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Dom-Chromatic Number of Comb Product of Path with Snake Graphs

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Abstract

The dom-coloring problem combines the core concepts of domination and coloring in Graph Theory. This combination offers an effective approach to explore and analyze complicated network structures that yield numerous applications in various fields. This paper deals with the determination of dom coloring set and its minimum cardinality. Let $G(V, E)$ be a graph. A proper vertex coloring of a graph G is an assignment of colors to its vertices so that no two adjacent vertices have the same color. A group of vertices with the same color forms a color class. A non-empty subset S of the vertex set V of G is said to be a dominating set if for every vertex v in $V - S$, there is a vertex u in S such that u is adjacent to v. A dominating set is said to be a dom coloring set if it contains at least one vertex from each color class. The minimum cardinality of a dom-coloring set is said to be its dom chromatic number and is denoted by $\gamma_{dc}(G)$. The comb product of two graphs G and H, denoted by $G \triangleright H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identifying the i-th copy of H at the vertex v to the i-th vertex of G. In this paper, the dom-chromatic number for the comb product of a path with triangular snake and double triangular snake graph, quadrilateral snake and double quadrilateral snake graph, and diamond snake graph are determined and proved that the bound is sharp for these graphs.

Keywords: Dominating set, Dom Coloring set, Dom chromatic number, Comb product.

1. INTRODUCTION

Graph theory is the study of networks formed by vertices and edges. It has emerged as an important mathematical tool for representing complex systems such as communication structures, social interactions, and transportation networks. Among the numerous parameters investigated in this field, domination and coloring hold a significant place due to their theoretical richness and practical relevance. Domination deals with selecting a small set of vertices that can monitor or influence the entire graph, while coloring ensures that adjacent vertices are assigned different colors to avoid conflicts. When these two ideas are combined, they give rise to the problem called dom-coloring problem, which focuses on achieving both efficient control and proper conflict-free organization within the same network. This motivates the study of dom-coloring parameters, as they capture both control and separation requirements within a single model. This concept gives rise to a parameter called the dom-chromatic number, defined as the minimum number of colors required such that each color class includes atleast one dominating vertex. Exploring these combined properties not only deepens theoretical understanding but also supports practical improvements in areas such as sensor networks, scheduling, routing, and resource allocation. Graph coloring originated in 1852 when Francis Guthrie raised the Four Color Problem while studying map coloring. Arthur Cayley later published the first academic work on the topic in 1879 [1]. Domination in graph theory was initially defined by Oystein Ore in 1962 in his work Theory of Graphs [13]. A survey paper was published by Cockayne and Hedetniemi in which they first introduced the domination number of a graph in 1977 [6]. In 2010, T. N. Janakiraman and M. Poobalaranjani introduced the dom-coloring problem by defining a dom-coloring set S, which is a dominating set whose induced subgraph has the same chromatic number as the original graph, i.e., $\chi(\langle S\rangle) = \chi(G)$ [8]. In 2017, Poobalaranjani M. and Kirithka S., proved that the dom-chromatic problem is NP-complete [14]. The dom-chromatic number of cycle-related graphs appeared in 2020 [16], and for splitting graphs of path and comb graphs in 2021 [9], for caterpillar, coconut tree, lobster, tadpole, pan and lollipop graphs, mobius ladder and circular ladder in 2022 [10,11], for wrapped butterfly & bloom graphs in 2023 [12]. In 1978, Godsil C.D., & McKay B.D. introduced the rooted product of graphs, which forms the theoretical basis for later comb-type constructions [7]. In 2019, domination theory was applied to the comb product [2] and in 2020, chromatic and domination parameters specifically studied for the comb product [3] by Arumugam S., Kala R., & Gayathri G. In practical applications, domination or coloring alone may not fully capture the complexity of a network. In this paper, the dom-chromatic number of the comb product of a path with triangular snake and double triangular snake graph, quadrilateral snake and double quadrilateral snake graph, and diamond snake graph.

2. PRELIMINARIES

This section focuses on some basic definitions and terminology of graphs.

Definition 2.1. [1] *A proper vertex coloring of G assigns different colors to adjacent vertices. A set of vertices sharing a color is a color class.*

Definition 2.2. [1] *The chromatic number $\chi(G)$) is the minimum number of colors needed to color G so that adjacent vertices differ.*

Definition 2.3. [13] *A non-empty subset $D' \subset V(G)$ is a dominating set if every $v \in V(G) - D'$ is adjacent to some $u \in D'$. If D' has the minimum vertices, then it is the minimum dominating set, whose size is the domination number denoted by $\gamma(G)$.*

Definition 2.4. [8] ***DC - condition*** *In a k -coloring of G, a dominating set D' is a dom-coloring set if $D' = \{v \in V(G)/v$ belongs to atleast one color class $C_i, 1 \leq i \leq k$,and $D' \cap C_i \neq \phi, \forall i\}$. This is the dom-coloring condition (DC-condition). The size of the smallest such set is the dom-chromatic number, $\gamma_{dc}(G)$.*

Definition 2.5. [7] *Let v be a vertex of graph H. The comb product of graphs between G and H, denoted by $G \triangleright H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identifying the i-th copy of H at the vertex v to the i-th vertex of G.*

Definition 2.6. [4] *A walk is a finite sequence $W = v_o e_1 v_1 e_2 v_2 \ldots, e_k v_k$, whose terms are alternately vertices $v_i, 0 \leq i \leq k$ and edges $e_i, 1 \leq i \leq k$ such that the ends of e_i are v_{i-1} and v_i. If the vertices $v_0, v_1, v_2, \ldots, v_k$ are distinct, then W is a path.*

Definition 2.7. [17] *A graph obtained by replacing every edge of a path P_n with a cycle C_3 is called a triangular snake graph which is denoted by $T_S(P_n)$. In a triangular snake $|V(T_S(P_n))| = 2n - 1$ and $|E(T_S(P_n))| = 3(n - 1)$, where n is the number of vertices in the path.*

Definition 2.8. [17] *When two triangular snake graphs share a common path P_n, the graph obtained is a double triangular snake graph which is denoted by $DT_S(P_n)$.*

Definition 2.9. [17] *A graph obtained by replacing every edge of a path P_n with a cycle C_4 is called a quadrilateral snake graph which is denoted by $Q_S(P_n)$. Here $|V(Q_S(P_n))| = 3n - 2$ and $|E(Q_S(P_n))| = 4(n - 1)$.*

Definition 2.10. [17] *When two quadrilateral snake graphs share a common path P_n, the graph obtained is a double quadrilateral snake graph which is denoted by $DQ_S(P_n)$.*

Definition 2.11. [15] *A diamond snake graph has n cycles each of length 4, in which every 2 adjacent cycles have exactly one vertex in common. It has with $3n + 1$ vertices and $4n$ edges and is denoted by $D_S(n)$.*

Theorem 2.1. [5] *For any graph G,*

$$\max\{\gamma(G), \chi(G)\} \leq \gamma_{dc}(G) \leq \gamma(G) + \chi(G) - 1.$$

3. MAIN RESULTS

This section deals with the comb product of path with ladder related graphs namely triangular snake and double triangular snake graph, quadrilateral snake and double quadrilateral snake graph, and diamond snake graph. Here, we determine the dom chromatic number for the resulting graphs. Also the bound is proved to be sharp.

