International Journal of Difference Equations
ISSN 0973-6069 Volume 9, Number 1 (2014), pp. 1-5

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Long Wave Length Soliton Solutions of Navier Stokes Equation

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Abstract

We have used Becker's field theoretic formulation of Navier Stokes equation to derive the equation of motion for a double well potential. We find Domain wall solutions. Double well profiles are found in Cahn-Hillard Navier Stokes system. We suggest that Domain walls exist in the Cahn-Hillard system. To investigate this further we compute the Free Energy of the Domain Walls. The plot of the Domain Wall Free Energy agrees very well the profiles computed by other authors. Finally we use the method of Sakaguchi and Malomed to obtain the long wavelength expansion of the Navier Stokes equation. In this limit we find conservation laws. This result implies that under certain conditions the infinite array of conservation laws is also applicable to the Navier Stokes equation.

Keywords—Soliton, Long Wave length, Navier Stokes Equation.

1. INTRODUCTION.

Recently R. J. Becker [3] has developed the Lagrangian approach to Navier Stokes equation. In Becker's formalism the velocity vector is defined in terms of the gradient of a scalar field ψ and curl of a vector field $\vec{A}$ as

$$\vec{v} = \nabla \psi + \nabla X \vec{A} \tag{1}$$

We consider a system where the fluid has a low velocity and a stream lined flow. Here we can safely assume

$$\vec{A} = 0 \tag{2}$$

The Lagrangian density for the Navier Stokes equation is then given according to the formalism of Becker by [3],

$$L = L_0 + L_I \tag{3}$$

$$L_0 = \dot{\psi}^2 + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 \tag{4}$$

where we have neglected the Diffusion term. The interaction term is taken as the Double well potential

$$L_I = -A\frac{\psi^2}{2} - B\frac{\psi^4}{4} \tag{5}$$

Double well potentials admit Soliton solutions. Solitons have been both observed in fluids [15] as well as predicted via calculations and simulations. From the Euler Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = 0 \tag{6}$$

Note that a double well potential admits bound states as has indeed been found in the long wavelength limit. One can derive equation of motion

$$\ddot{\psi} + A\psi + B\psi^3 - \psi'' = 0 \tag{7}$$

whose solution is as obtained by [12]

$$\psi = \tanh\left(\frac{x - vt}{\sqrt{2}}\right) \tag{8}$$

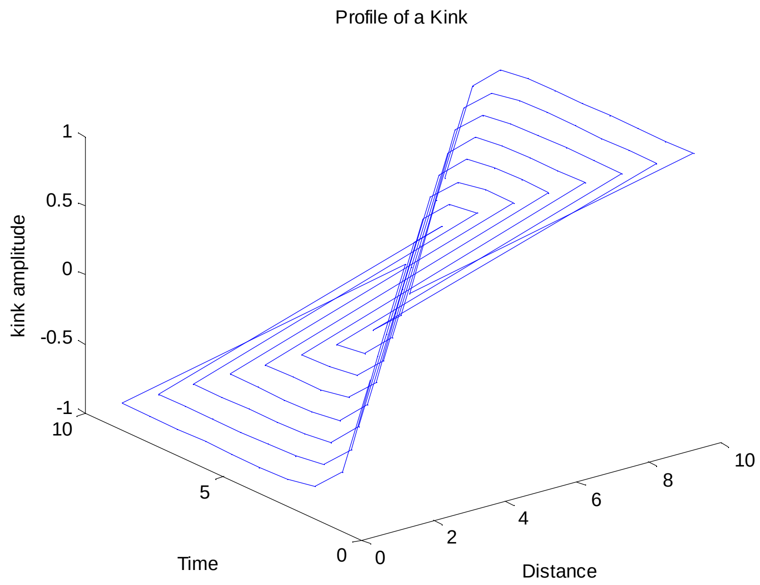

Fig. 1 Profile of Kink

This is also known as the domain wall solution (see fig. 1). Shapes in the form of such domains are seen near the sea shore when Tsunami occurs. In a sense this model is unrealistic as it discusses fluid flow in one dimension only. However a 3 dimensional model of this idea is found in the Cahn-Hillard Navier Stokes model. Here one finds a double well model in 3 dimensions. It is also of interest the thermodynamic equilibrium of such a one dimensional model as the Free Energy of the Cahn-Hillard has been simulated experimentally [11].

2. Equilibrium Behavior

We are interested in the thermodynamic equilibrium of the Navier Stokes system. The functional integral form of the Partition function is given by

$$Z = \int [\delta u \delta p] e^{-\beta H(p,u)} \qquad (9)$$

Scalpino, Sears and Ferrel [13] (SSF), Kac and Helfand [14] have shown how to evaluate (9). In particular Krumhansl and Schrieffer [12] have computed the Free Energy of Domain walls in the double well model. A plot of this result is shown in fig. 2 One sees that the Free energy plots in Fig. 2 are identical with the Free Energy plots of F. Boyer, C. Lapuerta, S. Minjeoud, B. Piar and M. Quintard [11] obtained after extensive simulation. This implies that double well model we have used is indeed correct.

