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Large Nonnegative Solutions to a Two Point Boundary Value Problem for a Class of Quasi-linear Ordinary Differential Equations*

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Abstract

In this paper, our main purpose is to establish the existence of large nonnegative solutions for the problem

$$-(\Phi_p(u'))' = \lambda f(u(x)), \quad 0 < x < 1$$

$$\lim_{x \to 0^+} u(x) = \infty = \lim_{x \to 1^-} u(x),$$

where $\lambda > 0$ is parameter and f is a smooth function. We give some new sufficient conditions for the existence of nonnegative solutions. The main results of the paper extend previously known results.

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1. Introduction

We consider the existence of large solutions of the problems

$$-(\Phi_p(u'))' = \lambda f(u(x)), \quad 0 < x < 1 \tag{1.1}$$

$$\lim_{x\to 0^+} u(x) = \infty = \lim_{x\to 1^-} u(x), \tag{1.2}$$

where $\lambda > 0$ is a positive parameter and f is a Lipschitz continuous function, and $\Phi_p(u) = |u|^{p-2}u$, $p > 1$.

This problem appears in the study of non-Newtonian fluids and non-Newtonian filtration. The quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

Large solutions of the problem

$$-\triangle u(x) = f(u(x)), \quad x \in \Omega \tag{1.3}$$

$$u|_{\partial\Omega} = \infty, \tag{1.4}$$

where Ω is bounded domain in $\mathbb{R}^N (N \geq 1)$ have been extensively studied, see [1, 5, 7, 12–17, 19–22]. A problem of this was first considered by Bieberbach [5] in 1916, where $f(u) = -e^u$ and $N = 2$. Bieberbach showed that if Ω is a bounded domain in $\mathbb{R}^2$ such that $\partial\Omega$ is a C^2 sub-mainfold of $\mathbb{R}^2$, then there exists a unique $u \in C^2(\Omega)$ such that $-\triangle u = -e^u$ in Ω and $|u(x) - \ln(d(x))^{-2}|$ is bounded on Ω. Here $d(x)$ denotes the distance from a point x to $\partial\Omega$. Rademacher [21], using the idea of Bieberbach, extended to smooth bounded domain in $\mathbb{R}^3$. In this case the problem plays an important role, when $N = 2$, in the theory of Riemann surfaces of constant negative curvature and in the theory of automorphic functions, and when $N = 3$, according to [21], in the study of the electric potential in a glowing hollow metal body. Lazer and McKenna [14] extended the results for Ω a bounded domain in $\mathbb{R}^N (N \geq 1)$ satisfying a uniform external sphere condition and the nonlinearity $f = f(x, u) = p(x)e^u$, where $p(x)$ is continuous and strictly negative on $\overline{\Omega}$. Lazer and McKenna [15] obtained similar results when $\triangle$ is replaced by the Monge–Ampere operator and Ω is a smooth, strictly convex, bounded domain. Similar results were also obtained for $f = p(x)u^a$ with $a > 1$. Posteraro [20], for $f(u) = -e^u$ and $N \geq 2$, proved estimates for the solution $u(x)$ of (1.3)–(1.4) and for the measure of Ω comparing this problem with a problem of the same type defined in a ball. In particular, when $N = 2$, Posteraro obtained an explicit estimate of the minimum of $u(x)$ in terms of the measure of Ω:

$$\min_{\Omega} u(x) \geq \ln(8\pi/|\Omega|).$$

Further, the case where $\triangle$ is replaced by the p-Laplacian has been discussed by Diaz and Letelier [7] when $f(u) = bu^a$ with $a > p - 1$ and $p > 2$.

For general nonlinearities $f(u)$ and in one space dimension, Anuradha et al. [1] and Shin-Hwa Wang [22] considered Problem (1.3)–(1.4). They mainly proved the existence and multiplicity of large nonnegative solutions basing on building a quadrature method for such large solutions. Very recently, H. Yang and Z. Yang [23] proved the existence and multiplicity of large nonnegative solutions for the problem (1.1)–(1.2) basing on building a quadrature method. In this paper, we further obtain some new the existence results of large nonnegative solutions to (1.1)–(1.2) under new conditions by using quadrature method, the extends and complementary to the cases considered in [1, 22, 23].