Theorem 3.1. *Let P_m be a path graph on m vertices and $T_S(P_n)$ be a triangular snake graph on $2n-1$ vertices, where $m = 2, 3, n = 2, 3$. Then $\gamma_{dc}(P_m \triangleright T_S(P_n)) = 3$.*

Proof. Let P_m be a path on m vertices labelled as $\{v_1, v_2, \ldots, v_m\}$ and $T_S(P_n)$ be a triangular snake on $2n - 1$ whose vertices labelled as $\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_n^{(i)}, w_1^{(i)}, w_2^{(i)}, \ldots, w_{n-1}^{(i)}\}, i = 1, 2, \ldots, m.$

Let $v_1^{(i)}, i = 1, 2, \ldots, m$ be the vertices of a triangular snake graph $T_S(P_n)$. The comb product of graphs between P_m and $T_S(P_n)$ denoted by $P_m \triangleright T_S(P_n)$, is a graph obtained by taking one copy of P_m and $|V(P_m)|$ copies of $T_S(P_n)$ and identifying the i-th copy of $T_S(P_n)$ at the vertex $v_1^{(i)}, 1 = 1, 2, \ldots, m$ to the i-th vertex of P_m with the vertex set as that of $T_S(P_n)$. The graph $P_m \triangleright T_S(P_n)$ has $m(2n-1)$ vertices and $m(3(n-1)) + m - 1$ edges.

Coloring and domination of $P_m \triangleright T_S(P_n)$ done by four cases:

Case(i): $m = 2, n = 2$

Color the vertices $\{v_1^{(1)}, v_2^{(i)}\}$ of $T_S(P_2)$ using two colors in the order 1, 2. Color the vertices $\{v_1^{(2)}, v_2^{(2)}\}$ of $T_S(P_2)$ using two colors in the order 2, 3. Color the vertex $\{w_1^{(1)},\}$ of $T_S(P_2)$ with the color 3. Color the vertices $\{w_1^{(2)}\}$ of $T_S(P_2)$ with the color 1.

Let $D = \{v_1^{(1), v_1^{(2)}}\}$ be a subset of $V(P_2 \triangleright T_S(P_2))$ such that every vertex in $V(P_2 \triangleright T_S(P_2)) - D$ therexist a vertex in D and are adjacent to the vertices in $V - D$. Hence, D is a minimum dominating set. This set, D doesn't satisfies DC-condition. Hence, choose $D' = D \cup \{v_2^{(2)}\}$ which contains minimum of one vertex from each color class of $V(P_2 \triangleright T_S(P_2))$.That is, D satisfies DC-condition. Therefore, D is a dom – coloring set with cardinality, $|D'| = 2 + 1 \implies \gamma_{dc}(P_2 \triangleright T_S(P_2)) = 3$.

Case(ii): $m = 2, n = 3$

Color the vertices $\{v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\}$ of $T_S(P_3)$ using three colors in the order 1, 2, 3. Color the vertices $\{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\}$ of $T_S(P_3)$ using three colors in the order 2, 3, 1. Color the vertices $\{w_1^{(1)}, w_2^{(1)}\}$ of $T_S(P_3)$ using two colors in the order 3, 1. Color the vertices $\{w_1^{(2)}, w_2^{(2)}\}$ of $T_S(P_3)$ using two colors in the order 1, 2.

Let $D = \{v_2^{(1), v_2^{(2)}}\}$ be a subset of $V(P_2 \triangleright T_S(P_3))$ such that every vertex in

$V(P_2 \triangleright T_S(P_3)) - D$ therexist a vertex in D and are adjacent to the vertices in $V - D$. Hence, D is a minimum dominating set. This set, D doesn't satisfies DC-condition. Hence, choose $D' = D \cup \{v_1^{(1)}\}$ which contains minimum of one vertex from each color class of $V(P_2 \triangleright T_S(P_3))$.That is, D satisfies DC-condition. Therefore, D is a dom – coloring set with cardinality, $|D'| = 2 + 1 \implies \gamma_{dc}(P_2 \triangleright T_S(P_3)) = 3$. Refer Figure 1.

Case(iii): $m = 3, n = 2$

Color the vertices $\{v_1^{(1)}, v_2^{(1)}\}$ of $T_S(P_2)$ using two colors in the order 1, 2. Color the vertices $\{v_1^{(2)}, v_2^{(2)}\}$ of $T_S(P_2)$ using two colors in the order 2, 3. Color the vertices $\{v_1^{(3)}, v_2^{(3)}\}$ of $T_S(P_2)$ using two colors in the order 3, 1.Color the vertex $\{w_1^{(1)}\}$ of $T_S(P_2)$ with the color 3. Color the vertices $\{w_1^{(2)}\}$ of $T_S(P_2)$ with the color 1. Color the vertices $\{w_1^{(3)}\}$ of $T_S(P_2)$ with the color 2.

Let $D = \{v_1^{(1), v_1^{(2)}, v_1^{(3)}}\}$ be a subset of $V(P_3 \triangleright T_S(P_2))$ such that every vertex in $V(P_3 \triangleright T_S(P_2)) - D$ therexist a vertex in D and are adjacent to the vertices in $V - D$. Hence, D is a minimum dominating set. This set, D satisfies DC-condition. Hence, D is a minimum dominating set. That is, D satisfies DC-condition. Therefore, D is a dom – coloring set with cardinality, $|D| = 3 \implies \gamma_{dc}(P_3 \triangleright T_S(P_2)) = 3$.

Case(iv): $m = 3, n = 3$

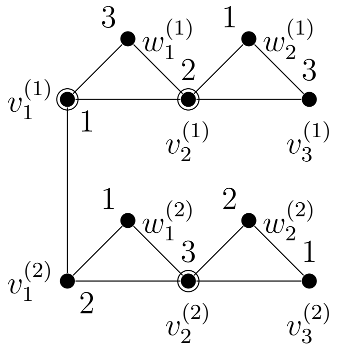
Color the vertices $\{v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\}$ of $T_S(P_3)$ using three colors in the order 1, 2, 3. Color the vertices $\{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\}$ of $T_S(P_3)$ using three colors in the order 2, 3, 1. Color the vertices $\{v_1^{(3)}, v_2^{(3)}, v_3^{(3)}\}$ of $T_S(P_3)$ using three colors in the order 3, 1, 2. Color the vertices $\{w_1^{(1)}, w_2^{(1)}\}$ of $T_S(P_3)$ using two colors in the order 3, 1. Color the vertices $\{w_1^{(2)}, w_2^{(2)}\}$ of $T_S(P_3)$ using two colors in the order 1, 2. Color the vertices $\{w_1^{(3)}, w_2^{(3)}\}$ of $T_S(P_3)$ using two colors in the order 2, 3.

