z_2)(w_1 + i_2 w_2) = (z_1 w_1 - z_2 w_2) + i_2(z_1 w_2 + w_1 z_2).$$

Furthermore, scalar multiplication with bicomplex numbers (which is carried out coordinate-wise) is expressed as

$$\alpha \cdot \xi = \alpha\big(\xi^- e_1 + \xi^+ e_2\big) = \alpha(\xi^-)e_1 + \alpha(\xi^+)e_2.$$

Idempotent Basis Representation: A notable property of $\mathbb{C}_2$ is the presence of idempotent elements. Let $e_1 = \frac{1+i_1 i_2}{2}$, $e_2 = \frac{1-i_1 i_2}{2}$, which satisfy $e_1^2 = e_1$, $e_2^2 = e_2$, and $e_1 e_2 = 0$. With these, every $\xi \in \mathbb{C}_2$ admits the unique decomposition

$$\begin{aligned}\xi &= (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2,\\ \xi &= \xi^- e_1 + \xi^+ e_2, \quad \xi^-, \xi^+ \in \mathbb{C}_1,\end{aligned}$$

where complex combinations $(z_1 - i_1 z_2)$ and $(z_1 + i_1 z_2)$ are called idempotent components of ξ [9] and denoted by ξ^- and ξ^+ respectively. These above notations are similar to the notations which are given by prof. R.K.Shrivastwa for convenience of analysis of $_2$ in 2008 [10]. This representation will be a central tool in the sequel, as it simplifies operations and serves as a natural setting for defining the Idempotent Inner Product (IIP).

Comparison between $\mathbb{C}_1$ and $\mathbb{C}_2$: The table below highlights key distinctions:

Feature	$\mathbb{C}_1$ **(Complex)**	$\mathbb{C}_2$ **(Bicomplex)**
Dimension over $\mathbb{R}$	2	4
Imaginary units	One (i_1)	Two (i_1, i_2) plus hyperbolic unit $i_1 i_2$
Zero divisors	None	Present
Idempotent elements	Trivial only 0,1	Four exactly: 0,1e_1, e_2
Conjugations	One	Multiple types (i_1-, i_2-, $(i_1 i_2)$-, etc.)
Geometric model	Plane $\mathbb{R}^2$	Four-dimensional space $\mathbb{R}^4$
Singular elements	Unique	Infinite, all elements of O_2

Norm on Bicomplex Space [1, 5, 6]: Let ξ be an arbitrary element in $_2$ then the norm of ξ is defined and expressed in their three popular form, as fallow

$$\begin{aligned}\|\xi\| &= (|x_1| + |x_2| + |x_3| + |x_4|)^{\frac{1}{2}}\\ &= (|z_1| + |z_2|)^{\frac{1}{2}}\\ &= \big(|\xi^-| + |\xi^+|\big)^{\frac{1}{2}}\end{aligned}$$

These three forms of ordinary (standard) norm are know as real form, complex form and idempotent form respectively on $_2$. The norm consistency of product of two arbitrary bicomplex numbers is as blow

$$\|\xi \cdot \eta\| \leq \sqrt{2}\ \|\xi\| \cdot \|\eta\|$$

Remark 1.2. *[6] By the above inequality of the norm,* $\mathbb{C}_2$ *forms a modified Banach algebra. This inequality is known as the* modified inequality of the norm of the product.

Some Special Subsets of $_2$ [7]: The principal ideals generated by e_1 and e_2 in $_2$ are defined as

$$\begin{array}{rl} \mathbb{I}_1 & \{\xi^- e_1 : \xi \in_2\} = \{\xi e_1 : \xi \in_2\}, \\ \mathbb{I}_2 & \{\xi^+ e_2 : \xi \in_2\} = \{\xi e_2 : \xi \in_2\}, \\ O_2 & \mathbb{I}_1 \cup \mathbb{I}_2, \\ H & \{\, x_1 + i_1 i_2 x_2 : x_1, x_2 \in_0 \}, \\ C(i_1) & \{\, x_1 + i_1 x_2 : x_1, x_2 \in_0 \}, \\ C(i_2) & \{\, x_1 + i_2 x_2 : x_1, x_2 \in_0 \}. \end{array}$$

Singular Elements of $_2$: Let $\xi = z_1 + i_2 z_2$ be a bicomplex number. Then there exists $\eta = w_1 + i_2 w_2 \in_2$ such that $\xi \cdot \eta = 1$. If such an η exists, ξ is called *non-singular*; otherwise, it is called *singular*. There are infinitely many elements in $_2$ that do not possess a multiplicative inverse. The set of all singular elements of $_2$ is denoted by O_2.

Remark 1.3. *[6, 7] A bicomplex number $\xi = z_1 + i_2 z_2$ is singular if and only if*

$$|z_1^2 + z_2^2| = 0.$$

Moreover, if $\xi = z_1 + i_2 z_2$ and $\eta = w_1 + i_2 w_2$ are elements of $_2$, then $\xi \cdot \eta$ is singular if and only if at least one of ξ or η is singular . And $\xi = 0$ if and only if $\xi^-, \xi^+ = 0$ both.

Conjugates in $_2$ [5]: In analogy with the notion of complex conjugation, a bicomplex number ξ admits three standard types of conjugates(ordinary conjugates), corresponding respectively to the independent units i_1, i_2, and $i_1 i_2$. These are denoted by $\overline{\xi}$, $\tilde{\xi}$, and $\overline{\tilde{\xi}}$, and are defined as follows:

$$\begin{aligned} \overline{\xi} &= (x_1 - i_1 x_2) + i_2(x_3 - i_1 x_4) = (\overline{\xi^+})e_1 + (\overline{\xi^-})e_2, \\ \tilde{\xi} &= (x_1 + i_1 x_2) - i_2(x_3 + i_1 x_4) = (\xi^+)e_1 + (\xi^-)e_2, \\ \overline{\tilde{\xi}} &= (x_1 - i_1 x_2) - i_2(x_3 - i_1 x_4) = (\overline{\xi^-})e_1 + (\overline{\xi^+})e_2. \end{aligned}$$

In the context of understanding conjugacy in $\mathbb{C}_2$, it is pertinent to mention that several new conjugates have also come to light, which are collectively referred to as non-ordinary conjugates (see, [3]).

Bicomplex Inner Product Spaces: The space $\mathbb{C}_2, {\mathbb{C}_2}^n$ form inner product spaces with the inner products given definitions below. We have studied inner product spaces formed by $\mathbb{C}_2$ and related spaces using various inner products, and specifically, we have defined idempotent norms on $\mathbb{C}_2$ and related spaces based on idempotent inner products (see, [4]). Motivated by this, it was decided to extend the theory of orthogonality in the classical manner. This study explores these very ideas.

Definition 1.1. [Idempotent Inner Product (IIP) on Bicoplex Space [4]:] Let f be a function defined from $_2 \times {}_2$ to $_1$, that is

$$f :_2 \times_2 \longrightarrow_1,$$

and $\forall\, \xi = \xi^- e_1 + \xi^+ e_2,\ \eta = \eta^- e_1 + \eta^+ e_2 \in \mathbb{C}_2$, then

$$\begin{aligned} f(\xi, \eta) &= f(\xi^- e_1 + \xi^+ e_2,\, \eta^- e_1 + \eta^+ e_2) \\ &= \xi^- \overline{\eta^-} + \xi^+ \overline{\eta^+} \in \mathbb{C}_1. \end{aligned}$$

where bar '$-$' denotes the i_1 conjugate of bicomplex number. Then this f is an inner product on $_2$ over $_1$, called idempotent inner product (IIP) and denote it by $\langle \bullet, \bullet \rangle_{ID}$.