2. Main Results

Let

$$F(s) = \int_0^s f(t)dt,$$

and

$$I = \Big\{ s \in \mathbb{R}^+ \cup \{0\} : \ f(s) < 0 \text{ and } F(s) > F(u) \ \text{ for all } u > s \\ \text{and } \int_s^\infty \frac{du}{(F(s) - F(u))^{1/p}} < \infty \Big\}.$$

From [23], we give the following lemma.

Lemma 2.1. Assume that $p > 1$. Let f be a Lipschitz continuous function in $\mathbb{R}$ except possibly at some point $s = 0$ where $f(s) = 0$ and f is continuous here. Then, given $\lambda > 0$, there exist solution u to (1.1)–(1.2) with $\inf_{x \in (0,1)} u(x) = \rho \in \mathbb{R}$ if and only if

$$G(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p} \int_\rho^\infty \frac{du}{(F(\rho) - F(u))^{1/p}} = \lambda^{1/p}, \quad \text{for } \rho \in I.$$

By a modification of the method given in [1, 22, 23], we obtain the following results.

Theorem 2.2. If there exists any solution to (1.1)–(1.2) for any $\lambda > 0$, then

$$\limsup_{u \to \infty} \left(-\frac{f(u)}{u^{p-1}} \right) = \infty \ \text{ and } \ \limsup_{u \to \infty} \left(-\frac{f(u)}{u^{p-1} (\ln u)^p} \right) = \infty.$$

The proof of Theorem 2.2 is similar to that of [23, Theorem 2], so we omit it here.

Remark 2.3. By Theorem 2.2, we also obtain

$$\limsup_{u \to \infty} \left(-\frac{f(u)}{u^{p-1} (\underbrace{\ln \ln \cdots \ln}_{n} u)^p} \right) = \infty.$$

Theorem 2.4. Assume one of the following conditions is satisfied:

(i) $f(u)$ satisfies

$$\liminf_{u\to\infty} \frac{-f(u)}{u^{p-1}(\ln u)^{p+1}} = L(0 < L \le \infty); \tag{2.1}$$

(ii) $f(u)$ satisfies

$$\liminf_{u\to\infty} \frac{-f(u)}{(u\underbrace{\ln\ln\cdots\ln}_{n-1} u \underbrace{\ln\ln\cdots\ln}_{n-2} u\cdots\ln u)^{p-1}(\underbrace{\ln\ln\cdots\ln}_{n} u)^{p+1}} = L$$
$$(0 < L \le \infty). \tag{2.2}$$

Then there exist solutions to Problem (1.1)–(1.2) for some $\lambda > 0$. Furthermore, $G(\rho)$ is well defined and continuous for all $\rho \in I$.

To prove Theorem 2.4, we need the following lemma.

Lemma 2.5. Let f satisfy Eq. (2.1) or (2.2) and $\rho \in [\rho_1, \rho_2] \subset I$. Then there exists $M > e^{p+1}$ such that

$$F(\rho) - F(u) \ge C_2(p-1)u^p(\ln u)^{p+1}, \quad \text{for } u > M,\ \rho \in [\rho_1, \rho_2], \tag{2.3}$$

where

$$C_2 = \frac{L}{2p(p+1)}, \quad \text{if } 0 < L < \infty;\ C_2 = \frac{1}{p(p+1)}, \text{ if } L = \infty;$$

or there exists $M > e^{e^{\cdot^{\cdot^{\cdot^{e^{p+1}}}}}}$ such that

$$F(\rho) - F(u) \ge C_3(p-1)u^p(\underbrace{\ln\ln\cdots\ln}_{n} u)^{p+1} \quad \text{for } u > M,\ \rho \in [\rho_1, \rho_2], \tag{2.4}$$

where

$$C_3 = \frac{L}{2p(p+1)}, \text{ if } 0 < L < \infty;\ C_3 = \frac{1}{p(p+1)}, \text{ if } L = \infty.$$