Let $D = \{v_2^{(1), v_2^{(2)}, v_2^{(3)}}\}$ be a subset of $V(P_3 \triangleright T_S(P_3))$ such that every vertex in $V(P_3 \triangleright T_S(P_3)) - D$ therexist a vertex in D and are adjacent to the vertices in $V - D$. Hence, D is a minimum dominating set. This set, D satisfies DC-condition. Hence, D is a minimum dominating set. That is, D satisfies DC-condition. Therefore, D is a dom – coloring set with cardinality, $|D| = 3 \implies \gamma_{dc}(P_3 \triangleright T_S(P_3)) = 3$. Refer Figure 2.

□

Theorem 3.2. *Let P_m be a path graph on m vertices and $T_S(P_n)$ be a triangular snake graph on $2n - 1$ vertices, where $m \geq 2, n \geq 4$. Then $\gamma_{dc}(P_m \triangleright T_S(P_n)) = m\lfloor \frac{n}{2} \rfloor$.*

Proof. Let P_m be a path on m vertices labelled as $\{v_i, 1 \leq i \leq m\}$ and $T_S(P_n)$ be a triangular snake on $2n - 1$ vertices, whose vertices labelled as



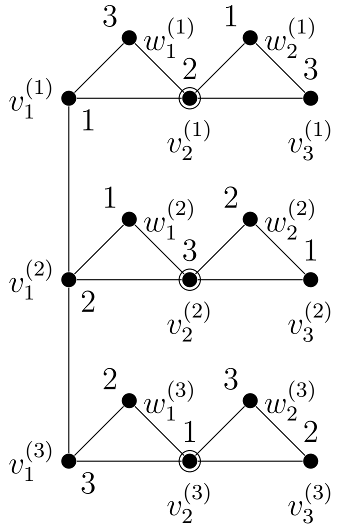


Figure 1: Encircled vertices for the dom coloring set $\{v_1^{(1)}, v_2^{(1)}, v_2^{(2)}\}, \gamma_{dc}(P_2 \triangleright T_S(P_3)) = 3$.

Figure 2: Encircled vertices for the dom coloring set $\{v_{12}^{(1)}, v_{22}^{(2)}, v_{32}^{(3)}\}, \gamma_{dc}(P_3 \triangleright T_S(P_3)) = 3$.

$\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_n^{(i)}, , w_1^{(i)}, w_2^{(i)}, w_3^{(i)}, \ldots, w_{(n-1)}^{(i)}\}, 1 \le i \le m.$

Let $v_1^{(i)}, 1 \le i \le m$ be the vertices of a double triangular snake graph $T_S(P_n)$. The comb product of graphs between P_m and $T_S(P_n)$ denoted by $P_m \triangleright T_S(P_n)$, is a graph obtained by taking one copy of P_m and $|V(P_m)|$ copies of $T_S(P_n)$ and identifying the i-th copy of $T_S(P_n)$ at the vertex $v_1^{(i)}, 1 \le i \le m$ to the i-th vertex of P_m with the vertex set as that of $T_S(P_n)$. The graph $P_m \triangleright T_S(P_n)$ has $m(2n-1)$ vertices and $m(3(n-1)) + m - 1$ edges.

Coloring of $P_m \triangleright T_S(P_n)$:

Color the vertices $\{v_1^{(3i-2)}, v_2^{(3i-2)}, v_3^{(3i-2)}, \ldots, v_n^{(3i-3)}\}, 1 \le i \le \lceil \frac{m}{3} \rceil$ of $(3i-2)^{th}$ copy of $T_S(P_n)$ using three colors in the order 1, 2, 3. Color the vertices $\{v_1^{(3i-1)}, v_2^{(3i-1)}, v_3^{(3i-1)}, \ldots, v_n^{(3i-1)}\}, 1 \le i \le \lfloor \frac{m+1}{3} \rfloor$ of $(3i-1)^{th}$ copy of $T_S(P_n)$ using three colors in the order 2, 3, 1. Color the vertices $\{v_1^{(3i)}, v_2^{(3i)}, v_3^{(3i)}, \ldots, v_n^{(3i)}\}, 1 \le i \le \lfloor \frac{m}{3} \rfloor$ of $(3i)^{th}$ copy of $T_S(P_n)$ using three colors in the order 3, 2, 1.

Color the vertices $\{w_1^{(3i-2)}, w_2^{(3i-2)}, w_3^{(3i-2)}, \ldots, w_{n-1}^{(3i-3)}\}, 1 \le i \le \lceil \frac{m}{3} \rceil$ of $(3i-2)^{th}$ copy of $T_S(P_n)$ using three colors in the order 3, 1, 2. Color the vertices $\{w_1^{(3i-1)}, w_2^{(3i-1)}, w_3^{(3i-1)}, \ldots, w_{n-1}^{(3i-1)}\}, 1 \le i \le \lfloor \frac{m+1}{3} \rfloor$ of $(3i-1)^{th}$ copy of $T_S(P_n)$ using three colors in the order 1, 2, 3. Color the vertices $\{w_1^{(3i)}, w_2^{(3i)}, w_3^{(3i)}, \ldots, w_{n-1}^{(3i)}\}, 1 \le i \le \lfloor \frac{m}{3} \rfloor$ of $(3i)^{th}$ copy of $T_S(P_n)$ using three colors in the order 2, 3, 1.

