

# Group analysis of differential equations: A new type of Lie symmetries

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**Abstract:** We construct a new type of symmetries using the regular Lie symmetries as the basis, which we call Modified symmetries. The contrast is that while Lie symmetries arise from point transformations, the Modified symmetries result from the transformations of the neighborhood of that point. The similarity is that as the neighborhood contracts to the central point, the two sets of symmetries become indistinguishable from one another, meaning the Modified symmetries will cease to exist if there were no Lie symmetries in the first place. The advantage is that the group invariant solutions are not affected by all these, because they result from ratios of the symmetries, and will therefore exist in the absence of Lie symmetries, i.e., zero symmetries. Zero symmetries lead to 0/0, and no further. With the Modified symmetries we get  $f(x, \omega)/g(x, \omega) = 0/0$  as  $\omega$  goes to zero, and there are numerous mathematical techniques through which this can be resolved.

We develop this concept using tensors and exterior calculus, and elaborate its application exhaustively through a numerical example.

**Keywords:**

## INTRODUCTION

Lie symmetry theoretical methods is a theory first proposed by Marius Sophus Lie (1842–1899), a Norwegian mathematician, through his now famous 1881 paper [1]. It makes the use of groups to analyze and solve differential equations.

Our view is that this theory, like other theories, has its challenges. It seeks to convert the differential equations into integral forms, from which integral rules are then applied. The conversion to integral forms works effortlessly, but the same can not be said about the integration. Furthermore, the resulting solutions do not always address all possible cases the equation is expected to describe. This we illustrate by means of an example in Section .

Section is on our suggestion on how the challenges discussed in Section can be addressed. We proposed new type of symmetries that we call Modified Lie symmetries. This we do through tensors, exterior calculus and group classification.

We demonstrate the usefulness and validity of the new symmetries in Section . This involves another analyses of the example introduced in Section through these new symmetries.

## AN ILLUSTRATIVE EXAMPLE

The heat equation

$$u_{xx} = u_t, \quad (1)$$

is a generally preferred example to illustrate Lie's theory, and showcase the solution it leads to, and that it can be reproduced through other methods. Which is true, but the reverse is not true, because in practice, be experiments or other mathematical techniques, other solutions have been observed, but cannot be arrived at through the pure regular Lie approach.

A full symmetry analysis of (1) can be found a wide range of texts in symmetry analysis, including the book by

Bluman and Kumei [2]. Here we provide a terse form to facilitate comparison with our own analyses, presented in Section . The symmetry generator used on (1) is of the form

$$G = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}, \quad (2)$$

and leads to the monomials

$$u_{tx} : T_x = 0, \quad (3)$$

$$u_t : T_t - T_{xx} - 2X_x = 0, \quad (4)$$

$$u_x : 2f_x - X_{xx} + X_t = 0, \quad (5)$$

$$u : f_{xx} - f_t = 0, \quad (6)$$

$$1 : g_{xx} - g_t = 0, \quad (7)$$

from which follows the symmetries

$$G_1 = \frac{t^2}{2} \frac{\partial}{\partial t} + \frac{xt}{2} \frac{\partial}{\partial x} - \left( \frac{x^2}{8} + \frac{t}{4} \right) u \frac{\partial}{\partial u}, \quad (8)$$

$$G_2 = t \frac{\partial}{\partial t} + \frac{x}{2} \frac{\partial}{\partial x}, \quad (9)$$

$$G_3 = \frac{\partial}{\partial t}, \quad (10)$$

$$G_4 = t \frac{\partial}{\partial x} - \frac{xu}{2} \frac{\partial}{\partial u}, \quad (11)$$

$$G_5 = \frac{\partial}{\partial x}, \quad (12)$$

$$G_6 = u \frac{\partial}{\partial u}, \quad (13)$$

$$G_\infty = g \frac{\partial}{\partial u}. \quad (14)$$

The symmetry  $G_\infty$  is called the infinite symmetry.

The solution follows from symmetry  $G_1$ . It leads to characteristic equations

$$\frac{2dt}{t^2} = \frac{2dx}{xt} = \frac{du}{-\left(\frac{x^2}{8} + \frac{t}{4}\right)u}, \quad (15)$$

from which the solution is found to be

$$u = e^{\frac{x^2}{4t}} \left( D_1 \frac{x}{t} + D_2 \right) \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}}, \quad (16)$$

where  $D_1$  and  $D_2$  are constants. Other symmetries also lead to solutions, but are too abstract to interpret practically.

## THEORETICAL BASIS

The primary distinction between the regular Lie symmetries and our Modified Lie symmetries, or simply Modified

symmetries, is that while Lie symmetries result from point transformations, ours result the transformations of the point and its neighborhood.

To begin, consider a point  $\mathbf{x} = (x^0; x^1; x^2; x^3)$  in Space-time, and

$$A = \left\{ \xi^0 \frac{\partial}{\partial x^0}; \xi^1 \frac{\partial}{\partial x^1}; \xi^2 \frac{\partial}{\partial x^2}; \xi^3 \frac{\partial}{\partial x^3} \right\}, \quad (17)$$

a set of Lie symmetries on a tangent space at the point, with the dual space

$$D = \{dx^0, dx^1, dx^2, dx^3\}. \quad (18)$$

The set  $A$  can be written in the notation

$$A = \{\xi^0 \partial x^0; \xi^1 \partial x^1; \xi^2 \partial x^2; \xi^3 \partial x^3\}, \quad (19)$$

to match  $D$ . We use these notations interchangeably.

In the neighborhood  $\mathbf{x}$ , is another tangent vector space(The envisaged Modified Lie symmetries)

$$\tilde{A} = \left\{ \tilde{\xi}^0 \partial \tilde{x}^0; \tilde{\xi}^1 \partial \tilde{x}^1; \tilde{\xi}^2 \partial \tilde{x}^2; \tilde{\xi}^3 \partial \tilde{x}^3 \right\}, \quad (20)$$

at point  $\tilde{\mathbf{x}}$ , with the dual space

$$\tilde{D} = \{d\tilde{x}^0; d\tilde{x}^1, d\tilde{x}^2, d\tilde{x}^3\}. \quad (21)$$

This point is in the neighborhood of  $\mathbf{x}$ . That is,

$$B(x_i - \tilde{x}_i) < \omega_i, \quad (22)$$

for some real number  $\omega_i$ , with  $\omega = (\omega_0, \omega_1, \omega_2, \omega_3)$ .