Proof. We only prove (2.3), the other case being similar. If f satisfies Eq. (2.1), it is easy to see that there exists a constant $M_3 > e^{p+1}$ such that

$$\begin{aligned} -f(u) &> p(p+1)Cu^{p-1}(\ln u)^{p+1} > h(u) \\ &= pC(u^{p-1}p(\ln u)^{p+1} + (p+1)u^{p-1}(\ln u)^p], \quad \text{for } u > M_3 > e^{p+1}, \end{aligned} \tag{2.5}$$

where C is defined in Eq. (2.3). Then, for $u > M_3$,

$$\begin{aligned} -F(u) &= -F(M_3) + \int_{M_3}^{u} -f(t)dt \ge -F(M_3) + \int_{M_3}^{u} h(t)dt \\ &= -F(M_3) + pC[u^p(\ln u)^{p+1} - M_3^p(\ln M_3)^{p+1}]. \end{aligned}$$

Let $\rho \in I$. Since I is open, there exist ρ_1 and $\rho_2 \in I$ and $[\rho_1, \rho_2] \subset I$. Let $K = -F(M_3) - pCM_3^p(\ln M_3)^{p+1} + \inf_{\rho\in[\rho_1,\rho_2]} F(\rho)$. We obtain

$$F(\rho) - F(u) \geq K + pCu^p(\ln u)^{p+1}, \quad \text{for } u > M_3, \ \rho \in [\rho_1, \rho_2]. \tag{2.6}$$

Now there exists a constant $M_4 > p$ such that

$$Cu^p(\ln u)^{p+1} \geq -K, \quad \text{for } u > M_4$$

which implies

$$K + pCu^p(\ln u)^{p+1} \geq C(p-1)u^p(\ln u)^{p+1} \ \text{ for } u > M_4. \tag{2.7}$$

Letting $M = \max\{M_3, M_4\} > e^{p+1}$, by Eqs. (2.6)–(2.7), we obtain

$$F(\rho) - F(u) \geq C(p-1)u^p(\ln u)^{p+1} \ \text{ for } u > M, \ \rho \in [\rho_1, \rho_2].$$

This completes the proof. ∎

Proof of Theorem 2.4. We only prove f satisfies Eq. (2.1), the other case being similar. Suppose that f satisfies Eq. (2.1), let $\rho \in I$. Since I is open, there exist $\rho_1, \rho_2 \in I$ such that $\rho \in (\rho_1, \rho_2)$ and $[\rho_1, \rho_2] \subset I$. Let C and M be as in Lemma 2.5. Note that

$$G(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p} \int_\rho^\infty \frac{du}{(F(\rho) - F(u))^{1/p}}$$

converges if and only if

$$\int_\rho^{\rho+\delta} \frac{du}{(F(\rho) - F(u))^{1/p}} < \infty, \quad 0 < \delta < \rho_2 - \rho$$

and

$$\int_M^\infty \frac{du}{(F(\rho) - F(u))^{1/p}} < \infty,$$

where we assume without loss of generality that $M > \rho_2$. Now, $[\rho_1, \rho_2] \subset I$ implies $L = \inf_{z\in[\rho_1,\rho_2]}(-f(z)) > 0$. This combined with the mean value theorem implies

$$F(\rho) - F(u) = -f(z)(u - \rho) \geq L(u - \rho) \ \ \forall u \in [\rho_1, \rho_2].$$

Since $\rho + \delta < \rho_2$, we have

$$\int_\rho^{\rho+\delta} \frac{du}{(F(\rho) - F(u))^{1/p}} \leq \frac{1}{L^{1/p}} \int_\rho^{\rho+\delta} \frac{du}{(u-\rho)^{1/p}} = \frac{p\delta^{(p-1)/p}}{L^{1/p}(p-1)} < \infty.$$

Also, using Lemma 2.5 we have

$$\int_M^\infty \frac{du}{(F(\rho) - F(u))^{1/p}} \leq \left(\frac{1}{C}\right)^{1/p} \int_M^\infty \frac{du}{u(\ln u)^{(p+1)/p}} = p\left(\frac{1}{C(p-1)\ln M}\right)^{1/p} < \infty.$$

Hence $G(\rho)$ is well defined for all $\rho \in I$, and by Lemma 2.1 there exists a solution to (1.1)–(1.2) for $\lambda = [G(\rho)]^p$ given and any $\rho \in I$.