Let

$$D = \begin{cases} \{v_2^{(i)}, v_4^{(i)}, v_6^{(i)}, \ldots, v_n^{(i)}\}, 1 \le i \le m & \text{for } \textit{even } n \\ \{v_2^{(i)}, v_4^{(i)}, v_6^{(i)}, \ldots, v_{(n-1)}^{(i)}\}, 1 \le i \le m & \text{for } \textit{odd } n \end{cases}$$

be a subset of $V(P_m \triangleright T_S(P_n))$ such that every vertex in $V(P_m \triangleright T_S(P_n)) - D$ therexist a vertex in D and are adjacent to the vertices in $V - D$. Hence, D is a minimum dominating set. This set, D satisfies DC-condition. Therefore, D is a dom – coloring set with cardinality, $|D| = m\lfloor \frac{n}{2} \rfloor \implies \gamma_{dc}(P_m \triangleright T_S(P_n)) = m\lfloor \frac{n}{2} \rfloor$. Refer Figure 3. □

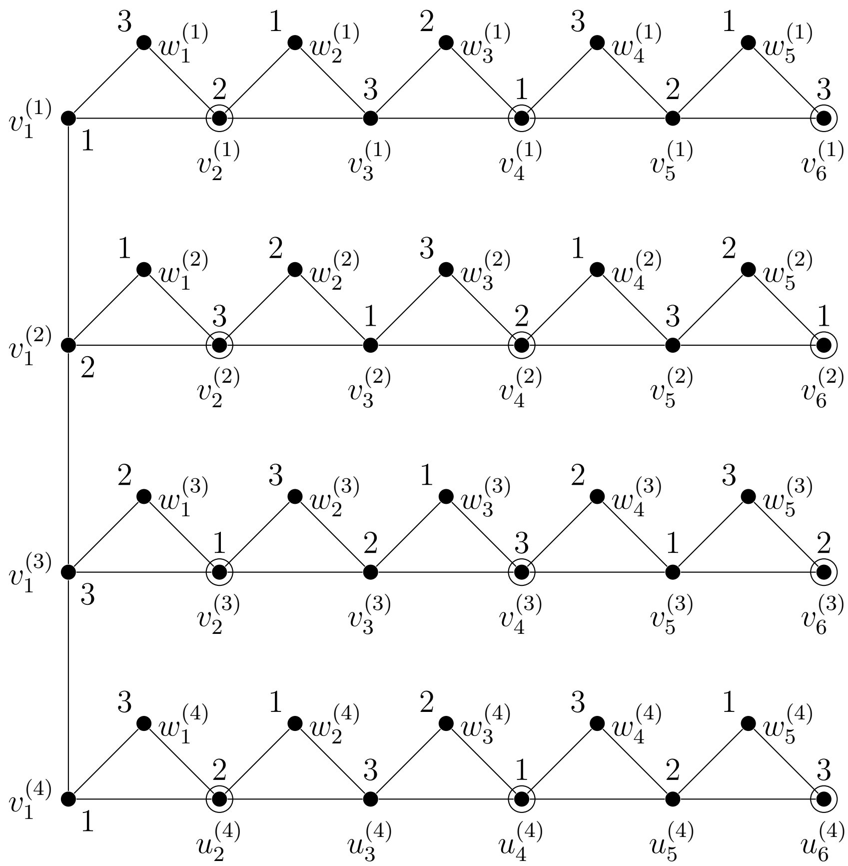


Figure 3: Encircled vertices for the dom coloring set $\{v_2^{(1)}, v_4^{(1)}, v_6^{(1)}, v_2^{(2)}, v_4^{(2)}, v_6^{(2)}, v_2^{(3)}, v_4^{(3)}, v_6^{(3)}, v_2^{(4)}, v_4^{(4)}, v_6^{(4)}\}, \gamma_{dc}(P_4 \triangleright T_S(P_6)) = 12$.

Theorem 3.3. *Let P_m be a path graph on m vertices and $DT_S(P_n)$ be a double triangular snake graph on $2n-1$ vertices, where $m = 2, 3, n = 2, 3$. Then $\gamma_{dc}(P_m \triangleright DT_S(P_n)) = 3$.*

Proof. The proof is as similar as Theorem 3.1. Refer Figure 4. □

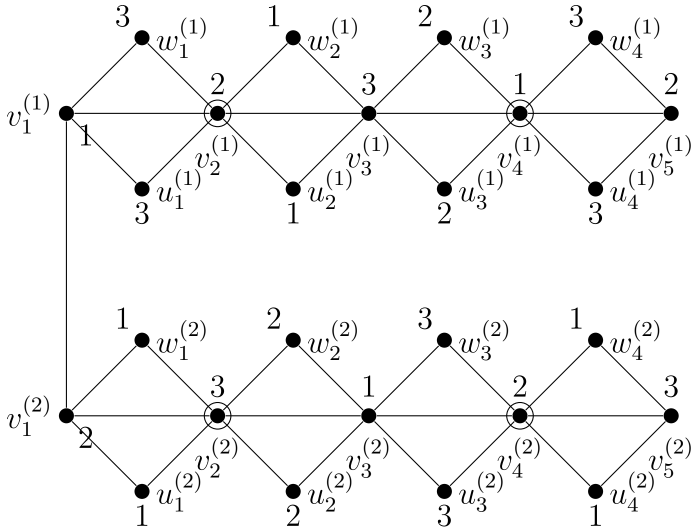


Figure 4: Encircled vertices for the dom coloring set $\{v_2^{(1)}, v_5^{(1)}, v_2^{(2)}, v_5^{(2)}\}$, $\gamma_{dc}(P_2 \triangleright DT_S(P_5)) = 4$.

Theorem 3.4. *Let P_m be a path graph on m vertices and $DT_S(P_n)$ be a double triangular snake graph on $3n-2$ vertices, where $m \geq 2, n \geq 4$. Then $\gamma_{dc}(P_m \triangleright DT_S(P_n)) = m\lfloor \frac{n}{2} \rfloor$.*

Proof. Let P_m be a path on m vertices labelled as $\{v_i, 1 \leq i \leq m\}$ and $DT_S(P_n)$ be a double triangular snake on $3n-2$ vertices, whose vertices labelled as $\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_n^{(i)}, w_1^{(i)}, w_2^{(i)}, w_3^{(i)}, \ldots, w_{(n-1)}^{(i)}, u_1^{(i)}, u_2^{(i)}, u_3^{(i)}, \ldots, u_{(n-1)}^{(i)}\}, 1 \leq i \leq m$.

Let $v_1^{(i)}, 1 \leq i \leq m$ be the vertices of a triangular snake graph $DT_S(P_n)$. The comb product of graphs between P_m and $DT_S(P_n)$ denoted by $P_m \triangleright DT_S(P_n)$, is a graph obtained by taking one copy of P_m and $|V(P_m)|$ copies of $DT_S(P_n)$ and identifying the i-th copy of $DT_S(P_n)$ at the vertex $v_1^{(i)}, 1 \leq i \leq m$ to the i-th vertex of P_m with the vertex set as that of $DT_S(P_n)$. The graph $P_m \triangleright DT_S(P_n)$ has $m(3n-2)$ vertices and $m(5(n-1)) + m - 1$ edges.