## Metric spaces approach

### One-dimensional tensor space

From  $v \in A$  and  $\tilde{v} \in \tilde{A}$ , we have the relation between the map  $\langle dx, v \rangle$  to  $\langle d\tilde{x}, \tilde{v} \rangle$  given by

$$(Q^I P_J \xi^J - \tilde{Q}^I \tilde{P}_J \tilde{\xi}^J) \delta_J^I = 0. \quad (23)$$

This is the formula for determining the Modified symmetries, for a one-dimensional tensor space. It follows from

$$\begin{aligned} \langle dx, v \rangle &= \langle Q^I dx^I, P_J \xi^J \partial x^J \rangle \\ &= Q^I P_J \xi^J \langle dx^I, \partial x^J \rangle \\ &= Q^I P_J \xi^J \delta_J^I, \end{aligned} \quad (24)$$

as per Einstein's notation. Similarly,

$$\langle d\tilde{x}, \tilde{v} \rangle = \tilde{Q}^I \tilde{P}_J \tilde{\xi}^J \delta_J^I. \quad (25)$$

Hence,  $\langle dx, v \rangle = \langle d\tilde{x}, \tilde{v} \rangle$  in the limit of  $\omega = |\mathbf{x} - \tilde{\mathbf{x}}|$  approaches zero.

### A demonstration on the use of equation (23)

As a demonstration of the procedure to be followed, we determine  $\tilde{\xi}$  from  $\xi$ :

- A single vector in A reduces (23) to

$$QP\xi - \tilde{Q}\tilde{P}\tilde{\xi} = 0, \quad (26)$$

or simply

$$P\xi - \tilde{P}\tilde{\xi} = 0. \quad (27)$$

- To determine the coefficients, we note that in the limit the drag is zero, we have (27) becoming to

$$(\tilde{P} - p)\xi = 0. \quad (28)$$

This means

$$\tilde{P} - P = 0 \quad (29)$$

or better still as

$$\tilde{P} - P = \delta = 0. \quad (30)$$

where  $\delta$  is an infinitesimal parameter.

- But the system is under-determined, meaning we need another equation so that both  $\tilde{P}$  and  $P$  can be determined. That comes from the commutator, applied on (27). That is,

$$[P\xi, \xi] - [\tilde{P}\tilde{\xi}, \xi] = 0, \quad (31)$$

so that

$$\frac{d}{dX} (\tilde{P} - P) = 0, \quad (32)$$

or

$$\frac{d}{dX} (\tilde{P} - P) = \epsilon, \quad (33)$$

where  $\epsilon$  is another infinitesimal parameter.

- The ratio of (30) and (33) gives

$$\frac{\frac{d}{dX} (\tilde{P} - P)}{\tilde{P} - P} = \frac{\epsilon}{\delta}, \quad (34)$$

so that

$$\tilde{P} - P = C \exp\left(\frac{\epsilon}{\delta} X\right), \quad (35)$$

or

$$\tilde{P} - P = \frac{\exp\left(\frac{\epsilon}{\delta} X\right)}{C}, \quad (36)$$

where  $C$  is an integration constant. Hence, or

$$\tilde{P} - P = \frac{\sin\left(i\frac{\epsilon}{\delta} X\right)}{iC} + \frac{\cos\left(i\frac{\epsilon}{\delta} X\right)}{C}. \quad (37)$$

That is,

$$\tilde{P} = P + \frac{\sin\left(i\frac{\epsilon}{\delta} X\right)}{iC} + \frac{\cos\left(i\frac{\epsilon}{\delta} X\right)}{C}. \quad (38)$$

The symmetry  $\tilde{v}$  is then

$$\tilde{v} = \left( QP + Q \frac{\sin\left(i\frac{\epsilon}{\delta}[X + \chi]\right)}{iC} \right) \xi \frac{\partial}{\partial Y}. \quad (39)$$

Because at the center  $\tilde{v} = v$ , we require  $QP = 1$ , so that

$$\tilde{v} = \left( 1 + Q \frac{\sin\left(i\frac{\epsilon}{\delta}[X + \chi]\right)}{iC} \right) \xi \frac{\partial}{\partial Y}. \quad (40)$$

By the same condition, we have

$$Q/(iC) = \omega \ll 1, \quad (41)$$

so that

$$\tilde{v} = \left( 1 + \omega \sin\left(i\frac{\epsilon}{\delta}[X + \chi]\right) \right) \xi \frac{\partial}{\partial Y}. \quad (42)$$

Hence, we have the symmetries

$$\tilde{v}_1 = \xi \frac{\partial}{\partial Y}, \text{ and } \tilde{v}_2 = \omega \sin\left(i\frac{\epsilon}{\delta}[X + \chi]\right) \xi \frac{\partial}{\partial Y}, \quad (43)$$

The symmetry  $\tilde{v}_1$  is already known,  $\tilde{v}_2$  is the Modified symmetry.

