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On Dirichlet Boundary Value Problem for Some Quasilinear Elliptic Systems

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Abstract

We consider the existence of positive solutions for the quasilinear elliptic system

$$\begin{cases} -\Delta u_1 = \lambda u_2^{\alpha_1}, & x \in \Omega, \\ -\Delta u_2 = \lambda u_3^{\alpha_2}, & x \in \Omega, \\ \quad \cdots & \\ -\Delta u_n = \lambda u_1^{\alpha_n}, & x \in \Omega, \\ u_i = 0, & x \in \partial\Omega, i = 1, 2, \ldots, n, \end{cases}$$

where λ is a positive parameter, Δ is the Laplacian operator, $\alpha_i > 0$ for $i = 1, 2, \ldots, n$, and Ω is a bounded domain in $\mathbb{R}^N (N > 1)$ with smooth boundary $\partial\Omega$. By using the method of sub-super solutions we prove the existence of positive solutions for each $\lambda > 0$.

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1. Introduction

In this paper we consider the existence of positive solutions for the quasilinear elliptic system of the form

$$\begin{cases} -\Delta u_1 = \lambda u_2^{\alpha_1}, & x \in \Omega, \\ -\Delta u_2 = \lambda u_3^{\alpha_2}, & x \in \Omega, \\ \quad \cdots & \\ -\Delta u_n = \lambda u_1^{\alpha_n}, & x \in \Omega, \\ u_i = 0, & x \in \partial\Omega, i = 1, 2, \ldots, n, \end{cases} \tag{1.1}$$

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where λ is a positive parameter, Δ is the Laplacian operator, $\alpha_i > 0$ for $i = 1, 2, \ldots, n$, and Ω is a bounded domain in $\mathbb{R}^N (N > 1)$ with smooth boundary $\partial\Omega$.

Systems of the form (1.1) arise in several contexts in biology and engineering (see [8]). It provides a simple model to describe, for instance, the interaction of n diffusing biological species. u_i for $i = 1, 2, \ldots, n$ represent the densities of n species. See [9] for details on the physical models involving more general elliptic system.

For the case $n = 2$, (1.1) has been studied by many authors. Especially, Dalmasso [2] proved the uniqueness and existence of positive solutions to (1.1) for the case $n = 2, 0 < \alpha_1\alpha_2 < 1$. See also [5,7], for related results. On the other hand, when $\alpha_1, \alpha_2 > 1$ with additional assumption on α_1 and α_2, existence of a positive solution to (1.1) has been studied by several authors (see e.g., [1,10] and the reference therein). In this paper, we show the existence of positive solutions for (1.1) by using the method of sub- and supersolutions, see [4]. We refer to [3,6] for additional results on elliptic systems.

2. Existence Results

To prove our existence results we use the method of sub-super solutions. To do so, we now define sub and super solutions of (1.1).

Definition 2.1. A pair of nonnegative functions $(\psi_1, \psi_2, \ldots, \psi_n), (z_1, z_2, \ldots, z_n)$ in the space $(C_0^2(\bar{\Omega}))^n$ is called a subsolution and supersolution of (1.1) if they satisfy $\psi_i(x) \leq z_i(x)$ in Ω for $i = 1, 2, \ldots, n$, and

$$-\Delta\psi_1 \leq \lambda\psi_2^{\alpha_1}, -\Delta\psi_2 \leq \lambda\psi_3^{\alpha_2}, \ldots, -\Delta\psi_n \leq \lambda\psi_1^{\alpha_n}, \qquad x \in \Omega,$$

and

$$-\Delta z_1 \geq \lambda z_2^{\alpha_1}, -\Delta z_2 \geq \lambda z_3^{\alpha_2}, \ldots, -\Delta z_n \geq \lambda z_1^{\alpha_n}, \qquad x \in \Omega.$$

We shall obtain the existence of a positive solution to system (1.1) by constructing a positive subsolution $(\psi_1, \psi_2, \ldots, \psi_n)$ and supersolution $(z_1, z_2, \ldots, z_n)$.

Theorem 2.2. Let $\alpha_1, \alpha_2, \ldots, \alpha_n < 1$. Then system (1.1) has a positive solution for each $\lambda > 0$.