Coloring of $P_m \triangleright DT_S(P_n)$:

Color the vertices $\{v_1^{(3i-2)}, v_2^{(3i-2)}, v_3^{(3i-2)}, \ldots, v_n^{(3i-3)}\}, 1 \leq i \leq \lceil \frac{m}{3} \rceil$ of $(3i-2)^{th}$ copy of $DT_S(P_n)$ using three colors in the order 1, 2, 3. Color the vertices $\{v_1^{(3i-1)}, v_2^{(3i-1)}, v_3^{(3i-1)}, \ldots, v_n^{(3i-1)}\}, 1 \leq i \leq \lfloor \frac{m+1}{3} \rfloor$ of $(3i-1)^{th}$ copy of $DT_S(P_n)$ using three colors in the order 2, 3, 1. Color the vertices $\{v_1^{(3i)}, v_2^{(3i)}, v_3^{(3i)}, \ldots, v_n^{(3i)}\}, 1 \leq i \leq \lfloor \frac{m}{3} \rceil$ of $(3i)^{th}$ copy of $DT_S(P_n)$ using three colors in the order 3, 2, 1.

Color the $\{w_1^{(3i-2)}, w_2^{(3i-2)}, w_3^{(3i-2)}, \ldots, w_{n-1}^{(3i-3)}, u_1^{(3i-2)}, u_2^{(3i-2)}, u_3^{(3i-2)}, \ldots, u_{n-1}^{(3i-3)}\}$, $1 \leq i \leq \lceil \frac{m}{3} \rceil$ vertices of $(3i-2)^{th}$ copy of $DT_S(P_n)$ using three colors in the order 3, 1,

2. Color the vertices $\{w_1^{(3i-1)}, w_2^{(3i-1)}, w_3^{(3i-1)}, \ldots, w_{n-1}^{(3i-1)}, u_1^{(3i-1)}, u_2^{(3i-1)}, u_3^{(3i-1)}, \ldots, u_{n-1}^{(3i-1)}\}, 1 \leq i \leq \lfloor \frac{m+1}{3} \rfloor$ of $(3i-1)^{th}$ copy of $DT_S(P_n)$ using three colors in the order 1, 2, 3. Color the vertices $\{w_1^{(3i)}, w_2^{(3i)}, w_3^{(3i)}, \ldots, w_{n-1}^{(3i)}, u_1^{(3i)}, u_2^{(3i)}, u_3^{(3i)}, \ldots, u_{n-1}^{(3i)}\}, 1 \leq i \leq \lfloor \frac{m}{3} \rceil$ of $(3i)^{th}$ of $DT_S(P_n)$ using three colors in the order 2, 3, 1.

Let

$$D = \begin{cases} \{v_2^{(i)}, v_4^{(i)}, v_6^{(i)}, \ldots, v_n^{(i)}\}, 1 \leq i \leq m & \text{for } \textit{even } n \\ \{v_2^{(i)}, v_4^{(i)}, v_6^{(i)}, \ldots, v_{(n-1)}^{(i)}\}, 1 \leq i \leq m & \text{for } \textit{odd } n \end{cases}$$

be a subset of $V(P_m \rhd DT_S(P_n))$ such that every vertex in $V(P_m \rhd DT_S(P_n)) - D$ therexist a vertex in D and are adjacent to the vertices in $V - D$. Hence, D is a minimum dominating set. This set, D satisfies DC-condition. Therefore, D is a dom – coloring set with cardinality, $|D| = m\lfloor \frac{n}{2} \rfloor \implies \gamma_{dc}(P_m \rhd DT_S(P_n)) = m\lfloor \frac{n}{2} \rfloor$. Refer Figure 5. $\square$

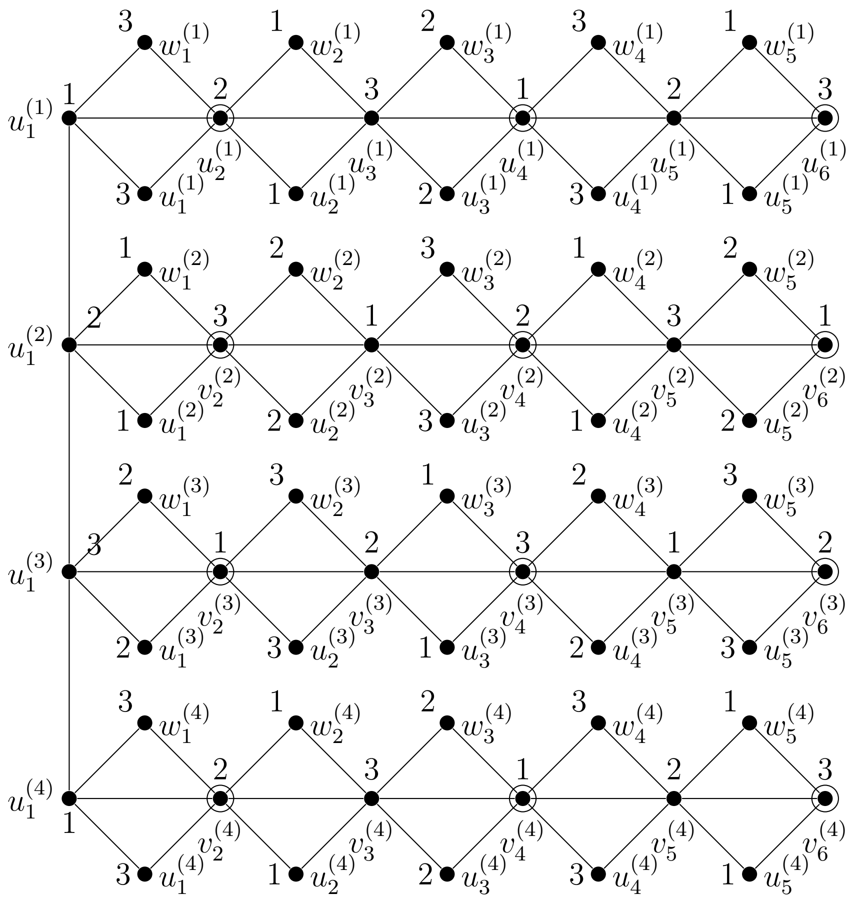


Figure 5: Encircled vertices for the dom coloring set $\{v_2^{(1)}, v_4^{(1)}, v_6^{(1)}, v_2^{(2)}, v_4^{(2)}, v_6^{(2)}, v_2^{(3)}, v_4^{(3)}, v_6^{(3)}, v_2^{(4)}, v_4^{(4)}, v_6^{(4)}\}, \gamma_{dc}(P_4 \rhd DT_S(P_6)) = 12.$

Theorem 3.5. *Let P_m be a path graph on m vertices and $Q_S(P_n)$ be a quadrilateral snake graph on $3n-2$ vertices, where $m,n \geq 2$. Then* $\gamma_{dc}(P_m \triangleright Q_S(P_n)) = n\lfloor \frac{m+1}{2} \rfloor + (n-1)[\lceil \frac{m-3}{2} \rceil + 1]$.