*Two-dimensional tensor product spaces*

The metric tensors,  $T$  and  $\tilde{T}$ , arise from the tensor product vector spaces

$$\{v \times v, D \otimes D\} \text{ and } \{\tilde{v} \times \tilde{v}, \tilde{D} \otimes \tilde{D}\}, \quad (44)$$

respectively. The action of the tensor

$$T = \tau_{ij} dx^i \otimes dx^j, \quad (45)$$

on the vectors  $\{v, u\} \subset A$ , as  $T(v, u)$ , gives

$$\begin{aligned} T(v, u) &= \tau_{ij} dx^i \otimes dx^j (v, u) \\ &= \tau_{ij} dx^i \otimes dx^j (v^k \partial_k, u^l \partial_l) \\ &= \tau_{ij} v^k u^l dx^i \otimes dx^j (\partial_k, \partial_l). \end{aligned} \quad (46)$$

That is,

$$\begin{aligned} T(v, u) &= \tau_{ij} v^k u^l \langle dx^i, \partial_k \rangle \langle dx^j, \partial_l \rangle \\ &= \tau_{ij} v^k u^l \delta_k^i \delta_l^j. \end{aligned} \quad (47)$$

Similarly, the action of  $\tilde{T}$  on  $\{\tilde{T}, (\tilde{v}, \tilde{u})\} \subset \tilde{A}$  gives

$$\tilde{T}(\tilde{v}, \tilde{u}) = \tilde{\tau}_{mn} \tilde{v}^p \tilde{u}^q \delta_p^m \delta_q^n = \tilde{\tau}_{mn} v^m u^n. \quad (48)$$

In the limit  $T = \tilde{T}$ , we have

$$(\tau_{ij} v^k u^l - \tilde{\tau}_{ij} \tilde{v}^k \tilde{u}^l) \delta_k^i \delta_l^j = 0, \quad (49)$$

which is the equation for determining Modified symmetries in tensor product spaces. This discussion can easily be extended to the system

$$\begin{aligned} \{v \times v \times v \times v\} \text{ with } \{D \otimes D \otimes D \otimes D\} \\ \text{and} \\ \{\tilde{v} \times \tilde{v} \times \tilde{v} \times \tilde{v}\} \text{ with } \{\tilde{D} \otimes \tilde{D} \otimes \tilde{D} \otimes \tilde{D}\}. \end{aligned}$$

### The approach through Exterior Calculus and Lie group classification

Exterior calculus allows us to choose parts of the neighborhoods from which the Modified symmetries should come from. Here we will focus on the ones on congruent surfaces to the tangent spaces at the center, which we call Smart symmetries. *Smart* because they reduce the evaluations of integral to limit properties that is simpler by comparison to the rules of integral calculus.

The Modified symmetries can be partitioned. Some of these symmetries arise in situations where regular Lie symmetries do not exist. Others are from neighborhood points at the intersection of congruent and parallel curves. While others are from neighborhoods that do not have central points. Just an empty space at the center.

The Lie derivative leads us to Modified symmetries on congruent surfaces. This requires the use of this derivative in limit form. That is,

$$\lim_{\omega_C \rightarrow 0} \left( \frac{\tilde{\xi} \frac{\partial}{\partial x^i} - \xi \frac{\partial}{\partial x^i}}{\omega_C} \right) = \mathfrak{f}(\tilde{\xi})_V, \quad (50)$$

subject to

$$\xi_{x^i x^i} = 0. \quad (51)$$

The parameter  $\omega_C$ , is called the Lie drag, while the choice for the condition (51) is to limit the order of the

differential equation that we are about to derive, out of the Lie derivative. The expression can be rewritten in the form

$$\lim_{\omega_C \rightarrow 0} \left( \tilde{\xi} \frac{\partial}{\partial x^i} - \xi \frac{\partial}{\partial x^i} \right) = \lim_{\omega_C \rightarrow 0} (\omega_C \mathfrak{f}(\xi)_V), \quad (52)$$

or simply

$$\tilde{\xi} \frac{\partial}{\partial x^i} - \xi \frac{\partial}{\partial x^i} = 0. \quad (53)$$

The exterior derivative leads to

$$\frac{\partial \Phi}{\partial x^*} - \frac{\partial \Phi}{\partial x} = 0, \quad (54)$$

with new variables to simplify. This then gives

$$\frac{\partial \Phi}{\partial x^*} - \left( \frac{dy}{dx} \right) \frac{\partial \Phi}{\partial y} = 0, \quad (55)$$

or

$$\frac{\partial^3 \Phi}{\partial x^{*3}} - \left( \frac{dx}{dy} \frac{\partial^2}{\partial x^{*2}} \left[ \frac{dy}{dx} \right] \right) \frac{\partial \Phi}{\partial x^*} = 0. \quad (56)$$

That is,

$$\frac{\partial^3 \Phi}{\partial x^{*3}} - \left( \frac{dx}{dy} \frac{d^3 y}{dx^3} \right) \frac{\partial \Phi}{\partial x^*} = 0. \quad (57)$$

It can be simplified to

$$u_{tt} + f(t)u = 0, \quad (58)$$

where  $t = x^*$ ,  $u(t) = \partial \Phi / \partial x^*$  and

$$f(t) = \frac{dx}{dy} \frac{d^3 y}{dx^3}. \quad (59)$$

### The group classification

The analysis and classification of differential equations using group theory goes back to Sophus Lie. The first systematic investigation of the problem of group classification was done by L.V. Ovsiannikov [3] in 1959 for nonlinear heat equation

$$u_t = [f(u)u_x] u_{xx}, \quad (60)$$

where  $f(u)$  is an arbitrary nonlinearity. Other works subsequent to that the analyses by Akhatov, Gazizov and Ibragimov [4] of the equation

$$u_t = [g(u_x)] u_{xx}, \quad (61)$$

where  $g(u_x)$  is an arbitrary nonlinearity. Dorodnitsyn [5]: so that

$$u_t = G(u)u_{xx} + \frac{dG(u)}{du}u_x^2 + g(u), \quad (62)$$

where  $G(u)$  and  $g(u)$  are arbitrary functions. There are many more.

