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Integral Operator Inequalities on Time Scales

George A. Anastassiou
University of Memphis
Department of Mathematical Sciences
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we present a wide range integral operator general inequalities on time scales under convexity. Our treatment is combined by using the diamond-alpha integral. When that fails in the fractional setting, we use the delta and nabla integrals. We give plenty of interesting applications.

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1 Introduction

We start with the definition of the Riemann–Liouville fractional integrals, see [20]. Let $[a, b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann–Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad (x > a), \tag{1.1}$$

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt, \quad (x < b), \tag{1.2}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We mention a basic property of the operators $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$, see also [23]. The result says that the fractional integral operators $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ are bounded in $L_p(a, b)$, $1 \leq p \leq \infty$, that is

$$\left\|I_{a+}^{\alpha} f\right\|_p \leq K \|f\|_p \quad , \quad \left\|I_{b-}^{\alpha} f\right\|_p \leq K \|f\|_p , \tag{1.3}$$

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where

$$K = \frac{(b-a)^{\alpha}}{\alpha \Gamma(\alpha)}. \tag{1.4}$$

Inequality (1.3), that is the result involving the left-sided fractional integral, was proved by H. G. Hardy in one of his first papers, see [16]. He did not write down the constant, but the calculation of the constant was hidden inside his proof.

So we are motivated by (1.3), and also [2, 5, 7, 19], and we will prove analogous properties on time scales. But first we need some background on time scales, see also [11].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The time scales calculus was initiated by S. Hilger in his PhD thesis in order to unify discrete and continuous analysis [17, 18]. Let $\mathbb{T}$ be a time scale with the topology that it inherits from the real numbers. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \tag{1.5}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}. \tag{1.6}$$

If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$, then we say that t is left-scattered. Points that are simultaneously right-scattered and left-scattered are said to be isolated. If $\sigma(t) = t$, then t is called right-dense; if $\rho(t) = t$, then t is called left-dense. The mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ defined by $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$ are called, respectively, the forward and backward graininess function.

Given a time scale $\mathbb{T}$, we introduce the sets $\mathbb{T}^k$, $\mathbb{T}_k$, and $\mathbb{T}_k^k$ as follows. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$.

Let $f : \mathbb{T} \to \mathbb{R}$ be a real valued function on a time scale $\mathbb{T}$. Then, for $t \in \mathbb{T}^k$, we define $f^{\Delta}(t)$ to be the number, if one exists, such that for all $\epsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \le \epsilon \left| \sigma(t) - s \right|. \tag{1.7}$$

We say that f is delta differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. Similarly, for $t \in \mathbb{T}_k$, we define $f^{\nabla}(t)$ to be the number, if one exists, such that for all $\epsilon > 0$, there is a neighborhood V of t such that for all $s \in V$

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s) \right| \le \epsilon \left| \rho(t) - s \right|. \tag{1.8}$$

We say that f is nabla differentiable on $\mathbb{T}_k$, provided that $f^{\nabla}(t)$ exists for all $t \in \mathbb{T}_k$.

For $f : \mathbb{T} \to \mathbb{R}$, we define the function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ by $f^{\sigma}(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, that is $f^{\sigma} = f \circ \sigma$. Similarly, we define the function $f^{\rho} : \mathbb{T} \to \mathbb{R}$ by $f^{\rho}(t) = f(\rho(t))$ for all $t \in \mathbb{T}$, that is, $f^{\rho} = f \circ \rho$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits finite at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous, provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits finite at all right-dense points in $\mathbb{T}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\,\Delta t = F(b) - F(a). \tag{1.9}$$

A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by $\int_a^b g(t)\,\nabla t = G(b) - G(a)$. For more details on time scales one can see [1, 11, 12].

Now we describe the diamond-α derivative and integral, referring the reader to [21, 22, 24–27] for more on this calculus.

Let $\mathbb{T}$ be a time scale and f differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}_k^k$, we define the diamond-α dynamic derivative $f^{\diamondsuit^{\alpha}}(t)$ by

$$f^{\diamondsuit^{\alpha}}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \ 0 \le \alpha \le 1. \tag{1.10}$$

Thus, f is diamond-α differentiable if and only if f is Δ and ∇ differentiable. The diamond-α derivative reduces to the standard Δ derivative for $\alpha = 1$, or the standard ∇ derivative for $\alpha = 0$. Also, it gives a "weighted derivative" for $\alpha \in (0, 1)$. Diamond-α derivatives have shown in computational experiments to provided efficient and balanced approximation formulae, leading to the design of more reliable numerical methods [24, 25].

Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}_k^k$. Then

(i) $f \pm g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}_k^k$ with

$$(f \pm g)^{\diamondsuit^{\alpha}}(t) = (f)^{\diamondsuit^{\alpha}}(t) \pm (g)^{\diamondsuit^{\alpha}}(t). \tag{1.11}$$

(ii) For any constant c, $cf : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}_k^k$ with

$$(cf)^{\diamondsuit^{\alpha}}(t) = c(f)^{\diamondsuit^{\alpha}}(t). \tag{1.12}$$

(iii) $fg : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}_k^k$ with

$$(fg)^{\diamondsuit^{\alpha}}(t) = (f)^{\diamondsuit^{\alpha}}(t)\, g(t) + \alpha f^{\sigma}(t)\, g^{\Delta}(t) + (1-\alpha) f^{\rho}(t)\, g^{\nabla}(t). \tag{1.13}$$

Let $a, t \in \mathbb{T}$, and $h : \mathbb{T} \to \mathbb{R}$. Then, the diamond-$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(\tau)\,\diamondsuit_{\alpha}\tau = \alpha \int_a^t h(\tau)\,\Delta\tau + (1-\alpha)\int_a^t h(\tau)\,\nabla\tau, \quad 0 \le \alpha \le 1. \tag{1.14}$$

We may notice the absence of an anti-derivative for the $\diamondsuit_\alpha$ combined derivative. For $t \in \mathbb{T}_k^k$, in general

$$\left(\int_a^t h(\tau) \diamondsuit_\alpha \tau \right)^{\diamondsuit^\alpha} \neq h(t). \tag{1.15}$$

Although the fundamental theorem of calculus does not hold for the $\diamondsuit_\alpha$-integral, other properties hold true. Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Then

(i) $\int_a^t \{f(\tau) \pm g(\tau)\} \diamondsuit_\alpha \tau = \int_a^t f(\tau) \diamondsuit_\alpha \tau \pm \int_a^t g(\tau) \diamondsuit_\alpha \tau;$

(ii) $\int_a^t cf(\tau) \diamondsuit_\alpha \tau = c \int_a^t f(t) \diamondsuit_\alpha \tau;$

(iii) $\int_a^t f(\tau) \diamondsuit_\alpha \tau = \int_a^b f(\tau) \diamondsuit_\alpha \tau + \int_b^t f(\tau) \diamondsuit_\alpha \tau;$

(iv) If $f(t) \geq 0$ for all t, then $\int_a^b f(t) \diamondsuit_\alpha t \geq 0;$

(v) If $f(t) \leq g(t)$ for all t, then $\int_a^b f(t) \diamondsuit_\alpha t \leq \int_a^b g(t) \diamondsuit_\alpha t;$

(vi) If $f(t) \geq 0$ for all t, then $f(t) \equiv 0$ if and only if $\int_a^b f(t) \diamondsuit_\alpha t = 0;$

(vii) $\int_a^b c \diamondsuit_\alpha t = c(b-a);$

(viii) $\left| \int_a^b f(t) \diamondsuit_\alpha t \right| \leq \int_a^b |f(t)| \diamondsuit_\alpha t.$

We would use Jensen's diamond-α integral inequalities.

Theorem 1.1 (Jensen's inequality, see [26]). *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and $c, d \in \mathbb{R}$. If $g \in C([a,b]_{\mathbb{T}}, (c,d))$ and $f \in C((c,d), \mathbb{R})$ is convex, then*

$$f\left(\frac{\int_a^b g(s) \diamondsuit_\alpha s}{b-a} \right) \leq \frac{\int_a^b f(g(s)) \diamondsuit_\alpha s}{b-a}. \tag{1.16}$$

Also we need the extended Jensen inequality on time scales via diamond-α integral.

Theorem 1.2 (Generalized Jensen's inequality, see [26]). *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, $c, d \in \mathbb{R}$, $g \in C([a,b]_{\mathbb{T}}, (c,d))$, and $h \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ with*

$$\int_a^b |h(s)| \diamondsuit_\alpha s > 0. \tag{1.17}$$

If $f \in C((c,d), \mathbb{R})$ *is convex, then*

$$f\left(\frac{\int_a^b |h(s)| g(s) \diamondsuit_\alpha s}{\int_a^b |h(s)| \diamondsuit_\alpha s}\right) \leq \frac{\int_a^b |h(s)| f(g(s)) \diamondsuit_\alpha s}{\int_a^b |h(s)| \diamondsuit_\alpha s}. \tag{1.18}$$

We further need the following inequality.