Also, G is continuous at ρ. This can be shown by defining

$$G_n(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p} \int_{\rho+1/n}^{\rho+n} \frac{du}{(F(\rho)-F(u))^{1/p}}, \quad \text{for } n \in \mathbb{N},\ \rho \in [\rho_1, \rho_2].$$

Since G_n is a proper integral of a continuous integrand for each $n \in \mathbb{N}$, G_n is continuous on $[\rho_1, \rho_2]$. We will have that G is continuous on $[\rho_1, \rho_2]$, and thus at ρ, if we can show that $G_n \to G$ uniformly as $n \to \infty$.

Since I is open, there exists a constant $\delta > 0$ such that $[\rho_1, \rho_2 + \delta] \subset I$ which implies $\overline{L} = \inf_{z \in [\rho_1, \rho_2+\delta]} (-f(z)) > 0$. Choose N_1 large enough so that $1/n \le \delta$ for all $n \ge N_1$. Thus, we obtain by the mean value theorem

$$2\left(\frac{p-1}{p}\right)^{1/p} \int_{\rho}^{\rho+1/n} \frac{du}{(F(\rho)-F(u))^{1/p}} \le \left(\frac{2(p-1)}{\overline{L}p}\right)^{1/p} \int_{\rho}^{\rho+1/n} \frac{du}{(u-\rho)^{1/p}}$$

$$= 2p\left(\frac{p-1}{p\overline{L}n}\right)^{1/p}, \quad \text{for all } n \ge N_1,\ \rho \in [\rho_1, \rho_2].$$

Also, choosing N_2 large enough so that $\rho + n \ge \rho_1 + n \ge M$ for all $n \ge N_2$, here $M > 0$ is as in Lemma 2.5, we have

$$\begin{aligned} &2\left(\frac{p-1}{p}\right)^{1/p} \int_{\rho+n}^{\infty} \frac{du}{(F(\rho)-F(u))^{1/p}} \\ &\quad \le 2\left(\frac{1}{pC}\right)^{1/p} \int_{\rho_1+n}^{\infty} \frac{du}{u(\ln u)^{(p+1)/p}} = 2p\left(\frac{1}{pC\ln(\rho_1+n)}\right)^{1/p}. \end{aligned}$$

Letting $N = \max\{N_1, N_2\}$, we have

$$|(G-G_n)(\rho)| \le 2p\left(\frac{p-1}{p\overline{L}n}\right)^{1/p} + 2p\left(\frac{1}{pC\ln(\rho_1+n)}\right)^{1/p}, \quad \text{for } n \ge N$$

which implies $\sup_{\rho \in [\rho_1, \rho_2]} |(G-G_n)(\rho)| \to 0$ as $n \to \infty$. Thus, $G_n \to G$ uniformly on $[\rho_1, \rho_2]$ as $n \to \infty$. Hence G is continuous on all of I. ■

Theorem 2.6. If $f(u)$ satisfies Eq. (2.1) or (2.2), then

$$G(\rho) \to 0^+ \text{ as } \rho \to \infty. \tag{2.8}$$

Proof. We only prove f satisfies Eq. (2.1), the other case being similar. Suppose that f satisfies Eq. (2.1). By Theorem 2.4, $G(\rho)$ exists and is continuous on I. Moreover, in

Eq. (2.1), it is known that, there exists a constant $M_3 > e^{e^{\cdot^{\cdot^{\cdot^{e^{p+1}}}}}}$ such that

$$-f(u) > p(p+1)Cu^{p-1}(\underbrace{\ln\ln\cdots\ln}_{n} u)^{p+1} > h(u) = pCu^{p-1}\left[p(\underbrace{\ln\ln\cdots\ln}_{n} u)^{p+1} \right.$$
$$\left. +(p+1)(\underbrace{\ln\ln\cdots\ln}_{n} u)^{p} \frac{1}{\underbrace{\ln\ln\cdots\ln}_{n-1} u \underbrace{\ln\ln\cdots\ln}_{n-2} u \cdots \ln u}\right], \quad \text{for } u > M_4, \tag{2.9}$$