Definition 1.2. [Idempotent Norm on Bicoplex Space] [4]: Let f be a function from $_2 \times_2$ to $_1$ such that f is an idempotent inner product then the value $\sqrt{f(\xi,\xi)_{ID}} = \{|\xi^-|^2 + |\xi^+|^2\}^{\frac{1}{2}}$ is called the **idempotent norm** and is denoted by $||\bullet||_{ID}$ on $_2$.

Remark 1.4. *[4] Since the idempotent norm of a bicomplex number ξ is*

$$\begin{aligned} ||\xi||_{ID} &= \sqrt{\langle \xi, \xi \rangle_{ID}} = \left(|\xi^-|^2 + |\xi^+|^2\right)^{\frac{1}{2}} \\ &= \sqrt{2}||\xi||. \end{aligned}$$

Thus, $||\xi||_{ID} = \sqrt{2}||\xi|| = \sqrt{2}||\xi||_R = \sqrt{2}||\xi||_C$.

Definition 1.3. [Real Inner Product(RIP) on Bicoplex Space] [4]: Let f be a function defined from $_2 \times_2$ into $_1$, that is,

$$f :\ _2 \times_2 \ \longrightarrow \ _1,$$

and if $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,\ \eta = y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4$, are two bicomplex numbers, then

$$\begin{aligned} f(\xi, \eta) &= f(x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,\ y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4) \\ &= x_1 y_1 + i_1 x_2 y_2 + i_2 x_3 y_3 + i_1 i_2 x_4 y_4 \\ &= \sum x_i y_i, \text{ where } \quad 1 \leq i \leq 4, \end{aligned}$$

is called Real Inner Product(RIP) on $\mathbb{C}_2$ and denote it by $\langle \bullet, \bullet \rangle_R$.

Remark 1.5. *[4] Since the real norm of a bicomplex number ξ is*

$$\begin{aligned} ||\xi||_R &= \sqrt{\langle \xi, \xi \rangle_R} = \left(\sum_{i=1}^{4} x_i^2 \right)^{\frac{1}{2}} \\ &= ||\xi||. \end{aligned}$$

Thus, the real norm coincides with the standard norm on $\mathbb{C}_2$.

Definition 1.4. [Complex Inner Product(CIP) on Bicomplex Space] [4]: Let f be a function from ${}_2{\times}_2$ into ${}_1$, and if $\xi = z_1 + i_2 z_2,\ \eta = w_1 + i_2 w_2$ are two bicomplex numbers, then

$$\begin{aligned} f(\xi, \eta) &= f(z_1 + i_2 z_2,\ w_1 + i_2 w_2) \\ &= z_1\, \overline{w_1}\ +\ z_2\, \overline{w_2},\ \text{is called Complex Inner Product(CIP) on } \mathbb{C}_2 \end{aligned}$$

and denoted by $\langle \bullet, \bullet \rangle_C$.

Remark 1.6. *[4] Since the complex norm of a bicomplex number ξ is*

$$\begin{aligned} ||\xi||_C &= \sqrt{\langle \xi, \xi \rangle_C} = \left(|z_1|^2 + |z_2|^2 \right)^{\frac{1}{2}} \\ &= ||\xi||. \end{aligned}$$

Thus, the complex norm coincides with the standard norm on $\mathbb{C}_2$.

Definition 1.5. [Idempotent Inner Products on $\mathbb{C}_2[\xi]_n$] [4]: Let $\mathbb{C}_2[\xi]_n$ represent the set of all bicomplex polynomials of degree at most n; that is,

$$\mathbb{C}_2[\xi]_n \;=\; \{P(\xi) : P(\xi) = a_0 + a_1\xi + \cdots + a_n\xi^n,\ \text{such that } a_0, a_1, \cdots, a_n \in \mathbb{C}_2\}$$

Let $P(\xi), Q(\xi)$ be the arbitrary elements of $\mathbb{C}_2[\xi]_n$ and define the algebraic operations of addition $(+)$ and scalar multiplication $(\cdot)$ on $\mathbb{C}_2[\xi]_n$ as follows:

$$\begin{aligned} P(\xi) + Q(\xi) &= (a_0 + b_0) + (a_1 + b_1)\xi + \cdots + (a_n + b_n)\xi^n, \\ \alpha \cdot P(\xi) &= (\alpha a_0) + (\alpha a_1)\xi + \cdots + (\alpha a_n)\xi^n. \end{aligned}$$

Then, the structure $(\mathbb{C}_2[\xi]_n, +, \cdot)$ forms a vector space over $\mathbb{C}_1$. We define a map f_1, f_2 and f_3 as follows:

$$f_1 : \mathbb{C}_2[\xi]_n \times \mathbb{C}_2[\xi]_n \longrightarrow \mathbb{C}_1 \text{ such that}$$
$$f_1(P(\xi), Q(\xi)) = \sum_{k=0}^{n} (a_k)^-\ \overline{(b_k)^-} + (a_k)^+\ \overline{(b_k)^+}.$$

Thus, f_1 defines inner product on $\mathbb{C}_2[\xi]_n$. We refer to this inner products induced by f_1 as idempotent inner product on $\mathbb{C}_2[\xi]_n$ and denote by $\langle \bullet, \bullet \rangle_{ID}$; that is,

$$f_1(P(\xi), Q(\xi)) \quad = \quad \langle P(\xi), Q(\xi) \rangle_{ID}.$$

Definition 1.6. Let $P = (\xi_1, \xi_2, \cdots, \xi_n)$ and $Q = (\eta_1, \eta_2, \cdots, \eta_n)$ be two arbitrary elements of ${\mathbb{C}_2}^n$, then, we define a map f as follows:

$$\begin{aligned} & f : {\mathbb{C}_2}^n \times {\mathbb{C}_2}^n \longrightarrow \mathbb{C}_1 \text{ such that} \\ & f(P, Q) = \sum_{k=1}^{n} (\xi_k)^- \, \overline{(\eta_k)^-} + (\xi_k)^+ \, \overline{(\eta_k)^+}. \end{aligned} \tag{1}$$

Definition 1.7. [Idempotent Inner Product on ${\mathbb{C}_2}^n$] [4]: Let f be a map of the type (1.6). Then, f defines an inner product on ${\mathbb{C}_2}^n$. We refer to this inner product as the idempotent inner product on ${\mathbb{C}_2}^n$, and denote it by $\langle\langle \bullet, \bullet \rangle\rangle_{ID}$; that is,

$$f(P, Q) = \langle\langle P, Q \rangle\rangle_{ID}.$$

Consequently, $({\mathbb{C}_2}^n, \langle\langle \bullet, \bullet \rangle\rangle_{ID})$ is an inner product space. The norm on ${\mathbb{C}_2}^n$ induced by the idempotent inner product is called the idempotent norm and is denoted $|||\bullet|||_{ID}$; that is,

$$\begin{aligned} |||P|||_{ID} &= \sqrt{\langle\langle P, P \rangle\rangle_{ID}}. \\ &= \sqrt{2}\big(||\xi_1||^2 + ||\xi_2||^2 + \cdots + ||\xi_n||^2\big)^{\frac{1}{2}} = \sqrt{2}|||P|||_R = \sqrt{2}|||P|||_C, \ (see, 1.4). \end{aligned}$$

Relevance to the Present Work: The notations and results summarized above provide the algebraic and geometric foundation upon which the present study is built. In particular, the decomposition $\xi = \xi^- e_1 + \xi^+ e_2$ plays a pivotal role in formulating the Idempotent Inner Product (IIP), and in deriving the results on orthogonality, orthonormality, and their geometric interpretations in $\mathbb{C}_2$ developed in later sections.