Proof. Let P_m be a path on m vertices labelled as $\{v_i, 1 \leq i \leq m\}$ and $Q_S(P_n)$ be a quadrilateral snake on $3n-2$ whose vertices labeled as $\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_n^{(i)}, w_1^{(i)}, w_2^{(i)}, w_3^{(i)}, \ldots, w_{2(n-1)}^{(i)}\}, 1 \leq i \leq m$.

Let $v_1^{(i)}, 1 \leq i \leq m$ be the vertices of a quadrilateral snake graph $Q_S(P_n)$. The comb product of graphs between P_m and $Q_S(P_n)$ denoted by $P_m \triangleright Q_S(P_n)$, is a graph obtained by taking one copy of P_m and $|V(P_m)|$ copies of $Q_S(P_n)$ and identifying the i-th copy of $Q_S(P_n)$ at the vertex $v_1^{(i)}, 1 \leq i \leq m$ to the i-th vertex of P_m with the vertex set as that of $Q_S(P_n)$. The graph $P_m \triangleright Q_S(P_n)$ has $m(3n-2)$ vertices and $m(4(n-1)) + m - 1$ edges.

Coloring of $P_m \triangleright Q_S(P_n)$:

Color the vertices $\{v_1^{(2i-1)}, v_2^{(2i-1)}, v_3^{(2i-1)}, \ldots, v_n^{(2i-1)}\}, 1 \leq i \leq \lfloor \frac{m+1}{2} \rfloor$ of $(2i-1)^{th}$ copy of $Q_S(P_n)$ using two colors in the order 1, 2. Color the vertices $\{v_1^{(2i)}, v_2^{(2i)}, v_3^{(2i)}, \ldots, v_n^{(2i)}\}, 1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$ of $(2i)^{th}$ copy of $Q_S(P_n)$ using two colors in the order 2, 1.

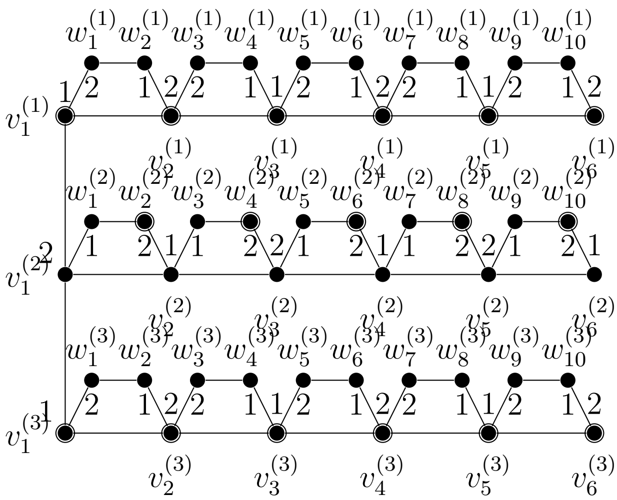


Figure 6: Encircled vertices for the dom coloring set $\{v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, v_4^{(1)}, v_5^{(1)}, v_6^{(1)}, , w_2^{(2)}, w_4^{(2)}, w_6^{(2)}, w_8^{(2)}, w_{10}^{(2)}, v_3^{(3)}, v_2^{(3)}, v_3^{(3)}, v_4^{(3)}, v_5^{(3)}, v_6^{(3)}\}, \gamma_{dc}(P_3 \triangleright Q_S(P_6)) = 17$.

Color the vertices $\{w_1^{(2i-1)}, w_2^{(2i-1)}, w_3^{(2i-1)}, \ldots, w_{2(n-1)}^{(2i-1)}\}, 1 \leq i \leq \lfloor \frac{m+1}{2} \rfloor$ of $(2i-1)^{th}$ copy of $Q_S(P_n)$ using two colors in the order 2, 1. Color the vertices

$\{w_1^{(2i)}, w_2^{(2i)}, w_3^{(2i)}, \ldots, w_{2(n-1)}^{(2i)}\}, 1 \leq i \leq \lceil \frac{m-3}{2} \rceil + 1$ of $(2i)^{th}$ copy of $Q_S(P_n)$ using two colors in the order 1, 2.

Let $D = \{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_n^{(i)}\}, 1 \leq i \leq \lfloor \frac{m+1}{2} \rfloor\} \cup \{w_2^{(i)}, w_4^{(i)}, w_6^{(i)}, \ldots, w_{2(n-1)}^{(i)}\}, 1 \leq i \leq \lceil \frac{m-3}{2} \rceil + 1$ be a subset of $V(P_m \triangleright Q_S(P_n))$ such that every vertex in $V(P_m \triangleright Q_S(P_n)) - D$ therexist a vertex in D and are adjacent to the vertices in $V - D$. Hence, D is a minimum dominating set. This set, D satisfies DC-condition. Therefore, D is a dom – coloring set with cardinality, $|D| = n\lfloor \frac{m+1}{2} \rfloor + (n-1)[\lceil \frac{m-3}{2} \rceil + 1] \implies \gamma_{dc}(P_m \triangleright Q_S(P_n)) = n\lfloor \frac{m+1}{2} \rfloor + (n-1)[\lceil \frac{m-3}{2} \rceil + 1]$. Refer Figure 6. □

Theorem 3.6. *Let P_m be a path graph on m vertices and $DQ_S(P_n)$ be a double quadrilateral snake graph on $5n - 4$ vertices, where $m, n \geq 2$. Then $\gamma_{dc}(P_m \triangleright DQ_S(P_n)) = mn$.*

Proof. Let P_m be a path on m vertices labelled as $\{v_i, 1 \leq i \leq m\}$ and $DQ_S(P_n)$ be a double quadrilateral snake on $5n - 4$ whose vertices labeled as $\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_n^{(i)}, w_1^{(i)}, w_2^{(i)}, w_3^{(i)}, \ldots, w_{2(n-1)}^{(i)}, u_1^{(i)}, u_2^{(i)}, u_3^{(i)}, \ldots, u_{2(n-1)}^{(i)}\}, 1 \leq i \leq m$.

Let $v_1^{(i)}, 1 \leq i \leq m$ be the vertices of a double quadrilateral snake graph $DQ_S(P_n)$. The comb product of graphs between P_m and $DQ_S(P_n)$ denoted by $P_m \triangleright DQ_S(P_n)$, is a graph obtained by taking one copy of P_m and $|V(P_m)|$ co9pies of $DQ_S(P_n)$ and identifying the i-th copy of $Q_S(P_n)$ at the vertex $v_1^{(i)}, 1 \leq i \leq m$ to the i-th vertex of P_m with the vertex set as that of $DQ_S(P_n)$. The graph $P_m \triangleright DQ_S(P_n)$ has $m(5n-4)$ vertices and $m(7(n-1)) + m - 1$ edges.