where C is defined in Eq. (2.4). Thus for $u > \rho \geq M_3$,

$$\begin{aligned} F(\rho) - F(u) &= \int_\rho^u -f(t)dt \geq \int_\rho^u h(t)dt \\ &= pC\left[u^p(\underbrace{\ln\ln\cdots\ln}_{n} u)^{p+1} - \rho^p(\underbrace{\ln\ln\cdots\ln}_{n} \rho)^{p+1}\right]. \end{aligned}$$

If we let $\rho^* = p\rho$, then for $u \geq \rho^* = p\rho$, we have

$$\begin{aligned} F(\rho) - F(u) &\geq pC\left[u^p(\underbrace{\ln\ln\cdots\ln}_{n} u)^{p+1} - \rho^p(\underbrace{\ln\ln\cdots\ln}_{n} \rho)^{p+1}\right] \\ &\geq ((p+1)C/2)u^p(\underbrace{\ln\ln\cdots\ln}_{n} u)^{p+1}, \end{aligned} \tag{2.10}$$

since

$$u^p(\underbrace{\ln\ln\cdots\ln}_{n} u)^{p+1} > (p\rho)^p(\underbrace{\ln\ln\cdots\ln}_{n} \rho)^{p+1}.$$

Then, by Eq. (2.10), for $\rho \geq M_3 > e^{e^{\cdot^{\cdot^{\cdot^{e^{p+1}}}}}}$,

$$\begin{aligned} G(\rho) &= 2\left(\frac{p-1}{p}\right)^{1/p} \int_\rho^\infty \frac{du}{(F(\rho)-F(u))^{1/p}} \\ &\leq 2\left(\frac{p-1}{p}\right)^{1/p} \left[\int_\rho^{\rho^*} \frac{du}{(F(\rho)-F(u))^{1/p}} \right. \\ &\quad \left. + \left(\frac{2p}{C(p+1)}\right)^{1/p} \int_{\rho^*}^\infty \frac{du}{u(\underbrace{\ln\ln\cdots\ln}_{n} u)^{(p+1)/p}}\right] \\ &\leq 2\left(\frac{p-1}{p}\right)^{1/p} \left[\int_\rho^{\rho^*} \frac{du}{[-f(z)(u-\rho)]^{1/p}} + \left(\frac{2p^{(p+1)}}{(p+1)C\ln\rho^*}\right)^{1/p}\right], \end{aligned}$$

where $z = z(u) \in (\rho, u)$ for each $u \in (\rho, \rho^*)$ exists by the mean value theorem. However, Eq. (2.9) implies

$$-f(z) \geq p^2 C z^{p-1}(\underbrace{\ln\ln\cdots\ln}_{n} z)^{p+1} > p^2 C \rho^{p-1}(\underbrace{\ln\ln\cdots\ln}_{n} \rho)^{p+1} \quad \text{for } z > \rho \geq M_3.$$

So we have

$$\begin{aligned} G(\rho) &\leq 2\left(\frac{p-1}{p}\right)^{1/p}\left[\int_\rho^{\rho^*}\frac{du}{[-f(z)(u-\rho)]^{1/p}} + \left(\frac{2p^{(p+1)}}{(p+1)C\ln\rho^*}\right)^{1/p}\right] \\ &< 2\left(\frac{p-1}{p}\right)^{1/p}\left[\left(\frac{1}{p^2C\rho^{p-1}(\underbrace{\ln\ln\cdots\ln}_{n}\rho)^{p+1}}\right)^{1/p}\int_\rho^{\rho^*}\frac{du}{(u-\rho)^{1/p}}\right. \\ &\quad \left. + \left(\frac{2p^{(p+1)}}{(p+1)C\ln\rho^*}\right)^{1/p}\right] \\ &= 2\left(\frac{p-1}{p}\right)^{1/p}\left[\left(\frac{p^{p-2}}{C(p-1)(\underbrace{\ln\ln\cdots\ln}_{n}\rho)^{p+1}}\right)^{1/p}\right. \\ &\quad \left. + \left(\frac{2p^{(p+1)}}{(p+1)C\ln p\rho}\right)^{1/p}\right] \to 0, \quad \text{as } \rho \to \infty. \end{aligned}$$

This completes the proof. ■

Theorem 2.7. Let f satisfy (2.1) or (2.2). If there exists a constant $s \geq 0$ such that $s \in I$ and f is nonincreasing on $[s, \infty)$, then G is strictly decreasing on $[s, \infty)$.