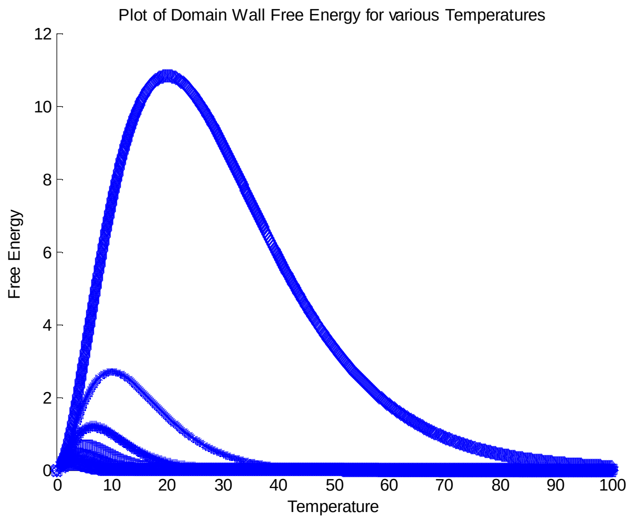

Fig. 2 Domain Wall Free Energy

3. Long Wave Length Solutions

H. Sakaguchi and B. A. Malomed in their seminal paper [1] proposed a novel expansion technique (11) to obtain the long wavelength Soliton solutions for the Gross-Pitavskii equation. By substituting the expansion (10) in the navier Stokes equation we obtain (by comparing coefficients) the effective equations in the long wavelength region. The effective equation is put in the form of a conservation law (15). The spatial component of the conservation law is an eigenvalue equation. For the KdV equation the eigenvalue equation is the Schrodinger equation. However for Navier Stokes some variable transformations are required to obtain the corresponding Schrodinger equation. The important point is that in the asymptotic limit the nonlinear differential equation becomes equivalent to the Schrodinger equation. In each case the Schrodinger equation is solved and the Green's function obtained from the eigen-functions.

4. NAVIER STOKES EQUATION.

Here again we concentrate on the scalar field whose gradient gives the velocity vector The equation of the scalar potential are

$$\ddot{\psi} - D\,\nabla^2\dot{\psi} - c^2\nabla^2\psi = 0 \qquad (10)$$

This equation reflects the interplay between diffusion and propagation. We look for solutions of the form

$$\psi(x,t) = \psi^{(0)}(x,t) + \psi^{(1)}(x,t)\cos(2x) + ... \qquad (11)$$

Equating the coefficients of cos(2x) we get

$$\psi_{tt}^{(1)} = D\left[\psi_{txx}^{(1)} - 4\psi_t^{(1)}\right] + c^2\left[\psi_{xx}^{(1)} - 4\psi^{(1)}\right] \qquad (12)$$

Navier Stokes Equation

$$\psi_{tt}^{(1)} + 4D\psi_t^{(1)} = D\psi_{txx}^{(1)} + c^2\psi_{xx}^{(1)} - 4c^2\psi^{(1)} \qquad (13)$$

Define

$$\psi_t^{(1)} + D\psi_t^{(1)} = \xi \qquad (14)$$

The conservation equation is

$$\xi_t + \left(\left(c^2 - Dv\right)\psi_{xx} - 4c^2\psi\right)_x = 0 \qquad (15)$$

$$\left(c^2 - Dv\right)\psi_{xx} = \left(4c^2\right)\psi \qquad (16)$$

OR $\psi_{xx} = \dfrac{\left(4c^2\right)\psi}{\left(c^2 - Dv\right)}$

Where $k^2 = \dfrac{\left(4c^2\right)}{\left(c^2 - Dv\right)}$ (17)

The above equation is an eigen-value equation. For $Dv > c^2$ the solution is

$$\psi \sim \sin(kx + \delta) \qquad (18)$$

This corresponds to the bound state. Bound states in Navier Stokes equations have been reported [11]. For $Dv < c^2$ the solution is

$$\psi = Ae^{kx} + Be^{-kx} \qquad (19)$$

For slowly diffusing media if the velocity is larger than the speed of sound in the media then the disturbance will propagate as sinusoidal waves. In the opposite limit the disturbance will propagate as decaying exponentials.

5. Conclusion.

We have used Becker's field theoretic formulation of Navier Stokes equations to obtain the Lagrangian corresponding to a double well potential. The double well potential has domain wall solutions. The profile of the Free Energy for Domains (as computed by Krumhansl and Schreiffer) is in good agreement with Free Energy profile as computed by [11] via computer simulations. Thus the Cahn-Hillard model for two or three phase flow thus can be accounted for Domain wall solutions. Further we have used the formulation of Sakaguchi and Malomed [1] to derive the long wavelength approximation to the Navier Stokes equation. In the long wavelength approximation we have derived the conservation law. This result implies that only in the long wavelength limit the array of infinite conservation laws are applicable to the Navier Stokes equation.

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