Proof. Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions and ϕ_1 denote the corresponding eigenfunction, satisfying $\phi_1(x) > 0$ in Ω, $|\nabla\phi_1| > 0$ on $\partial\Omega$ and $||\phi_1||_\infty = 1$. We shall verify that $(\psi_1, \psi_2, \ldots, \psi_n) = \underbrace{(\psi, \psi, \ldots, \psi)}_{n \text{ times}}$, where $\psi = \frac{k}{2}\phi_1^2$, is a subsolution of (1.1), where $k > 0$ is small and specified later. A calculation shows that

$$-\Delta\psi = -\frac{k}{2}\Delta\phi_1^2 = -k\left(|\nabla\phi_1|^2 + \phi_1\Delta\phi_1\right) = k\left(\lambda_1\phi_1^2 - |\nabla\phi_1|^2\right).$$

Since $\phi_1 = 0$ and $|\nabla\phi_1| > 0$ on $\partial\Omega$, there is $\delta > 0$ such that

$$\lambda_1\phi_1^2 - |\nabla\phi_1|^2 \le 0, \qquad x \in \bar{\Omega}_\delta,$$

with $\bar{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \le \delta\}$. This implies that

$$k\left(\lambda_1\phi_1^2 - |\nabla\phi_1|^2\right) \le 0 \le \lambda\psi^{\alpha_1}, \qquad x \in \bar{\Omega}_\delta.$$

Next, we note that $\phi_1(x) \ge \eta > 0$ in $\Omega_0 = \Omega\backslash\bar{\Omega}_\delta$ for some $\eta > 0$. Since $\alpha_1 < 1$, there is $k_0 > 0$ such that if $k \in (0, k_0)$ we have

$$k^{1-\alpha_1}\lambda_1\phi_1^2 \le \lambda(1/2)^{\alpha_1}\eta^{2\alpha_1} \le \lambda(1/2)^{\alpha_1}\phi_1^{2\alpha_1}, \quad x \in \Omega_0.$$

Hence

$$-\Delta\psi = k\left(\lambda_1\phi_1^2 - |\nabla\phi_1|^2\right) \le \lambda\psi^{\alpha_1}, \quad x \in \Omega_0.$$

Thus

$$-\Delta\psi \le \lambda\psi^{\alpha_1}, \qquad x \in \Omega.$$

Similarly, we have

$$-\Delta\psi \le \lambda\psi^{\alpha_2}, \ldots, -\Delta\psi \le \lambda\psi^{\alpha_n} \quad x \in \Omega,$$

i.e., $(\psi_1, \psi_2, \ldots, \psi_n) = \underbrace{(\psi, \psi, \ldots, \psi)}_{n \text{ times}}$, is a subsolution of (1.1).

Next, let ζ be the positive solution of the problem

$$\begin{cases} -\Delta\zeta = 1, & x \in \Omega, \\ \zeta = 0, & x \in \partial\Omega. \end{cases}$$

Let

$$(z_1, z_2, \ldots, z_n) = (C_1\zeta, C_2\zeta, \ldots, C_n\zeta),$$

where $C_1, C_2, \ldots, C_n > 0$ are large numbers to be chosen later. We shall verify that $(z_1, z_2, \ldots, z_n)$ is a supersolution of (1.1). A calculation shows that

$$-\Delta z_1 = C_1.$$

Similarly we have

$$-\Delta z_2 = C_2, \ldots, -\Delta z_n = C_n.$$

Let $l = ||\zeta||_\infty$. It is easy to prove that there exist positive large constants $C_1, C_2, \ldots, C_n$ such that

$$\begin{aligned} C_1 &= \lambda(C_2 l)^{\alpha_1}, \\ C_2 &= \lambda(C_3 l)^{\alpha_2}, \\ &\cdots \\ C_n &= \lambda(C_1 l)^{\alpha_n}, \end{aligned}$$

Then we have

$$C_1 = \lambda(C_2 l)^{\alpha_1} \geq \lambda(C_2\zeta)^{\alpha_1} = \lambda z_2^{\alpha_1}.$$

Similarly we have

$$C_2 \geq \lambda z_3^{\alpha_2}, \ldots, C_n \geq \lambda z_1^{\alpha_n},$$

and therefore

$$-\Delta z_1 \geq \lambda z_2^{\alpha_1}, -\Delta z_2 \geq \lambda z_3^{\alpha_2}, \ldots, -\Delta z_n \geq \lambda z_1^{\alpha_n}, \qquad x \in \Omega,$$

i.e., $(z_1, z_2, \ldots, z_n)$ is a supersolution of (1.1) with $z_i \geq \psi_i$ $(i = 1, 2, \ldots, n)$ in Ω for large $C_1, C_2, \ldots, C_n$. Thus, by the comparison principle, there exists a solution $(u_1, u_2, \ldots, u_n)$ of (1.1) with $\psi_i \leq u_i \leq z_i$ for $i = 1, 2, \ldots, n$. This completes the proof of Theorem 2.2. ■

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