Coloring of $P_m \triangleright DQ_S(P_n)$:

Color the vertices $\{v_1^{(2i-1)}, v_2^{(2i-1)}, v_3^{(2i-1)}, \ldots, v_n^{(2i-1)}\}, 1 \leq i \leq \lfloor \frac{m+1}{2} \rfloor$ of $(2i-1)^{th}$ copy of $DQ_S(P_n)$ using two colors in the order 1, 2. Color the vertices $\{v_1^{(2i)}, v_2^{(2i)}, v_3^{(2i)}, \ldots, v_n^{(2i)}\}, 1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$ of $(2i)^{th}$ copy of $DQ_S(P_n)$ using two colors in the order 2, 1. Color the vertices $\{w_1^{(2i-1)}, w_2^{(2i-1)}, w_3^{(2i-1)}, \ldots, w_{2(n-1)}^{(2i-1)}, u_1^{(2i-1)}, u_2^{(2i-1)}, u_3^{(2i-1)}, \ldots, u_{2(n-1)}^{(2i-1)}\}$, $1 \leq i \leq \lfloor \frac{m+1}{2} \rfloor$ of $(2i-1)^{th}$ copy of $DQ_S(P_n)$ using two colors in the order 2, 1.

Color the vertices $\{w_1^{(2i)}, w_2^{(2i)}, w_3^{(2i)}, \ldots, w_{2(n-1)}^{(2i)}, u_1^{(2i)}, u_2^{(2i)}, u_3^{(2i)}, \ldots, u_{2(n-1)}^{(2i)}\}, 1 \leq i \leq \lceil \frac{m-3}{2} \rceil + 1$ of $(2i)^{th}$ copy of $DQ_S(P_n)$ using two colors in the order 1, 2.

Let $D = \{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_n^{(i)}\}, 1 \leq i \leq m$ be a subset of $V(P_m \triangleright DQ_S(P_n))$ such that every vertex in $V(P_m \triangleright DQ_S(P_n)) - D$ therexist a vertex in D and are adjacent to the vertices in $V - D$. Hence, D is a minimum dominating set. This set, D satisfies DC-condition. Therefore, D is a dom – coloring set with cardinality,

$|D| = mn \implies \gamma_{dc}(P_m \triangleright DQ_S(P_n)) = mn$. Refer Figure 7. □

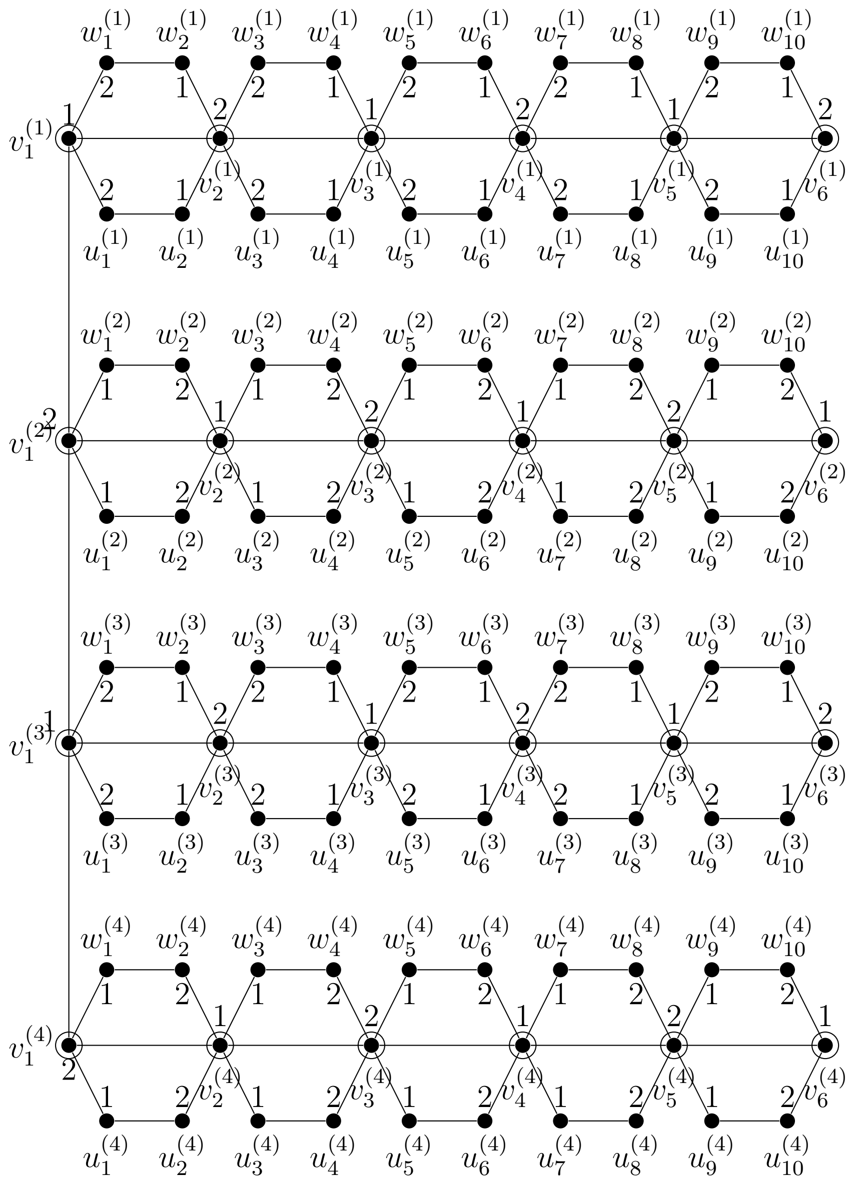


Figure 7: Encircled vertices for the dom coloring set $\{v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, v_4^{(1)}, v_5^{(1)}, v_6^{(1)}, v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, v_4^{(2)}, v_5^{(2)}, v_6^{(2)}, v_1^{(3)}, v_2^{(3)}, v_3^{(3)}, v_4^{(3)}, v_5^{(3)}, v_6^{(3)}, v_1^{(4)}, v_2^{(4)}, v_3^{(4)}, v_4^{(4)}, v_5^{(4)}, v_6^{(4)}\}$, $\gamma_{dc}(P_4 \triangleright DQ_S(P_6)) = 24$.