We seek here a continuous group of equivalence transformations for equation (58) through the generator

$$Y = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \mu \frac{\partial}{\partial f}, \quad (63)$$

where  $\mu$  is sought as a function of  $t, u, f$  and  $u'$ .

$$Y^2 = Y + \zeta^2 \frac{\partial}{\partial u''} \quad (64)$$

with prolongation

$$\begin{aligned} \zeta^2 = & \eta_{tt} + (2\eta_{tu} - \xi_{tt})u' + (\eta_{uu} - 2\xi_{tu})u'^2 \\ & - u'^3\xi_{uu} + (\eta_u - 2\xi_t - 3u'\xi_u)u'' + \mu^t \frac{\partial}{\partial f_t} \\ & + \mu^u \frac{\partial}{\partial f_u} + \mu^{u'} \frac{\partial}{\partial f_{u'}}, \end{aligned} \quad (65)$$

where

$$\begin{aligned} \mu^t = & \tilde{D}_t(\mu) - f_t \tilde{D}_t(\xi) - f_u \tilde{D}_t(\eta) - \\ & f_{u'} \tilde{D}_t(\zeta^1), \end{aligned} \quad (66)$$

$$\begin{aligned} \mu^u = & \tilde{D}_u(\mu) - f_t \tilde{D}_u(\xi) - f_u \tilde{D}_u(\eta) - \\ & f_{u'} \tilde{D}_u(\zeta^1), \end{aligned} \quad (67)$$

$$\begin{aligned} \mu^{u'} = & \tilde{D}_{u'}(\mu) - f_t \tilde{D}_{u'}(\xi) - f_u \tilde{D}_{u'}(\eta) - \\ & f_{u'} \tilde{D}_{u'}(\zeta^1), \end{aligned} \quad (68)$$

where

$$\tilde{D}_t = \frac{\partial}{\partial t} + f_t \frac{\partial}{\partial f}, \quad (69)$$

$$\tilde{D}_u = \frac{\partial}{\partial u} + f_u \frac{\partial}{\partial f}, \quad (70)$$

$$\tilde{D}_{u'} = \frac{\partial}{\partial u'} + f_{u'} \frac{\partial}{\partial f}. \quad (71)$$

The invariance

$$Y^2(u_{tt} + f(t)u) \Big|_{u_{tt}+f(t)u=0} = 0 \quad (72)$$

leads to

$$\begin{aligned} & \eta_{tt} + (2\eta_{tu} - \xi_{tt})u' + (\eta_{uu} - 2\xi_{tu})u'^2 \\ & - u'^3\xi_{uu} + (\eta_u - 2\xi_t - 3u'\xi_u)f_u \\ & \xi f'u + \eta f + \mu u = 0, \end{aligned} \quad (73)$$

$$u'^3 : \xi_{uu} = 0, \quad (74)$$

$$u'^2 : \eta_{uu} - 2\xi_{tu} = 0, \quad (75)$$

$$u' : 2\eta_{tu} - \xi_{tt} = 0, \quad (76)$$

$$\begin{aligned} 1 : & \eta_{tt} + (\eta_u - 2\xi_t)f_u + \xi f'u \\ & + \eta f + \mu u = 0, \end{aligned} \quad (77)$$

From (74) we have

$$\xi = ua + b, \quad (78)$$

so that (75) gives

$$\eta = u^2 a' + uc + d, \quad (79)$$

where  $a, b, c$  and  $d$  are functions of  $t$ . Substituting  $\xi$  and  $\eta$  into (76):

$$4ua'' + 2c' - ua'' - b'' = 0, \quad (80)$$

from which

$$a = A_1 t + A_2, \quad (81)$$

and

$$2c = b' + B_1 t. \quad (82)$$

Hence,

$$\xi = u(A_1 t + A_2) + b, \quad (83)$$

and

$$\eta = u^2 A_1 + \frac{u}{2}(b' + B_1) + d, \quad (84)$$

together with the classifying result

$$\begin{aligned} & uc'' + d'' + (c - 2b')fu \\ & + [ua + b]f'u + (u^2 a' + uc + d)f + \mu u = 0 \end{aligned} \quad (85)$$

where  $A_1, A_2, B_1$  and  $B_2$  are constants. The generators for the principal algebra are

$$Y_1 = ut \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial u} \quad (86)$$

$$Y_2 = \frac{u}{2} \frac{\partial}{\partial u} \quad (87)$$

$$Y_b = b \frac{\partial}{\partial t} \quad (88)$$

$$Y_{b'} = \frac{u}{2} b' \frac{\partial}{\partial u} \quad (89)$$

$$Y_d = d \frac{\partial}{\partial u} \quad (90)$$

$$Y_f = \mu \frac{\partial}{\partial f}, \quad (91)$$

with

$$f = -\frac{d''}{d}, \quad (92)$$

and

$$\begin{aligned} \mu &= -(uc'' + d'' + (c - 2b')fu \\ &+ [ua + b]f'u + (u^2a' + uc + d)f)/u. \end{aligned} \quad (93)$$

## APPLICATIONS

The monomial (3) can be expressed in the form

$$T_{xx} = 0, \quad (94)$$

thus satisfying the condition (51), used to develop the procedure for determining smart Modified symmetries. This procedure leads to

$$\begin{aligned} \tilde{G}_1 &= \frac{2e^{\omega^2 t}}{\omega^4(\omega^2 - 1)} \cos(\omega x/i) \frac{\partial}{\partial t} \\ &+ \frac{ie^{\omega^2 t}}{\omega^3(\omega^2 - 1)} \sin(\omega x/i) \frac{\partial}{\partial x} \\ &- \frac{e^{\omega^2 t}}{2} \cos(\omega x/i) u \frac{\partial}{\partial u}, \end{aligned} \quad (95)$$

$$\begin{aligned} \tilde{G}_2 &= -\frac{2\phi e^{\omega^2 t}}{\omega^4(\omega^2 + 1)} \sin(\omega x/i) \frac{\partial}{\partial t} \\ &+ \frac{i\phi e^{\omega^2 t}}{\omega^3(\omega^2 + 1)} \cos(\omega x/i) \frac{\partial}{\partial x} \\ &- \frac{\phi e^{\omega^2 t}}{2} \sin(\omega x/i) u \frac{\partial}{\partial u}, \end{aligned} \quad (96)$$

$$\begin{aligned} \tilde{G}_3 &= -2\phi t \sin(\omega x/i) \frac{\partial}{\partial t} \\ &+ \frac{i\phi}{\omega} \cos(\omega x/i) \frac{\partial}{\partial x}, \end{aligned} \quad (97)$$

$$\begin{aligned} \tilde{G}_4 &= 2t \cos(\omega x/i) \frac{\partial}{\partial t} \\ &+ \frac{i}{\omega} \sin(\omega x/i) \frac{\partial}{\partial x}, \end{aligned} \quad (98)$$