The proof of Theorem 2.7 is similar to the proof of [1, Theorem 3.4], so we omit it here.

Theorem 2.8. Let $f(u)$ satisfy Eq. (2.1) or (2.2) and $0 \in I$. Then

$$G(0) < \infty.$$

The proof of Theorem 2.8 is quite similar to that of Theorem 2.4, so we omit it here.

Theorem 2.9. Let f satisfy (2.1) or (2.2). If $f(0) < 0$ and f is nonincreasing on $[0, \infty)$, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*]$ there exists a unique nonnegative solution to (1.1)–(1.2) and for all $\lambda > \lambda^*$ there exist no nonnegative solutions to (1.1)–(1.2).

Proof. Let f satisfy (2.1) (f satisfy (2.2) being similar). Since $f(0) < 0$ and f is non-increasing on $[0, \infty)$, $I = [0, \infty)$. Thus, $G(\rho)$ exists and is continuous for all

$\rho \in (0, \infty)$ by Theorem 2.4. Also, by Theorem 2.7, G is strictly decreasing on $[0, \infty)$ so that

$$G(\rho) \leq G(0) \quad \text{for all} \quad \rho \geq 0. \tag{2.11}$$

By Theorem 2.6, we have $G(0) \to 0^+$ as $\rho \to \infty$. Combining these results we have that given an $r \in (0, G(0))$, there exists a unique $\rho \in [0, \infty)$ such that $G(\rho) = r$. So, given $0 < \lambda \leq \lambda^* = [G(0)]^p$, there exists a unique $\rho \geq 0$ such that $G(\rho) = \lambda^{1/p}$, and thus there will exist a unique nonnegative solution (1.1)–(1.2) for $\lambda \in (0, \lambda^*]$. For $\lambda > \lambda^*$, there will not exist $\rho \geq 0$ such that $G(\rho) = \lambda^{1/p} > G(0)$ because of (2.11), and thus there will exist no nonnegative solutions to (1.1)–(1.2) for $\lambda > \lambda^*$. ■

Theorem 2.10. In addition to Eq. (2.2), suppose that $f(u)$ satisfies $f(u) < 0$ on $[0, \infty)$ and $F(u) = \int_0^u f(t)dt$ and $H(u) = \int_0^u h(t)dt$ is an elementary function. Assume that there exist a function $h(u)$ and constants $b > k > 0$, $c < d = f(b) < 0$ satisfying

$$\begin{aligned} &f(u) \leq h(u) = \frac{(d-c)u}{k} + c, \quad 0 \leq u < k, \\ &f(u) \leq h(u) = d, \quad k \leq u \leq b, \\ &f(u) \leq h(u) = f(u), \quad u > b. \end{aligned} \tag{2.12}$$

Suppose that there exist $n, m \in \mathbb{N}$ such that

$$T = \overline{C_1} - A_1 - A_2 - \overline{B_1} > 0, \tag{2.13}$$

where

$$\overline{C_1} = \frac{b}{2^n} \sum_{i=1}^{2^n} \frac{1}{(F(b) - F((i/2^n + 1)b))^{1/p}}, \tag{2.14}$$

$$\begin{aligned} A_1 = \int_0^k \frac{du}{(-H(u))^{1/p}} &= k(-cx + 1/2(c-d)k)^{-1/p} \\ &\times \left(1 + \frac{(-c+d)k}{2cx}\right)^{1/p} (2cxk - ck^2 + dk^2)^{1/p} \\ &(2cxk + (-c+d)k^2)^{-1/p} \text{ Hypergeometric2F1} \left[\frac{1}{2}, \frac{1}{p}, \frac{3}{2}, -\frac{(-c+d)k}{2cx}\right], \end{aligned} \tag{2.15}$$