2. ORTHOGONALITY ON BICOMPLEX SPACE

Analogous to the classical theory of orthogonality in $\mathbb{C}_1$, the notion of orthogonality can also be introduced in the bicomplex space $\mathbb{C}_2$. In this section, specific concepts related to orthogonality have been examined with respect to the Idempotent Inner Product; furthermore, results regarding complex orthogonality both consistent and inconsistent have been systematically analyzed. This will prove significant in understanding the complexities of the space for future research.

2.1. Orthogonal Elements in $\mathbb{C}_2$

Definition 2.1. [Orthogonal Element]: Let $\mathbb{C}_2$ be an inner product space equipped with an idempotent inner product (IIP). An element $\xi \in \mathbb{C}_2$ is said to be *orthogonal* to an element $\eta \in \mathbb{C}_2$ if $\langle \xi, \eta \rangle_{ID} = 0 = \langle \eta, \xi \rangle_{ID}$, denoted by $\xi \perp \eta$. Equivalently, $\xi \perp \eta \iff \eta \perp \xi \iff \langle \xi, \eta \rangle_{ID} = 0$.

Example 2.2. Illustration through various examples:

1. The zero element is orthogonal to any element in $_2$, that is $0 \perp \xi,\ \forall\ \xi \in_2$.
2. $(3e_1 + 3e_2) \perp (-i_1e_1 + i_1e_2)$. Since $3 \cdot \overline{-i_1} + 3 \cdot \overline{i_1} = 0$, (the idempotent inner product).
3. $\big(2 + 3i_1\big)e_1 \perp \big(5 - 6i_1\big)e_2$, because $(2 + 3i_1) \cdot \overline{0} + 0 \cdot \big(\overline{5 - 6i_1}\big) = 0$.
4. $\big(2 + 3i_1 - i_2 + i_1i_2\big) \not\perp \big(3 - 4i_1 - 2i_2 + i_1i_2\big)$, since $(3 + 4i_1) \cdot \big(\overline{4 - 2i_1}\big) + (1 + 2i_1) \cdot \big(\overline{2 - 6i_1}\big) = -6 + 32i_1 \neq 0$.
5. Consider $(1, 0), (0, 1) \in_2^2$. The idempotent inner product of these two elements is $\langle (1, 0)e_1 + (1, 0)e_2,\ (0, 1)e_1 + (0, 1)e_2 \rangle_{ID} = 1 \cdot \overline{0} + 0 \cdot \overline{1} + 1 \cdot \overline{0} + 0 \cdot \overline{1} = 0$.
6. $e_1 \perp e_2$, showing that the two non-trivial idempotent elements are mutually orthogonal.

Theorem 2.3. *If one pair of corresponding idempotent components of two orthogonal vectors in the bicomplex vector space $_2$ are orthogonal, then the other pair of idempotent components are also orthogonal.*

Proof. Let $(\xi, \eta)_{ID}$ be a pair of orthogonal vectors in $_2$. By definition 2.1, $\langle \xi, \eta \rangle_{ID} = 0$. Expressing this relation in terms of the idempotent components, we obtain

$$\langle \xi, \eta \rangle_{ID} = \langle \xi^-, \eta^- \rangle + \langle \xi^+, \eta^+ \rangle = 0. \quad \text{Suppose that} \quad \xi^- \perp \eta^-, \text{that is, } \langle \xi^-, \eta^- \rangle = 0.$$

It then follows that, $0 + \langle \xi^+, \eta^+ \rangle = 0$, which implies $\langle \xi^+, \eta^+ \rangle = 0 \Longrightarrow \xi^+ \perp \eta^+$. Thus, if one pair of idempotent components of two orthogonal vectors are orthogonal, the other pair must also be orthogonal. This completes the proof. □

Theorem 2.4. *Let $\xi, \eta \in_2$. If a non-zero vector ξ belongs to $\mathbb{I}_1$ and satisfies $\langle \xi, \eta \rangle_{ID} = 0$, then $\eta \in \mathbb{I}_2$. Conversely, if a non-zero $\eta \in \mathbb{I}_2$ and $\langle \xi, \eta \rangle_{ID} = 0$, then $\xi \in \mathbb{I}_1$.*

Proof. Suppose $\xi, \eta \in_2$ and $0 \neq \xi \in \mathbb{I}_1$. Then $\xi = \xi^- e_1$, and by assumption $\langle \xi, \eta \rangle_{ID} = 0$. Hence,

$$\langle \xi^- e_1, \eta^- e_1 + \eta^+ e_2 \rangle_{ID} = 0$$

using IIP,

$$\langle \xi^-, \eta^- \rangle + \langle 0 \cdot \eta^+ \rangle = 0 \implies \xi^- \cdot \overline{\eta^-} = 0.$$

Since $\xi^- \neq 0$, it follows that $\overline{\eta^-} = 0 \implies \eta^- = 0$. Thus, $\eta = \eta^+ e_2 \in \mathbb{I}_2$.
Conversely, let $0 \neq \eta \in \mathbb{I}_2$. Then $\eta = \eta^+ e_2$ and assume $\langle \xi, \eta \rangle_{ID} = 0$. Therefore,

$$\begin{aligned} & \langle \xi^- e_1 + \xi^+ e_2,\ 0 \cdot e_1 + \eta^+ e_2 \rangle_{ID} = 0 \\ \implies \quad & \xi^- \cdot \overline{0} + \xi^+ \cdot \overline{\eta^+} = 0 \\ \implies \quad & \xi^+ \cdot \overline{\eta^+} = 0. \end{aligned}$$

Since $\eta^+ \neq 0$, we must have $\xi^+ = 0$. Hence, $\xi = \xi^- e_1 \in \mathbb{I}_1$. Thus, the claim holds in both directions, and the theorem is proved. □

Theorem 2.5. *Let $\xi = \xi^- e_1 + \xi^+ e_2$ and $\eta = \eta^- e_1 + \eta^+ e_2$ be elements of $\mathbb{C}_2$. If $\xi^- \perp \eta^-$ and $\xi^+ \perp \eta^+$ in $\mathbb{C}_1$, then $\xi \perp \eta$. The converse does not hold in general. However, the converse is valid whenever at least one of ξ or η belongs to O_2, the set of singular elements in $\mathbb{C}_2$.*

Proof. Assume that $\xi^- \perp \eta^-$ and $\xi^+ \perp \eta^+$ in $\mathbb{C}_1$ therefore $\langle \xi^-, \eta^- \rangle = 0$ and $\langle \xi^+, \eta^+ \rangle = 0$. Using the idempotent representation of bicomplex numbers, we obtain

$$\langle \xi, \eta \rangle_{ID} = \langle \xi^-, \eta^- \rangle e_1 + \langle \xi^+, \eta^+ \rangle e_2.$$

Hence, $\langle \xi, \eta \rangle = 0 \cdot e_1 + 0 \cdot e_2 = 0$. Therefore, $\xi \perp \eta$.
To show that the converse is not true in general, consider $\xi^- = 3 + 4i_1, \xi^+ = 3 + 4i_1$ and $\eta^- = 4 - 2i_1, \eta^+ = -4 + 2i_1$. Then

$$\begin{aligned} \langle \xi^-, \eta^- \rangle &= (3 + 4i_1(4 + 2i_1) = 4 + 22i_1 \neq 0, \\ \text{and} \quad \langle \xi^+, \eta^+ \rangle &= (3 + 4i_1)(-4 - 2i_1) = -4 - 22i_1 \neq 0. \end{aligned}$$

However, $\langle \xi, \eta \rangle_{ID} = (4 + 22i_1)e_1 + (-4 - 22i_1)e_2 = 0$. Thus, $\xi \perp \eta$, although neither $\xi^- \perp \eta^-$ nor $\xi^+ \perp \eta^+$.
Now suppose that $\xi \in O_2$ and $\xi \perp \eta$.