$$\begin{aligned} \tilde{G}_5 &= 2\phi e^{-t} \sin(\omega x/i) \frac{\partial}{\partial t} \\ &+ \frac{i\phi}{\omega} e^{-t} \cos(\omega x/i) \frac{\partial}{\partial x}, \end{aligned} \quad (99)$$

$$\begin{aligned} \tilde{G}_6 &= 2e^t \cos(\omega x/i) \frac{\partial}{\partial t} \\ &+ \frac{i}{\omega} e^t \sin(\omega x/i) \frac{\partial}{\partial x}, \end{aligned} \quad (100)$$

$$\tilde{G}_7 = \frac{\partial}{\partial t}, \quad (101)$$

$$\tilde{G}_8 = u \frac{\partial}{\partial u}. \quad (102)$$

The last defining equation leads to an infinite symmetry generator.

$$\tilde{G}_\infty = g(t, x) \frac{\partial}{\partial u}. \quad (103)$$

## CONSTRUCTION OF SOLUTIONS

In Section , the symmetry  $G_1$  led to a single solution. We demonstrate here that its modified component  $\tilde{G}_1$ , leads to more than one, and are all practical. In addition, solutions resulting from the other Modified symmetries also lead to practical results, which can be reproduced through other techniques.

*Invariant solutions through the symmetry  $\tilde{G}_1$*

The characteristic equations that arise from the symmetry  $\tilde{G}_1$  lead to

$$\lambda = -\omega^2 t - 2 \ln |\sin(\omega x/i)|, \quad (104)$$

and

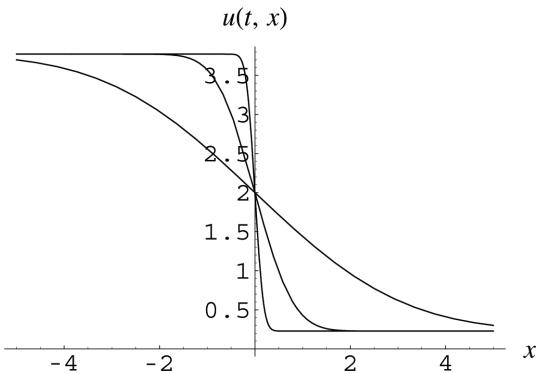
$$\eta = e^{\frac{\omega^2}{2} t} |\sin(\omega x/i)| \quad (105)$$

where  $\eta = \exp(-\lambda/2)$ , so that

$$u = e^{(\omega^4(\omega^2 - 1)t/4)} \phi(\eta), \quad (106)$$

$$\begin{aligned} u_t &= \frac{\omega^4(\omega^2 - 1)}{4} e^{(\omega^4(\omega^2 - 1)t/4)} \phi \\ &+ \frac{\omega^2}{2} \eta e^{(\omega^4(\omega^2 - 1)t/4)} \dot{\phi}. \end{aligned} \quad (107)$$

$$\begin{aligned} u_{xx} &= \omega^2 e^{(\omega^4(\omega^2 - 1)t/4)} \ddot{\phi} \left( e^{\omega^2 t} - \eta^2 \right) \\ &+ \omega^2 \eta e^{(\omega^4(\omega^2 - 1)t/4)} \dot{\phi}. \end{aligned} \quad (108)$$



**FIGURE 1.** Plot of the solution in (112) for equation (1).

Substituting  $u_t$  and  $u_{xx}$  into (1) gives

$$\ddot{\phi} = \frac{1}{2} \frac{\eta}{\eta^2 - 1}, \quad (109)$$

so that

$$\int_{\eta_1}^{\eta_2} \frac{d}{d\eta} (\ln \dot{\phi}) d\eta = \frac{1}{2} \int_{\eta_1}^{\eta_2} \frac{\tilde{\eta}}{\tilde{\eta}^2 - 1} d\tilde{\eta}. \quad (110)$$

which leads to

$$u = e^{(\omega^4(\omega^2-1)t)/4} \left[ F_0 \int_{\eta_1}^{\eta_2} e^{\frac{-x^2}{4t}} d\tilde{\eta} \right]. \quad (111)$$

The first solution through  $\tilde{G}_1$

When  $F_0 = -iA/\omega$  and  $\omega = 0$  inside the integral in (111), we get

$$u = Ae^{(\omega^4(\omega^2-1)t)/4} \int_{x_1}^{x_2} e^{\frac{-x^2}{4t}} dx. \quad (112)$$

The plot of this result is given in Figure 1. What is in Figure 2 is the same solution obtained through other means by Fassari and Rinaldi [6].

The second solutions through  $\tilde{G}_1$ : Olsen's result.

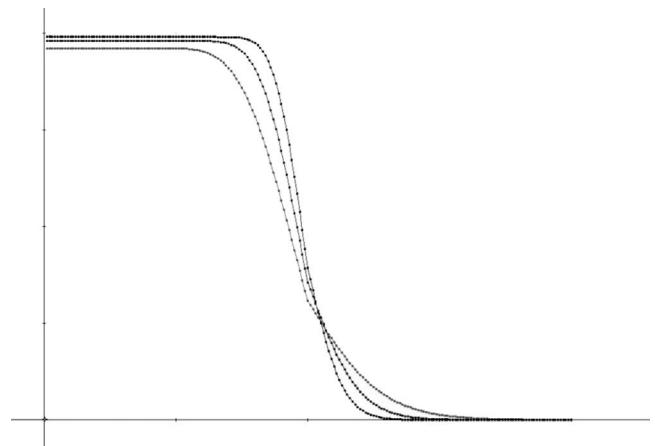
The second solution for (1) is

$$u = \frac{1}{\sqrt{(\omega^2-1)t}} \frac{F_0 \int_{\eta_1}^{\eta_2} e^{\frac{-x^2}{4t}} d\tilde{\eta}}{\omega^2}. \quad (113)$$