Theorem 3.7. *Let P_m be a path graph on m vertices and $D_S(n)$ be a diamond snake graph on $3n+1$ vertices, where $m, n \geq 2$. Then $\gamma_{dc}(P_m \triangleright D_S(n)) = (n+1)\lfloor \frac{m+1}{2} \rfloor + n[\lceil \frac{m-3}{2} \rceil + 1]$.*

Proof. Let P_m be a path on m vertices labelled as $\{v_i, 1 \leq i \leq m\}$

and $D_S(n)$ be a diamond snake on $3n+1$ whose vertices labeled as $\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_{n+1}^{(i)}, w_1^{(i)}, w_2^{(i)}, w_3^{(i)}, \ldots, w_n^{(i)}, u_1^{(i)}, u_2^{(i)}, u_3^{(i)}, \ldots, u_n^{(i)}\}, 1 \leq i \leq m.$

Let $v_1^{(i)}, 1 \leq i \leq m$ be the vertices of a diamond snake graph $D_S(n)$. The comb product of graphs between P_m and $D_S(n)$ denoted by $P_m \rhd D_S(n)$, is a graph obtained by taking one copy of P_m and $|V(P_m)|$ copies of $D_S(n)$ and identifying the i-th copy of $D_S(n)$ at the vertex $v_1^{(i)}, 1 \leq i \leq m$ to the i-th vertex of P_m with the vertex set as that of $D_S(n)$. The graph $P_m \rhd D_S(n)$ has $m(3n+1)$ vertices and $4mn+m-1$ edges.

Coloring of $P_m \rhd D_S(n)$:

Color the vertices $\{v_1^{(2i-1)}, v_2^{(2i-1)}, v_3^{(2i-1)}, \ldots, v_{n+1}^{(2i-1)}\}, 1 \leq i \leq \lceil \frac{m}{2} \rceil$ of $(2i-1)^{th}$ copy of $D_S(n)$ with color 1. Color the vertices $\{v_1^{(2i)}, v_2^{(2i)}, v_3^{(2i)}, \ldots, v_{n+1}^{(2i)}\}, 1 \leq i \leq \lceil \frac{m-1}{2} \rceil$ of $(2i)^{th}$ copy of $D_S(n)$ with color 2.

Color the vertices $\{w_1^{(2i-1)}, w_2^{(2i-1)}, w_3^{(2i-1)}, \ldots, w_n^{(2i-1)}, u_1^{(2i-1)}, u_2^{(2i-1)}, u_3^{(2i-1)}, \ldots, u_n^{(2i-1)}\}, 1 \leq i \leq \lceil \frac{m}{2} \rceil$ of $(2i-1)^{th}$ copy of $D_S(n)$ with color 2. Color the vertices $\{w_1^{(2i)}, w_2^{(2i)}, w_3^{(2i)}, \ldots, w_n^{(2i)}, u_1^{(2i)}, u_2^{(2i)}, u_3^{(2i)}, \ldots, u_n^{(2i)}\}, 1 \leq i \leq \lceil \frac{m-1}{2} \rceil$ of $(2i)^{th}$ copy of $D_S(n)$ with color 1.

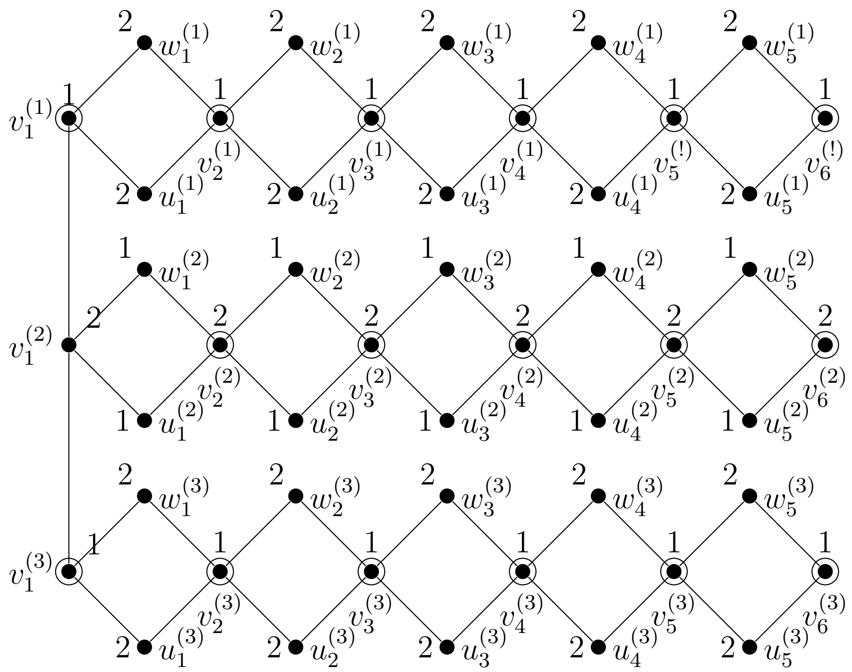


Figure 8: Encircled vertices for the dom coloring set $\{v_1, v_{12}^{(1)}, v_{13}^{(1)}, v_{14}^{(1)}, v_{15}^{(1)}, v_{16}^{(1)}, v_{22}^{(2)}, v_{23}^{(2)}, v_{24}^{(2)}, v_{25}^{(2)}, v_{26}^{(2)}, v_3, v_{32}^{(3)}, v_{33}^{(3)}, v_{34}^{(3)}, v_{35}^{(3)}, v_{36}^{(3)}\}, \gamma_{dc}(P_3 \rhd D_S(5)) = 17.$

Let $D = \{v_1^{(2i-1)}, v_2^{(2i-1)}, v_3^{(2i-1)}, \ldots, v_{n+1}^{(2i-1)}\}, 1 \leq i \leq \lfloor \frac{m+1}{2} \rfloor \cup \{v_2^{(2i)}, v_3^{(2i)}, v_4^{(2i)}, \ldots, v_{n+1}^{(2i)}\}, 1 \leq i \leq \lceil \frac{m-3}{2} \rceil + 1$ be a subset of $V(P_m \triangleright D_S(n))$ such that every vertex in $V(P_m \triangleright D_S(n)) - D$ therexist a vertex in D and are adjacent to the vertices in $V - D$. Hence, D is a minimum dominating set. This set, D satisfies DC-condition. Therefore, D is a dom – coloring set with cardinality, $|D| = (n+1)\lfloor \frac{m+1}{2} \rfloor + n[\lceil \frac{m-3}{2} \rceil + 1] \implies \gamma_{dc}(P_m \triangleright D_S(n)) = (n+1)\lfloor \frac{m+1}{2} \rfloor + n[\lceil \frac{m-3}{2} \rceil + 1]$. Refer Figure 8.

□

4. APPLICATION

The dom-chromatic number of the comb product of a path with a graph can be applied to Zomato's food delivery network, where the path represents sequential delivery zones and the attached graphs represent local restaurant clusters. Dominating vertices correspond to key delivery hubs ensuring coverage, while coloring avoids scheduling and routing conflicts. This model helps in optimizing delivery efficiency and resource allocation across urban regions.

5. CONCLUSIONS AND FUTURE WORK

The two major areas of Graph Theory domination and coloring are integrated to study the dom-chromatic number of the comb product of path with path, cycle, complete, star and wheel graphs. Exploring the dom-chromatic number in this study can also be extended to other graph operations, such as the lexicographic and tensor products, which remains the open problems for future research.

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