Moreover, Since ξ is singular, one of its idempotent components must vanish. Therefore, either $\xi^- = 0$ or $\xi^+ = 0$. If $\xi^+ = 0$, then $\xi = \xi^- e_1$. Consequently, $0 = \langle \xi, \eta \rangle_{ID} = \langle \xi^- e_1, \eta^- e_1 + \eta^+ e_2 \rangle_{ID}$. Using the relation $e_1 e_2 = 0$, we get $\langle \xi, \eta \rangle_{ID} = \langle \xi^-, \eta^- \rangle e_1$. Hence, $\langle \xi^-, \eta^- \rangle = 0$, that is, $\xi^- \perp \eta^-$. Similarly, if $\xi^- = 0$, then $\xi = \xi^+ e_2$, and $0 = \langle \xi, \eta \rangle_{ID} = \langle \xi^+, \eta^+ \rangle e_2$, which implies $\langle \xi^+, \eta^+ \rangle = 0$. Therefore, $\xi^+ \perp \eta^+$. Hence, the converse holds whenever at least one of the elements is singular. □

Example 2.6. Let $(\xi^-, \xi^+) = (2,3)$ and $(\eta^-, \eta^+) = (-3,2)$. Then $\langle \xi, \eta \rangle_{ID} = \xi^-\eta^- + \xi^+\eta^+ = 2(-3) + 3(2) = 0$ Hence, $\xi \perp \eta$. However, $\langle \xi^-, \eta^- \rangle = -6 \neq 0$, and $\langle \xi^+, \eta^+ \rangle = 6 \neq 0$. Therefore, neither $\xi^- \perp \eta^-$ nor $\xi^+ \perp \eta^+$, showing once again that the converse of Theorem 3.5 does not hold in general.

2.2. Orthogonal set in $\mathbb{C}_2$

Definition 2.7. [Orthogonal set in $\mathbb{C}_2$]: Let $\mathbb{C}_2(\mathbb{C}_1)$ be an inner product space equipped with IIP(idempotent inner product, see, 1.1). A subset $\{\xi_1, \xi_2 ... \xi_n\}$, where $\xi_i \in \mathbb{C}_2$, is said to be *orthogonal* if $\langle \xi_i, \xi_j \rangle = 0$ whenever $i \neq j$, that is, every pair of distinct vectors in the set is mutually orthogonal. Moreover, if each vector of the set has unit **idempotent norm**, i.e., $\sqrt{\langle \xi_i, \xi_i \rangle_{ID}} = 1$ for all i, from 1 to n, then the set $\{\xi_1, \xi_2 ... \xi_n\}$ is called an *orthonormal set* in $\mathbb{C}_2$.

Example 2.8. Consider the set $\{(1,0),(0,1)\}$ whose elements in $\mathbb{C}_2^2$.This set is orthogonal, but not orthonormal in $\mathbb{C}_2^2$ with idempotent inner product defined over $\mathbb{C}_2^2$ (by, 1.7), whereas it forms an orthonormal set in $\mathbb{C}_1$ with usual inner product. Indeed,

$$\langle\langle (1e_1 + 1e_2, 0e_1 + 0e_2), (0e_1 + 0e_2, 1e_1 + 1e_2) \rangle\rangle_{ID} = 1 \cdot \overline{0} + 1 \cdot \overline{0} + 0 \cdot \overline{1} + 0 \cdot \overline{1} = 0$$

which shows that the two vectors are orthogonal. However, for the norm we have

$$\begin{aligned} \langle\langle (1,0),(1,0) \rangle\rangle_{ID} &= \langle\langle (1e_1 + 1e_2, 0e_1 + 0e_2), (1e_1 + 1e_2, 0e_1 + 0e_2) \rangle\rangle_{ID} \\ &= 1 \cdot \overline{1} + 1 \cdot \overline{1} + 0 \cdot \overline{0} + 0 \cdot \overline{0} = 2. \end{aligned}$$

Therefore the norm $|||P|||_{ID} = \sqrt{2} \neq 1$, by 1.7. Thus, the set $\{(1,0),(0,1)\}$ fails to be orthonormal in $\mathbb{C}_2^2$.

Remark 2.1. *The monomial set*

$$\{1, \xi, \xi^2, \ldots, \xi^n\}$$

exhibits different orthogonality properties in the polynomial spaces over $\mathbb{C}_1$ and $\mathbb{C}_2$. Consider first the polynomial inner product space of degree at most n over $\mathbb{C}_1$, equipped with the coefficient inner product. Then

$$\langle \xi^i, \xi^j \rangle = \delta_{ij},$$

and consequently the monomials are mutually orthogonal with

$$\langle \xi^i, \xi^i \rangle = 1, \qquad 0 \leq i \leq n.$$

Hence,

$$\{1, \xi, \xi^2, \dots, \xi^n\}$$

forms an orthonormal set.

Now consider the bicomplex polynomial space $\mathbb{C}_2[\xi]_n$ of degree at most n, equipped with the idempotent inner product(IIP), defined by

$$\left\langle \sum_{i=0}^{n} a_i \xi^i, \sum_{i=0}^{n} b_i \xi^i \right\rangle_{ID} = \sum_{i=0}^{n} a_i^- \overline{b_i^-} + \sum_{i=0}^{n} a_i^+ \overline{b_i^+}. \quad (from, 1.5).$$

For $i \neq j$, we still have

$$\langle \xi^i, \xi^j \rangle_{ID} = 0,$$

so the monomials remain mutually orthogonal. However,

$$\begin{aligned} \langle \xi^i, \xi^i \rangle_{ID} &= \langle (\xi^-)^i, (\xi^-)^i \rangle + \langle (\xi^+)^i, (\xi^+)^i \rangle \\ &= 1 \cdot \overline{1} + 1 \cdot \overline{1} = 2. \end{aligned}$$

Therefore idempotent norm,

$$\|\xi^i\|_{ID} = \sqrt{2}, \qquad 0 \leq i \leq n,$$

and hence

$$\{1, \xi, \xi^2, \dots, \xi^n\}$$

forms an orthogonal set, but not an orthonormal set, in $\mathbb{C}_2[\xi]$ with respect to the Idempotent Inner Product.

Example 2.9. The sets $\{e_1, e_2\}$, $\{-e_1, -e_2\}$, $\{e_1, -e_2\}$ and $\{-e_1, e_2\}$ are orthonormal with respect to the idempotent inner product (IIP) in $\mathbb{C}_2$.

Example 2.10. The following sets are orthonormal with respect to IIP in $\mathbb{C}_2$ (see, 1.1):

1. $\{i_1 e_1,\ i_1 e_2\}$
2. $\left\{\left(\frac{1}{\sqrt{2}} + i_1 \frac{1}{\sqrt{2}}\right) e_1, \left(\frac{1}{\sqrt{2}} + i_1 \frac{1}{\sqrt{2}}\right) e_2\right\}$
3. $\left\{\left(\frac{1}{\sqrt{3}} + i_1 \cdot \frac{2}{\sqrt{3}}\right) e_1,\ \left(\frac{1}{\sqrt{3}} + i_1 \cdot \frac{2}{\sqrt{3}}\right) e_2\right\}$
4. $\left\{\left(\frac{\sqrt{1}}{\sqrt{n}} + i_1 \cdot \frac{\sqrt{n-1}}{\sqrt{n}}\right) e_1,\ \left(\frac{\sqrt{1}}{\sqrt{n}} + i_1 \cdot \frac{\sqrt{n-1}}{\sqrt{n}}\right) e_2\right\}$

Example 2.11. The set $\left\{\frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_2,\ \frac{i}{\sqrt{2}} e_1 - \frac{i}{\sqrt{2}} e_2\right\}$ is an orthonormal set in $\mathbb{C}_2$.