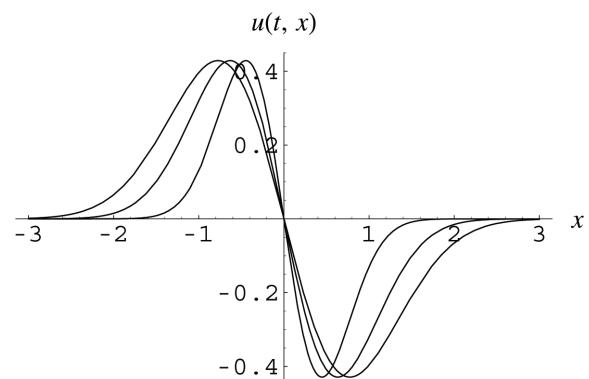
The L'Hopital's principle reduces it to

$$u = F_0 \frac{x}{2\sqrt{t}} e^{\frac{-x^2}{4t}}. \quad (114)$$

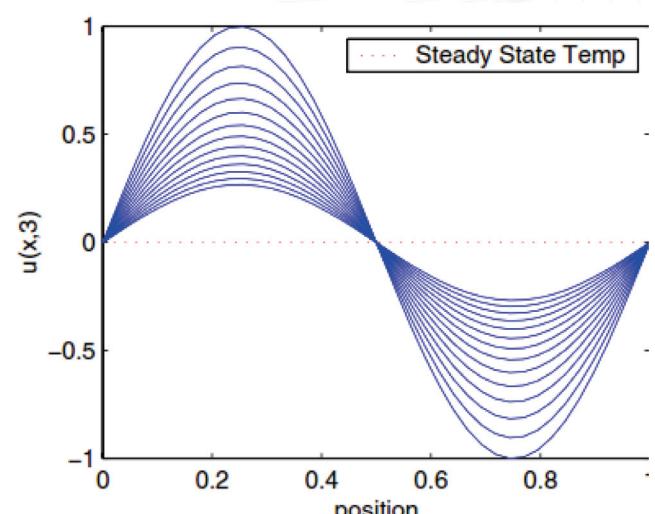
This solution is sketched in Figure 3. A similar result by Richards and Abrahamsen [7] is in Figure 4.



**FIGURE 2.** Plot of the solution obtained by Fassari and Rinaldi [6] for equation (1), similar to the one in Figure 1.



**FIGURE 3.** Plot of the solution in (114) for equation (1).



**FIGURE 4.** Plot of the solution obtained by Richards and Abrahamsen [7] for equation (1), similar to the one in Figure 3.

### The third solution through $\tilde{G}_1$

A third solution assumes the form

$$u = F_0 \frac{x}{2t^{3/2}} e^{\frac{-x^2}{4t}}. \quad (115)$$

This result is the same as the second component in Bluman's solution with  $C_2 = F_0/2$ .

### Invariant solutions through the symmetry $\tilde{G}_2$

The characteristic equation of  $\tilde{G}_2$  leads to

$$\eta = e^{\frac{\omega^2}{2}t} |\cos(\omega x/i)|, \quad (116)$$

and

$$u = e^{(\omega^4(\omega^2+1)t/4)} \phi(\eta), \quad (117)$$

$$\begin{aligned} u_t &= \frac{\omega^4(\omega^2+1)}{4} e^{(\omega^4(\omega^2+1)t/4)} \phi \\ &\quad + \frac{\omega^2}{2} \eta e^{(\omega^4(\omega^2+1)t/4)} \dot{\phi}, \end{aligned} \quad (118)$$

$$\begin{aligned} u_{xx} &= \omega^2 e^{(\omega^4(\omega^2+1)t/4)} \ddot{\phi} (e^{-\omega^2 t} - \eta^2) \\ &\quad + \omega^2 \eta e^{(\omega^4(\omega^2+1)t/4)} \dot{\phi}. \end{aligned} \quad (119)$$

Substituting the expression for  $u_t$  and  $u_{xx}$  into (1) gives

$$\frac{\ddot{\phi}}{\dot{\phi}} = \frac{1}{2} \frac{\eta}{\eta^2 - 1}. \quad (120)$$

That is,

$$\int \frac{d}{d\eta} (\ln \dot{\phi}) d\eta = F_0 + \frac{1}{2} \int \frac{\eta}{\eta^2 - 1} d\eta, \quad (121)$$

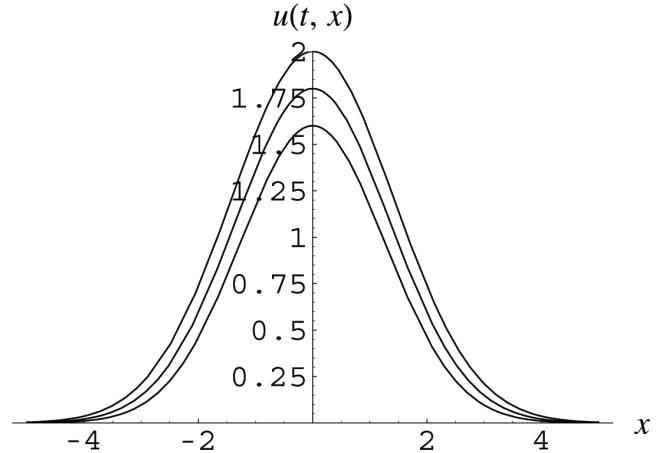
where  $F_0$  is a constant, leading to

$$u = e^{(\omega^4(\omega^2-1)t/4)} \left[ F_1 + F_0 \int e^{\frac{-x^2}{4t}} d\eta \right]. \quad (122)$$