$$\begin{aligned} A_2 = \int_k^b \frac{du}{(-H(u))^{1/p}} &= \frac{2^{-1+1/p}((-c-d)k)^{-1/p}(-ck-dk)p}{d(-1+p)} \\ &- \frac{-d(b-k) + 1/2(-c-d)k)^{-1/p}(-2bd - ck + dk)p}{2d(-1+p)}, \end{aligned} \tag{2.16}$$

$$\overline{B_1} = \frac{b}{2^m} \sum_{i=0}^{2^m - 1} \frac{1}{(-H((i/2^m + 1)b))^{1/p}}, \tag{2.17}$$

where Hypergeometric$_2F_1[a, b, c, z]$ is the hypergeometric function $_2F_1(a, b, c, z)$. The $_2F_1$ function has series expansion $_2F_1(a, b; c; z) = \sum_{k=0}^{\infty}(a)_k(b)_k/(c)_k/k!$. Then

(i) $G_f(b) > G_f(0)$ and $\lim_{\rho\to\infty} G_f(\rho) = 0$, here

$$G_f(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p}\int_\rho^\infty \frac{du}{(F(\rho) - F(u))^{1/p}}.$$

(ii) $G_f(\rho)$ has at least one critical point, a local maximum, on $(0, \infty)$.

(iii) Let

$$\underline{\lambda} = (\min_{\rho\in[0,b]} G_f(\rho))^p \quad \text{and} \quad \overline{\lambda} = (\max_{\rho\in[0,\infty)} G_f(\rho))^p,$$

then problem (1.1)–(1.2) has at least two nonnegative solutions for $\underline{\lambda} < \lambda < \overline{\lambda}$, at least one nonnegative solution for $0 < \lambda \le \underline{\lambda}$ and $\lambda = \overline{\lambda}$, and no nonnegative solutions for $\lambda > \overline{\lambda}$.

Proof. The proof is similar to the proof of [22, Theorem 2.6], so we omit the detail. ■

3. Example

Example 3.1. Consider the problem

$$\begin{cases} -(\Phi_p(u'))' = -\lambda e^u \\ \lim_{x\to 0^+} u(x) = \infty = \lim_{x\to 1^-} u(x) \end{cases}$$

This example for which $f(u) = -e^u$ demonstrates Theorem 2.9. Note that $F(u) = -e^u + 1$ implies

$$G(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p}\int_\rho^\infty \frac{du}{(e^u - e^\rho)^{1/p}}.$$

Letting $w = e^{u/p}$ we obtain

$$G(\rho) = 2p\left(\frac{p-1}{p}\right)^{1/p}\int_{e^{\rho/p}}^\infty \frac{dw}{w(w^p - e^\rho)^{1/p}}.$$

Letting $w = e^{\rho/p}\sec^{2/p}\theta$ we obtain

$$\begin{aligned} G(\rho) &= 2p\left(\frac{p-1}{p}\right)^{1/p} e^{-\rho/p}\int_0^{\pi/2} \tan^{(p-2)/p}\theta d\theta \\ &= p\pi e^{-\rho/p}\left(\frac{p-1}{p}\right)^{1/p} \csc(\pi/p). \end{aligned}$$

From $\lim_{\rho \to \infty} G(\rho) = 0^+$, $\lim_{\rho \to 0^+} G(\rho) = \infty$, and $G(\rho)$ is strictly decreasing on $(0, \infty)$. These results imply that G is a bijective mapping from $(0, \infty)$ onto $(0, \infty)$. Thus, given $\lambda > 0$ there exist a unique $\rho > 0$ such that $G(\rho) = \lambda^{1/p}$. Hence, by Theorem 2.8, there is a unique nonnegative solution to (1.1)–(1.2) for each $\lambda > \lambda^*$. Hence, $\lambda = p^p(\pi)^p e^{-\rho}((p-1)/p)(\csc(\pi/p))^p$ and $\lambda^* = p^{p-1}(\pi)^p(p-1)(\csc(\pi/p))^p$.

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