Proposition 2.1. *If $\{\xi, \eta\}$ is to be an orthonormal set in $\mathbb{C}_2$ with IIP (1.1), then each of the vectors ξ and η must be of unit **idempotent norm**. In particular,*

$$\langle \xi, \eta \rangle_{ID} = \begin{cases} 0, & \xi \neq \eta, \\ 1, & \xi = \eta. \end{cases}$$

Proof. For $\xi = \xi^- e_1 + \xi^+ e_2$, this condition requires

$$\begin{aligned} \langle \xi^- e_1 + \xi^+ e_2,\ \xi^- e_1 + \xi^+ e_2 \rangle_{ID} &= 1, \\ \xi^- \,\overline{\xi^-} + \xi^+ \,\overline{\xi^+} &= 1, \\ |\xi^-|^2 + |\xi^+|^2 &= 1, \\ \left(|\xi^-|^2 + |\xi^+|^2\right)^{1/2} &= 1, \\ \Longrightarrow \quad ||\xi||_{ID} &= 1. \end{aligned}$$

Thus, orthonormality in $\mathbb{C}_2$ under IIP requires each element to have unit idempotent norm. $\square$

Remark 2.2. *In $\mathbb{C}_2$, every orthonormal set is necessarily an orthogonal set.*

Definition 2.12. [Unit element]: Let $\xi \in \mathbb{C}_2$. An element ξ is called a unit element in $\mathbb{C}_2$ if its **idempotent norm** is equal to 1.

Example 2.13. For the idempotent basis element e_1, we compute

$$\begin{aligned} ||e_1||_{ID} &= \sqrt{2}\, \|e_1\| \\ &= \sqrt{2} \cdot \tfrac{1}{\sqrt{2}} = 1. \end{aligned}$$

Hence e_1 is a unit element. Similarly, e_2 also has idempotent norm 1, so it is a unit element in $\mathbb{C}_2$. It is noteworthy that if the usual norm is chosen instead of the idempotent norm, the idempotent elements e_1 and e_2 do not become unit elements.This could prove to be quite significant in the future.

Example 2.14. The element $\xi = \pm\frac{1}{\sqrt{2}}$ is a unit element in $\mathbb{C}_2$ with respect to the IIP inner product.

Example 2.15. The element $\xi = \sin\frac{\pi}{4} \cdot e_1 + \cos\frac{\pi}{4} \cdot e_2$, also satisfies $||\xi||_{ID} = 1$, hence it is a unit element.

Example 2.16. Consider $P = \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{C}_2^2$. Then, using the idempotent inner product (IIP), we have

$$\langle\langle P, P\rangle\rangle_{ID} = \left\langle\left\langle \frac{1}{2} \cdot e_1 + \frac{1}{2} \cdot e_2,\ \frac{1}{2} \cdot e_1 + \frac{1}{2} \cdot e_2 \right\rangle\right\rangle_{ID}.$$

Expanding the product, we obtain

$$\langle\langle P, P\rangle\rangle_{ID} = \frac{1}{2}\cdot\overline{\frac{1}{2}} + \frac{1}{2}\cdot\overline{\frac{1}{2}} + \frac{1}{2}\cdot\overline{\frac{1}{2}} + \frac{1}{2}\cdot\overline{\frac{1}{2}} = 1.$$

Hence, P has idempotent norm 1 and is a unit element in $\mathbb{C}_2^2$.

Definition 2.17. [Mutually orthogonal subsets in $\mathbb{C}_2$]:. A subset A of $\mathbb{C}_2$ is said to be *mutually orthogonal* to a subset B of $\mathbb{C}_2$ if $\langle\xi, \eta\rangle_{ID} = 0, \quad \forall\, \xi \in A,\ \forall\, \eta \in B$, and this relation is denoted by $A \perp B \quad$ or $\quad B \perp A$.

Example 2.18. The zero set is mutually orthogonal to the whole space:

$$\{0\} \perp \mathbb{C}_2.$$

Example 2.19. The idempotent subspaces are mutually is orthogonal; that is :

$$\mathbb{I}_1 \perp \mathbb{I}_2$$

Theorem 2.20. *Two mutually orthogonal sets which intersect non-trivially (that is; the intersection of subspaces is a super set of the set $\{0\}$) can never be orthonormal in $\mathbb{C}_2$.*

Proof. Let A and B be two mutually orthogonal sets in $\mathbb{C}_2$ such that $A \cap B \supsetneq \{0\}$. Then, there exists an element $\zeta \in A \cap B$ such that $\zeta \neq 0$. By the definition 2.17, we have $\langle\zeta, \zeta\rangle_{ID} = 0$, which contradicts the requirement for orthonormal condition; that is $\sqrt{\langle\zeta, \zeta\rangle}_{ID} = 1$. Hence, the sets A and B cannot be orthonormal in $\mathbb{C}_2$. □

Theorem 2.21. *Let $\xi = \xi^- e_1 + \xi^+ e_2, \quad \eta = \eta^- e_1 + \eta^+ e_2 \in \mathbb{C}_2$, and let the sets $\{\xi^-, \eta^-\}$ and $\{\xi^+, \eta^+\}$ be orthonormal in $\mathbb{C}_1$. Then, the set $\{\xi, \eta\}$ is orthogonal but never orthonormal in $\mathbb{C}_2$ over IIP.*

Proof. Let $\xi = \xi^- e_1 + \xi^+ e_2, \quad \eta = \eta^- e_1 + \eta^+ e_2 \in \mathbb{C}_2$, with $\{\xi^-, \eta^-\}$ and $\{\xi^+, \eta^+\}$ orthonormal in $\mathbb{C}_1$. Consider the inner product:

$$\begin{aligned}
\langle\xi, \eta\rangle_{ID} &= \langle\xi^- e_1 + \xi^+ e_2,\, \eta^- e_1 + \eta^+ e_2\rangle_{ID} \\
&= \langle\xi^-, \eta^-\rangle + \langle\xi^+, \eta^+\rangle \\
&= 0 + 0 = 0.
\end{aligned}$$

Since $\{\xi^-, \eta^-\}$ *and* $\{\xi^+, \eta^+\}$ are orthogonal in $\mathbb{C}_1$, showing that ξ and η are orthogonal. However, the norm of ξ is

$$\begin{aligned}
||\xi||_{ID}{}^2 &= \langle\xi, \xi\rangle_{ID} \\
&= \langle\xi^-, \xi^-\rangle + \langle\xi^+, \xi^+\rangle \\
&= 1 + 1 = 2 \\
\text{since } \{\xi^-, \eta^-\} \text{ and } & \{\xi^+, \eta^+\} \text{ are orthonormal in } \mathbb{C}_1 \\
&= |\xi^-|^2 + |\xi^+|^2 = 2 \\
||\xi||_{ID} &= \sqrt{2} = ||\eta||_{ID},
\end{aligned}$$

which implies that ξ and η are not of unit idempotent norm. Therefore, the set $\{\xi, \eta\}$ is orthogonal but not orthonormal. This completes the proof. $\square$

2.3. Three Forms of the Idempotent Norm and Geometric Interpretation

Three Forms of the Idempotent Norm. The idempotent norm of a bicomplex number ξ admits the following equivalent representations:

$$\begin{aligned} ||\xi||_{ID} &= \sqrt{2}\,\|\xi\| = \sqrt{2}\left(\frac{|\xi^-|^2 + |\xi^+|^2}{2}\right)^{1/2}, && (2) \\ &= \sqrt{2}\left(|Z_1|^2 + |Z_2|^2\right)^{1/2}, && (3) \\ &= \sqrt{2}\left(\sum_{i=1}^{4} x_i^2\right)^{1/2}. && (4) \end{aligned}$$

Equation (2) represents the idempotent form, Equation (3) represents the complex form, and Equation (4) represents the real form of the Idempotent norm.