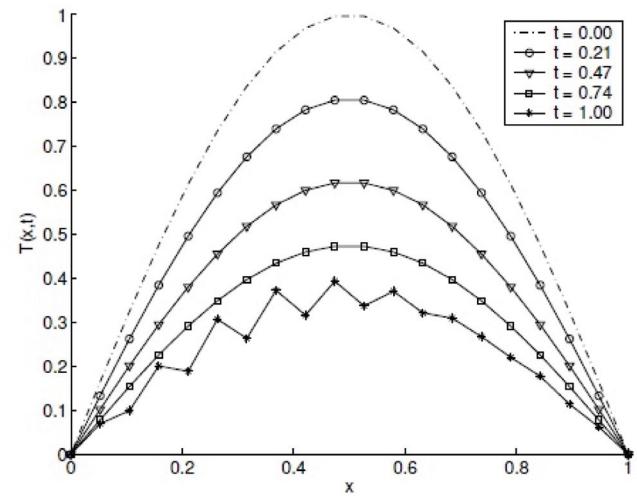
### The first solution through $\tilde{G}_2$

When  $\omega = 0$  and  $F_0 = -A/\omega$  in (122), we get

$$u = F_1 + A e^{\frac{-x^2}{4t}}. \quad (123)$$



**FIGURE 5.** Plot of the solution in (123) for equation (1).



**FIGURE 6.** Plot of the solution by Gerald Recktenwald, similar to the one in Figure 5.

*The second solution through  $\tilde{G}_2$ : Bluman's second result.*

The second solution through  $\tilde{G}_2$  follows a similar procedure as was for  $\tilde{G}_1$ , leading to

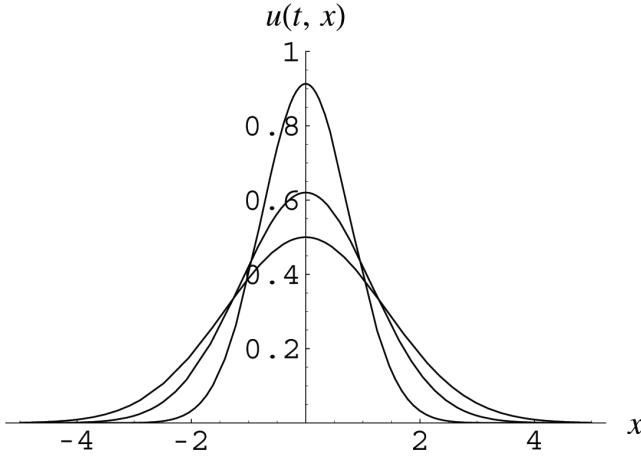
$$u = \frac{A}{2\sqrt{t}} e^{\frac{-x^2}{4t}}. \quad (124)$$

This result is the same as the first component in Bluman's solution with  $C_1 = 1/2$ . It is sketched in Figure 7. A similar result by Balluffi, Allen and Carter [8] is in Figure 8.

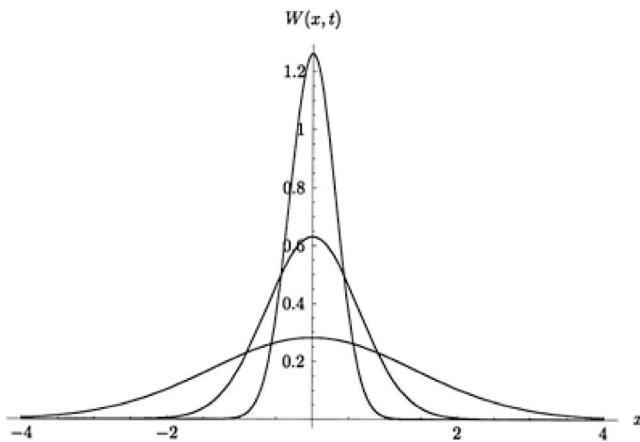
*The third solution through  $\tilde{G}_2$ : Ibragimov's result*

Like the second solution, a third solution takes the form

$$u = \frac{A}{2t^{3/2}} e^{\frac{-x^2}{4t}}. \quad (125)$$



**FIGURE 7.** Plot of the solution in (124) for equation (1).



**FIGURE 8.** Plot of the solution obtained by Balluffi, Allen and Carter [8] for equation (1), similar to the one in Figure 7.

This result is the same is a special case of Ibragimov's solution with  $n = 3$ .

#### Other solutions through $\tilde{G}_2$

One other solution that follows from  $\tilde{G}_2$ , through a different manipulation of the limits, is

$$u = F_1 + e^{\omega^4(\omega^2+1)} F_0 \int_{\eta_1}^{\eta_2} e^{\pm \frac{x^2}{2(x^2-t^2)}} d\tilde{\eta}. \quad (126)$$

#### A first couple of solutions through $\tilde{G}_2$

A simple pair of solutions results from (126) when  $\omega$  goes to zero requires. That is,

$$u = F_1 + F_0 \int_{\eta_1}^{\eta_2} e^{-\frac{x^2}{2(x^2-t^2)}} d\tilde{\eta} \quad (127)$$

and

$$u = F_1 + F_0 \int_{\eta_1}^{\eta_2} e^{\frac{x^2}{2(x^2-t^2)}} d\tilde{\eta}. \quad (128)$$

#### A second couple of solutions through $\tilde{G}_2$

Setting  $\eta_1 = \eta$  and  $\eta_2 = \eta + \omega$  in (126) and letting  $\omega$  go to zero requires that  $F_0 = -A/\omega$  for some constant  $A$ . This invokes L'hospital's principle, so that

$$u = F_1 + A e^{-\frac{x^2}{2(x^2-t^2)}} \quad (129)$$

and

$$u = F_1 + A e^{\frac{x^2}{2(x^2-t^2)}}. \quad (130)$$

#### A third couple of solutions through $\tilde{G}_2$

As was the case for  $\tilde{G}_1$ , the limits in Appendix B can be used to create more solutions. The following pair results:

$$u = F_1 + \frac{A}{\sqrt{t}} e^{-\frac{x^2}{2(x^2-t^2)}} \quad (131)$$

and

$$u = F_1 + \frac{A}{\sqrt{t}} e^{\frac{x^2}{2(x^2-t^2)}}. \quad (132)$$

#### A fourth couple of solutions through $\tilde{G}_2$

Continuing with the argument started in the preceding section leads to the fourth couple of solutions:

$$u = F_1 + \frac{A}{t^{3/2}} e^{-\frac{x^2}{2(x^2-t^2)}} \quad (133)$$

and

$$u = F_1 + \frac{A}{t^{3/2}} e^{\frac{x^2}{2(x^2-t^2)}}. \quad (134)$$

It is apparent from these calculations that though  $\tilde{G}_2$  are largely of the family