Geometric Interpretation.
Consider an arbitrary element of the idempotent subspace $I_1 = \{\xi^- e_1 \mid \xi^- \in \mathbb{C}_1\} \subset \mathbb{C}_2$. Then

$$\begin{aligned} \left|\left|\xi^- e_1\right|\right|_{ID} &= \sqrt{2}\,\|\xi^- e_1\| \\ &= \sqrt{2}\,\|\xi^- e_1 + 0 \cdot e_2\| \\ &= \sqrt{2}\left(\frac{|\xi^-|^2 + |0|^2}{2}\right)^{1/2} \\ &= \sqrt{2}\left(\frac{|\xi^-|^2}{2}\right)^{1/2} \\ &= \sqrt{2} \cdot \frac{1}{\sqrt{2}}\left(|\xi^-|^2\right)^{1/2} \\ &= |\xi^-|. \end{aligned}$$

Thus, every element of I_1 preserves its distance from the origin under the natural embedding $\xi^- \longmapsto \xi^- e_1$. Consequently, there exists a natural injective mapping

$$\begin{aligned} f : S_r(\mathbb{C}_1) &\longrightarrow S_r(\mathbb{C}_2), \\ f(\xi^-) &= \xi^- e_1, \end{aligned}$$

where

$$\begin{aligned} S_r(\mathbb{C}_1) &= \{\xi^- \in \mathbb{C}_1 : |\xi^-| = r\}, \\ S_r(\mathbb{C}_2) &= \{\xi \in \mathbb{C}_2 : ||\xi||_{ID} = r\}. \end{aligned}$$

To verify that f is injective, let

$$f(\xi^-) = f(\eta^-).$$

Then

$$\begin{aligned} \xi^- e_1 &= \eta^- e_1 \\ \Rightarrow \quad \xi^- &= \eta^-. \end{aligned}$$

Hence, f is injective.
Similarly, the same geometric interpretation holds for the idempotent subspace $I_2 = \{\xi^+ e_2 \mid \xi^+ \in \mathbb{C}_1\} \subset \mathbb{C}_2$.

2.4. Orthogonal Complements of Subspaces in $\mathbb{C}_2$:

Definition 2.22. [Orthogonal Complement]: Let W be a subspace of the inner product space $\mathbb{C}_2$ with idempotent inner product. The orthogonal complement of W is defined by

$$W^\perp = \{\xi \in \mathbb{C}_2 \mid \langle \xi, w \rangle_{ID} = 0 \; \forall \, w \in W\}.$$

Remark 2.3. *Let $\xi = \xi^- e_1 + \xi^+ e_2$ and $w = w^- e_1 + w^+ e_2 \in W$. Then, by the definition of the idempotent inner product,*

$$\langle \xi, w \rangle_{ID} = \langle \xi^-, w^- \rangle_{\mathbb{C}_1} + \langle \xi^+, w^+ \rangle_{\mathbb{C}_1}.$$

Consequently,

$$W^\perp = \left\{ \xi = \xi^- e_1 + \xi^+ e_2 \;\middle|\; \langle \xi^-, w^- \rangle_{\mathbb{C}_1} + \langle \xi^+, w^+ \rangle_{\mathbb{C}_1} = 0, \; \forall \, w \in W \right\}.$$

Example 2.23. Let the subspaces I_1 and I_2 of $\mathbb{C}_2$ be defined by $I_1 = \{\xi e_1 \mid \xi \in \mathbb{C}_2\}, I_2 = \{\xi e_2 \mid \xi \in \mathbb{C}_2\}$. Then I_1 and I_2 are orthogonal complements of each other, that is, $(I_1)^\perp = I_2$, $(I_2)^\perp = I_1$.

Theorem 2.24. *Let W be a subspace of $\mathbb{C}_2$. Then the orthogonal complement $W^\perp$ is a subspace of $\mathbb{C}_2$.*

Proof. Since $\langle 0, w \rangle_{ID} = 0$ for every $w \in W$, we have $0 \in W^\perp$. Thus, $W^\perp$ is non-empty. Let $\eta_1, \eta_2 \in W^\perp$ and $\alpha, \beta \in \mathbb{C}_1$. Then, for every $w \in W$,

$$\begin{aligned} \langle \alpha\eta_1 + \beta\eta_2, w \rangle_{ID} &= \alpha \langle \eta_1, w \rangle_{ID} + \beta \langle \eta_2, w \rangle_{ID} \\ &= 0. \end{aligned}$$

Hence, $\alpha\eta_1 + \beta\eta_2 \in W^\perp$. Therefore, $W^\perp$ is closed under addition and scalar multiplication. Consequently, $W^\perp$ is a subspace of $\mathbb{C}_2$. This completes the proof. □

The following theorem establishes the orthogonal decomposition of the bicomplex inner product space $(\mathbb{C}_2, \langle \bullet, \bullet \rangle_{ID})$ and is the bicomplex analogue of the classical orthogonal decomposition theorem for finite-dimensional inner product spaces.

Theorem 2.25. *Let W be a subspace of $\mathbb{C}_2$. Then $\mathbb{C}_2$ is the direct su of subspaces W and $W^\perp$; that is*

$$\mathbb{C}_2 = W \oplus W^\perp.$$

Proof. To prove the result, it is sufficient to show that

$$\mathbb{C}_2 = W + W^\perp \qquad \text{and} \qquad W \cap W^\perp = \{0\}.$$

Since $\mathbb{C}_2$ is a finite-dimensional inner product space over $\mathbb{C}_1$, every subspace is closed. Hence, by the orthogonal decomposition theorem for finite-dimensional inner product spaces,

$$\mathbb{C}_2 = W + W^\perp.$$

Now let $\xi \in W \cap W^\perp$. Then $\xi \in W$ and $\xi \in W^\perp$. Therefore,

$$\langle \xi, \xi \rangle_{ID} = 0.$$

Since the Idempotent Inner Product is positive definite, it follows that

$$\xi = 0.$$

Thus,

$$W \cap W^\perp = \{0\}.$$

Hence,

$$\mathbb{C}_2 = W \oplus W^\perp.$$

□

Definition 2.26. Let W be a subspace of $\mathbb{C}_2$. Define

$$W^- = \{\xi^- : \xi \in W\}, \qquad W^+ = \{\xi^+ : \xi \in W\}.$$

Further, define

$$W^- e_1 = \{\xi^- e_1 : \xi \in W\}, \qquad W^+ e_2 = \{\xi^+ e_2 : \xi \in W\}.$$

Remark 2.4. *Since W is a subspace of $\mathbb{C}_2$, each of*

$$W^-, \quad W^+, \quad W^- e_1, \quad W^+ e_2$$

is a subspace of $\mathbb{C}_2$.

Corollary 2.27. *Let W be a subspace of $\mathbb{C}_2$. Then*

$$W^-e_1 + W^+e_2 \supseteq W.$$

Proof. Let $\xi \in W$. Then, by the idempotent representation,

$$\xi = \xi^-e_1 + \xi^+e_2.$$

Since $\xi^- \in W^-$ and $\xi^+ \in W^+$, it follows that

$$\xi^-e_1 \in W^-e_1 \quad \text{and} \quad \xi^+e_2 \in W^+e_2.$$

Hence,

$$\xi = \xi^-e_1 + \xi^+e_2 \in W^-e_1 + W^+e_2.$$

Therefore,

$$W \subseteq W^-e_1 + W^+e_2,$$

or equivalently,

$$W^-e_1 + W^+e_2 \supseteq W.$$

□

Definition 2.28. The orthogonal complements of W^- and W^+ are defined by

$$\begin{aligned}(W^-)^\perp &= \{\xi \in \mathbb{C}_2 : \langle \xi, w^- \rangle_{ID} = 0,\ \forall\, w^- \in W^-\},\\ (W^+)^\perp &= \{\xi \in \mathbb{C}_2 : \langle \xi, w^+ \rangle_{ID} = 0,\ \forall\, w^+ \in W^+\}.\end{aligned}$$

Further, define

$$(W^-)^\perp e_1 + (W^+)^\perp e_2 = \left\{\xi = \xi^-e_1 + \xi^+e_2 \;\middle|\; \xi^- \in (W^-)^\perp,\ \xi^+ \in (W^+)^\perp\right\}.$$

Remark 2.5. *By the preceding theorem 2.24, $(W^-)^\perp$ and $(W^+)^\perp$ are subspaces of $\mathbb{C}_2$.*

Theorem 2.29. *Let W^- and W^+ be the subspaces introduced in Definition 2.26. Then*

$$(W^-)^\perp e_1 + (W^+)^\perp e_2$$

is a subspace of $\mathbb{C}_2$. Moreover,

$$(W^-)^\perp e_1 \cap (W^+)^\perp e_2 = \{0\}.$$

Proof. Let $\xi = \xi^- e_1 + \xi^+ e_2$ and $\eta = \eta^- e_1 + \eta^+ e_2$ be arbitrary elements of $(W^-)^\perp e_1 + (W^+)^\perp e_2$, where $\xi^-, \eta^- \in (W^-)^\perp$ and $\xi^+, \eta^+ \in (W^+)^\perp$. Then, for every $w^- \in W^-$ and $w^+ \in W^+$,

$$\begin{aligned}\langle \xi^-, w^- \rangle &= 0, \qquad \langle \eta^-, w^- \rangle = 0,\\ \langle \xi^+, w^+ \rangle &= 0, \qquad \langle \eta^+, w^+ \rangle = 0.\end{aligned}$$

By the linearity of the inner product,

$$\langle \xi^- + \eta^-, w^- \rangle = 0, \qquad \langle \xi^+ + \eta^+, w^+ \rangle = 0,$$

which implies that $\xi^- + \eta^- \in (W^-)^\perp$ and $\xi^+ + \eta^+ \in (W^+)^\perp$. Hence,

$$\begin{aligned}\xi + \eta &= (\xi^- + \eta^-)e_1 + (\xi^+ + \eta^+)e_2\\ &\in (W^-)^\perp e_1 + (W^+)^\perp e_2.\end{aligned}$$

Now let $\alpha \in \mathbb{C}_1$. Then

$$\langle \alpha\xi^-, w^- \rangle = 0, \qquad \langle \alpha\xi^+, w^+ \rangle = 0,$$

which implies that $\alpha\xi^- \in (W^-)^\perp$ and $\alpha\xi^+ \in (W^+)^\perp$. Consequently,

$$\begin{aligned}\alpha\xi &= (\alpha\xi^-)e_1 + (\alpha\xi^+)e_2\\ &\in (W^-)^\perp e_1 + (W^+)^\perp e_2.\end{aligned}$$

Thus, $(W^-)^\perp e_1 + (W^+)^\perp e_2$ is a subspace of $\mathbb{C}_2$.
Now let $\xi \in (W^-)^\perp e_1 \cap (W^+)^\perp e_2$. Then there exist $\eta^- \in (W^-)^\perp$ and $\zeta^+ \in (W^+)^\perp$ such that $\xi = \eta^- e_1 = \zeta^+ e_2$. Therefore,

$$\eta^- e_1 - \zeta^+ e_2 = 0.$$

Since e_1 and e_2 are orthogonal idempotents, it follows that $\eta^- = 0$ and $\zeta^+ = 0$. Hence, $\xi = 0$, and therefore

$$(W^-)^\perp e_1 \cap (W^+)^\perp e_2 = \{0\}.$$

$\square$

Corollary 2.30. *Let W be a subspace of $\mathbb{C}_2$, and let $W^\perp$ denote its orthogonal complement in the bicomplex inner product space $\mathbb{C}_2$. If the idempotent components W^- and $(W^\perp)^-$ are orthogonal in $\mathbb{C}_1$, and similarly W^+ and $(W^\perp)^+$ are orthogonal in $\mathbb{C}_1$, then W and $W^\perp$ are orthogonal in $\mathbb{C}_2$. However, the converse implication need not hold in general.*

Proof. Let $\xi = \xi^- e_1 + \xi^+ e_2 \in W^\perp$ and $w = w^- e_1 + w^+ e_2 \in W$. By hypothesis,

$$\langle \xi^-, w^- \rangle = 0, \qquad \langle \xi^+, w^+ \rangle = 0.$$

Therefore,

$$\langle \xi, w \rangle_{ID} = \langle \xi^-, w^- \rangle + \langle \xi^+, w^+ \rangle = 0.$$

Hence, W and $W^\perp$ are orthogonal in $\mathbb{C}_2$.
To see that the converse need not hold, let $w = e_1 + e_2$, so that $w^- = 1$ and $w^+ = 1$, and let $\xi = i_1 e_1 - i_1 e_2$, so that $\xi^- = i_1$ and $\xi^+ = -i_1$. Then

$$\begin{aligned} \langle \xi, w \rangle_{ID} &= \langle \xi^-, w^- \rangle + \langle \xi^+, w^+ \rangle \\ &= i_1 + (-i_1) = 0. \end{aligned}$$

Hence, ξ and w are orthogonal in $\mathbb{C}_2$. However, $\langle \xi^-, w^- \rangle = i_1 \neq 0$, which implies that $\xi^- \notin (W^-)^\perp$. Similarly, $\langle \xi^+, w^+ \rangle = -i_1 \neq 0$, and hence $\xi^+ \notin (W^+)^\perp$. Therefore, the converse implication does not hold. □

Proposition 2.2. *Let W be a subspace of the inner product space $\mathbb{C}_2$. Then $(W^\perp)^- \subseteq (W^-)^\perp$. In general, the reverse inclusion $(W^-)^\perp \subseteq (W^\perp)^-$ does not hold.*

Proof. Let $\xi \in (W^\perp)^-$, and let $\eta \in W^\perp$ be such that $\xi = \eta^-$. Since $\eta \in W^\perp$, we have $\langle \eta, w \rangle_{ID} = 0$ for every $w \in W$.
Now let $w = w^- e_1 + w^+ e_2 \in W$. Then

$$\langle \eta, w \rangle_{ID} = \langle \eta^-, w^- \rangle + \langle \eta^+, w^+ \rangle = 0.$$

Hence,

$$\langle \eta^-, w^- \rangle = 0, \qquad \forall\, w^- \in W^-.$$

Therefore, $\eta^- \in (W^-)^\perp$. Since $\xi = \eta^-$, it follows that $\xi \in (W^-)^\perp$. Consequently,

$$(W^\perp)^- \subseteq (W^-)^\perp.$$

Example Consider the subspace $W = \operatorname{span}\{e_1 + e_2\} \subset \mathbb{C}_2$. Then $W^- = \mathbb{C}_1$, and hence $(W^-)^\perp = \{\xi^+ e_2 \mid \xi^+ \in \mathbb{C}_1\}$. On the other hand, $W^\perp = \{0\}$, which implies that $(W^\perp)^- = \{0\}$. Therefore,

$$(W^-)^\perp \not\subseteq (W^\perp)^-.$$

This completes the proof. □

Theorem 2.31. *Let W be a subspace of $\mathbb{C}_2$, and define*

$$W^-e_1 = \{\xi^-e_1 \mid \xi \in W\}, \qquad W^+e_2 = \{\xi^+e_2 \mid \xi \in W\}.$$

Then

$$(W^-)^\perp e_1 + (W^+)^\perp e_2 = \big(W^-e_1 + W^+e_2\big)^\perp.$$

Proof. Let $\eta = \eta^-e_1 + \eta^+e_2 \in (W^-)^\perp e_1 + (W^+)^\perp e_2$. Then $\eta^- \in (W^-)^\perp$ and $\eta^+ \in (W^+)^\perp$. Hence, $\langle \eta^-, w^- \rangle = 0$ for all $w^- \in W^-$, and $\langle \eta^+, w^+ \rangle = 0$ for all $w^+ \in W^+$.