$$u = f(t, x) e^{\pm \frac{x^2}{2(x^2-t^2)}}. \quad (135)$$

### Applications: Heat conduction in thin plates

As mentioned in the Introduction, there are many methods used in practice to solve (1), an equation that finds application in a number of different situations. The backward heat equation

$$u_{xx} = -u_t, \quad (136)$$

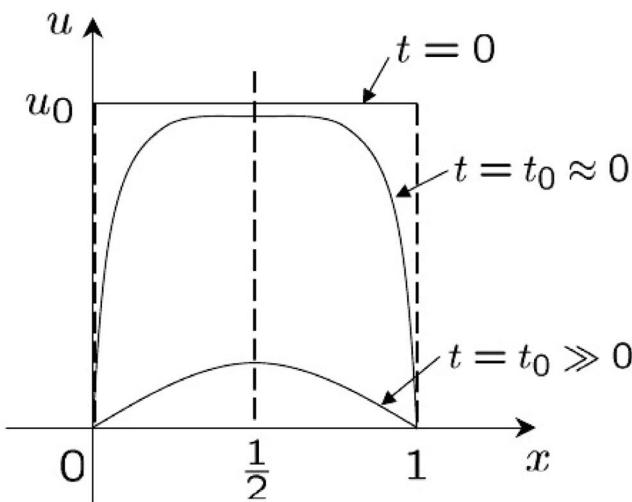
too, does arise in practice. Unfortunately, without analytical solutions, one could end up applying one of the two equations to a situation to which it does not apply.

For example, in a study on heat conduction in thin plates, Hancork [9] deduced solutions for (1) presented in Figure 9. These we unpack in Figures 10, 11 and 12 using (129). Unfortunately, practical results indicate it is (136) which is applicable to this situation. This we deduce from the fact that impractical singularities arise when  $u$  is plotted against  $t$  when (129) is used, but disappear when this expression assumes the form

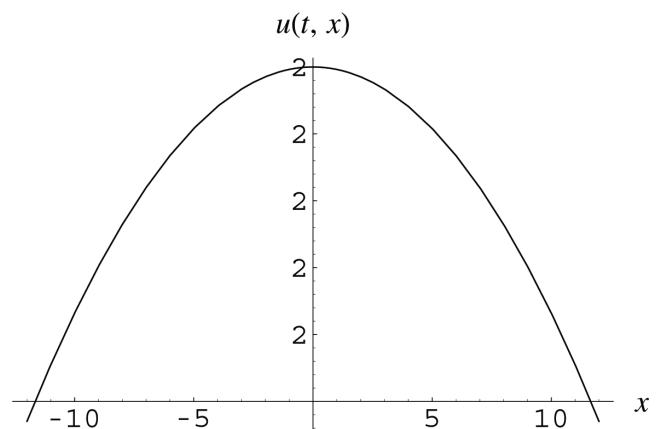
$$u = F_1 + Ae^{-\frac{x^2}{2(x^2+t^2)}}, \quad (137)$$

satisfying both (136). These are clearly in the family of the form

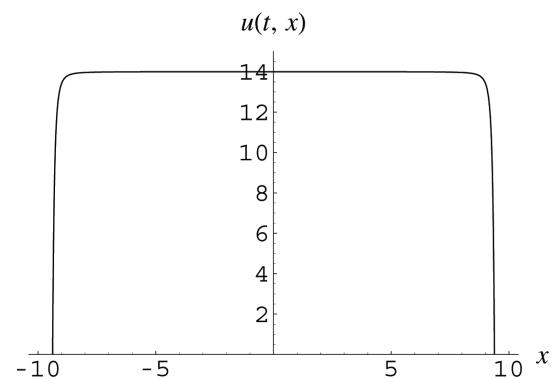
$$u = f(t, x)e^{\pm\frac{x^2}{2(x^2+t^2)}} \quad (138)$$



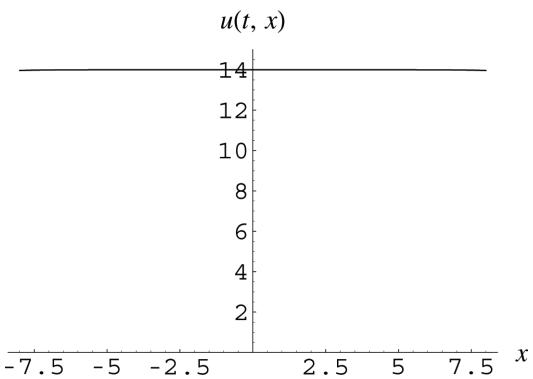
**FIGURE 9.** Plot of the solution obtained by Hancork [9] for equation (1) for cases  $t = 0$ ,  $t \approx 0$  and  $t \gg 0$ , all stacked onto the same sketch.



**FIGURE 10.** Plot of the solution in (129) for equation (1), similar to the one in Figure 9 for  $t = t_0 >> 0$ .



**FIGURE 11.** Plot of the solution in (129) for equation (1), similar to the one in Figure 9 for  $t = t_0 \approx 0$ .



**FIGURE 12.** Plot of the solution in (129) for equation (1), similar to the one in Figure 9 for  $t = t_0 = 0$ .

## DISCUSSION AND CONCLUSION

In this contribution, New type of Lie symmetries were suggested, called Modified Lie symmetries, simply because they involve a modification of the old symmetries, and

when you remove this modification you return to the regular Lie symmetries. Tensors and exterior calculus were used to develop procedures for determining them. The motivation was that although the regular Lie symmetries do yield some results, more are possible if we extend our search away from the center, and to the neighborhood.

The result in (42) suggests that  $Q = 0$  when  $\tilde{v} = 0$ , but that obviously follows from the initial assumptions. The bigger picture is that each Modified symmetry has an own daughter Modified symmetry, and this implies that (42) depicts an infinitesimal case of what is essentially an infinite polynomial-ed symmetry.

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