Now let $w = w^-e_1 + w^+e_2 \in W^-e_1 + W^+e_2$. Then

$$\langle \eta, w \rangle_{ID} \;=\; \langle \eta^-, w^- \rangle + \langle \eta^+, w^+ \rangle = 0.$$

Therefore, $\eta \in \big(W^-e_1 + W^+e_2\big)^\perp$. Thus,

$$(W^-)^\perp e_1 + (W^+)^\perp e_2 \subseteq \big(W^-e_1 + W^+e_2\big)^\perp.$$

Conversely, let $\eta = \eta^-e_1 + \eta^+e_2 \in \big(W^-e_1 + W^+e_2\big)^\perp$. Then $\langle \eta, w \rangle_{ID} = 0$ for every $w = w^-e_1 + w^+e_2 \in W^-e_1 + W^+e_2$. That is,

$$\langle \eta^-, w^- \rangle + \langle \eta^+, w^+ \rangle = 0.$$

Choosing $w^+ = 0$, we obtain $\langle \eta^-, w^- \rangle = 0$ for all $w^- \in W^-$. Hence, $\eta^- \in (W^-)^\perp$.
Similarly, choosing $w^- = 0$, we get $\langle \eta^+, w^+ \rangle = 0$ for all $w^+ \in W^+$. Therefore, $\eta^+ \in (W^+)^\perp$.
Consequently, $\eta \in (W^-)^\perp e_1 + (W^+)^\perp e_2$. Hence,

$$\big(W^-e_1 + W^+e_2\big)^\perp \subseteq (W^-)^\perp e_1 + (W^+)^\perp e_2.$$

Therefore,

$$(W^-)^\perp e_1 + (W^+)^\perp e_2 = \big(W^-e_1 + W^+e_2\big)^\perp.$$

This completes the proof. □

Example 2.32. Consider the subspace $W = \operatorname{span}\{e_1 + e_2\} \subset \mathbb{C}_2$. Then $W^- = \mathbb{C}_1$ and $W^+ = \mathbb{C}_1$. Consequently, $(W^-)^\perp = \{0\}$ and $(W^+)^\perp = \{0\}$. Therefore, $(W^-)^\perp e_1 + (W^+)^\perp e_2 = \{0\}$. On the other hand, $W^-e_1 + W^+e_2 = \mathbb{C}_2$. Hence, $\big(W^-e_1 + W^+e_2\big)^\perp = (\mathbb{C}_2)^\perp = \{0\}$. Thus, $(W^-)^\perp e_1 + (W^+)^\perp e_2 = \big(W^-e_1 + W^+e_2\big)^\perp$. Hence, the required equality holds.

Theorem 2.33. *Let W be a subspace of $\mathbb{C}_2$. Then*

$$\big(W^-e_1 + W^+e_2\big)^\perp \subseteq W^\perp.$$

In particular, $\big(W^-e_1 + W^+e_2\big)^\perp$ is a subspace of $W^\perp$.

Proof. Let $\eta = \eta^- e_1 + \eta^+ e_2 \in \left(W^- e_1 + W^+ e_2\right)^\perp$. Then $\langle \eta, w \rangle_{ID} = 0$ for every $w = w^- e_1 + w^+ e_2 \in W^- e_1 + W^+ e_2$. By Corollary 2.27, $W \subseteq W^- e_1 + W^+ e_2$. Hence, $\langle \eta, w \rangle_{ID} = 0$ for every $w \in W$. It follows that $\eta \in W^\perp$. Therefore,

$$\left(W^- e_1 + W^+ e_2\right)^\perp \subseteq W^\perp.$$

Since $\left(W^- e_1 + W^+ e_2\right)^\perp$ is an orthogonal complement, it is a subspace of $\mathbb{C}_2$. Hence, by the above inclusion, $\left(W^- e_1 + W^+ e_2\right)^\perp$ is a subspace of $W^\perp$. This completes the proof. $\square$

The following identity is a standard consequence of the orthogonal decomposition theorem 2.25.

Proposition 2.3. *Let W be a subspace of $\mathbb{C}_2$, and let $W^\perp$ denote the orthogonal complement of W. Then $(W^\perp)^\perp = W$.*

Proof. We first show that $W \subseteq (W^\perp)^\perp$. Let $w \in W$. Since every element of $W^\perp$ is orthogonal to every element of W, we have $\langle w, \eta \rangle_{ID} = 0 \; \forall \eta \in W^\perp$. Hence, $w \in (W^\perp)^\perp$. Therefore, $W \subseteq (W^\perp)^\perp$.

Next, we prove the reverse inclusion. Let $\xi \in (W^\perp)^\perp$. Since $\mathbb{C}_2 = W \oplus W^\perp$, there exist unique elements $w \in W, u \in W^\perp$ such that $\xi = w + u$. Now, because $\xi \in (W^\perp)^\perp$, we have

$$\langle \xi, \eta \rangle_{ID} = 0 \qquad \forall \eta \in W^\perp.$$

In particular, choosing $\eta = u \in W^\perp$, we obtain $0 = \langle \xi, u \rangle_{ID} = \langle w + u, u \rangle_{ID}$. By linearity of the inner product,

$$\langle \xi, u \rangle_{ID} = \langle w, u \rangle_{ID} + \langle u, u \rangle_{ID}$$

. Since $w \in W$ and $u \in W^\perp$, it follows that $\langle w, u \rangle_{ID} = 0$. Therefore, $\langle u, u \rangle_{ID} = 0$. By positive definiteness of the inner product, $u = 0$. Hence, $\xi = w \in W$. Thus, $(W^\perp)^\perp \subseteq W$. Consequently, $(W^\perp)^\perp = W$. $\square$

CONCLUSION

In this paper, the concept of orthogonality in the bicomplex space $\mathbb{C}_2$ has been developed through the Idempotent Inner Product, providing a natural extension of the classical notion from the complex space $\mathbb{C}_1$. The associated notions of orthogonal and orthonormal sets, unit elements, and orthogonal complements have been established and examined with the aid of suitable examples and counterexamples, leading to a clearer understanding of their algebraic and geometric behavior.

A geometric interpretation of orthogonal complements in the four-dimensional real model of $\mathbb{C}_2$ has also been presented. In addition, an injective mapping from a circle in $\mathbb{C}_1$ to a ball of the same radius in $\mathbb{C}_2$ has been obtained, revealing a meaningful geometric connection between these spaces. Collectively, the results presented in this paper extend the theory of bicomplex inner product spaces and provide a firm basis for further investigations in bicomplex functional analysis, geometry, and related mathematical structures.

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