Theorem 1.3 (Hölder's Inequality, see [14])**.** *For continuous functions* $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$*, we have:*

$$\int_a^b |f(t) g(t)| \diamondsuit_\alpha t \leq \left[\int_a^b |f(t)|^p \diamondsuit_\alpha t\right]^{\frac{1}{p}} \left[\int_a^b |g(t)|^q \diamondsuit_\alpha t\right]^{\frac{1}{q}}, \tag{1.19}$$

where $p > 1$*, and* $q = \frac{p}{p-1}$.

We obtain the following result.

Theorem 1.4 (Generalization of Hölder's inequality)**.** *Let* $f_i \in C([a,b]_{\mathbb{T}}, \mathbb{R})$*,* $i = 1, \ldots, n$*, and* $p_i > 1$ *such that* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$*. Then*

$$\int_a^b \prod_{i=1}^{n} |f_i(t)| \diamondsuit_\alpha t \leq \prod_{i=1}^{n} \left(\int_a^b |f_i(t)|^{p_i} \diamondsuit_\alpha t\right)^{\frac{1}{p_i}}. \tag{1.20}$$

Proof. We use (1.19) and the induction hypothesis, exactly as in [13]. □

Comment 1.5. By Tietze's extension theorem of general topology, we easily derive that a continuous function f of $\prod_{i=1}^{n}([a_i, b_i] \cap \mathbb{T}_i)$ (where $\mathbb{T}_i$, $i = 1, \ldots, n \in \mathbb{N}$ are time scales) is bounded, since its continuous extension F on $\prod_{i=1}^{n}[a_i, b_i] \subseteq \mathbb{R}^n$ is bounded, $n \in \mathbb{N}$.

Comment 1.6. It is regarding the univariate functions. Based on [15], we see that the Cauchy Time scales delta Δ and nabla ∇ integrals are equal to definite Riemann time scales Δ, ∇ integrals, respectively. Thus, the diamond-α-Cauchy integral (1.14) is a diamond α-Riemann integral over continuous functions. Of course the last integral exists, since continuous functions are Riemann Δ and ∇-integrable, and it is equal to the corresponding α-Lebesgue integral, by [15]. In particular, the dominated and bounded convergence theorems hold true with respect to the Lebesgue-Δ, ∇ measures.

Comment 1.7. Let $\mathbb{T}_1$, $\mathbb{T}_2$ be time scales and $f : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to \mathbb{R}$ be continuous. By [8, 9], we get that f is Riemann Δ and ∇-integrable over $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ and $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$, respectively. Hence by [8, 9], f is Lebesgue Δ and ∇-integrable there. Thus by Fubini's theorem, we ge

$$\int_a^b \left(\int_c^d f(x,y)\,\Delta y \right) \Delta x = \int_c^d \left(\int_a^b f(x,y)\,\Delta x \right) \Delta y, \tag{1.21}$$

and

$$\int_a^b \left(\int_c^d f(x,y)\,\nabla y \right) \nabla x = \int_c^d \left(\int_a^b f(x,y)\,\nabla x \right) \nabla y. \tag{1.22}$$

We define ($\alpha \in [0,1]$)

$$\int_a^b \left(\int_c^d f(x,y)\,\Diamond_\alpha y \right) \Diamond_\alpha x$$
$$\alpha \int_a^b \left(\int_c^d f(x,y)\,\Delta y \right) \Delta x + (1-\alpha) \int_a^b \left(\int_c^d f(x,y)\,\nabla y \right) \nabla x. \tag{1.23}$$

One can generalize (1.23) for multiple integrals. So for f continuous, we get the $\Diamond_\alpha$-Fubini's theorem main property:

$$\int_a^b \left(\int_c^d f(x,y)\,\Diamond_\alpha y \right) \Diamond_\alpha x = \int_c^d \left(\int_a^b f(x,y)\,\Diamond_\alpha x \right) \Diamond_\alpha y. \tag{1.24}$$

We notice the following.

Remark 1.8. Let $\mathbb{T}_1$, $\mathbb{T}_2$ be time scales and $f : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to \mathbb{R}$ be continuous. Consider

$$\begin{aligned} g(x) &= \int_c^d f(x,y)\,\Diamond_\alpha y \\ &= \alpha \int_c^d f(x,y)\,\Delta y + (1-\alpha) \int_c^d f(x,y)\,\nabla y, \end{aligned}$$

$\alpha \in [0,1]$, $\forall\, x \in [a,b]_{\mathbb{T}_1}$. We prove that g is continuous on $[a,b]_{\mathbb{T}_1}$. Let $x_n \to x$, where $\{x_n\}_{n\in\mathbb{N}}$, $x \in [a,b]_{\mathbb{T}_1}$ then $f(x_n,y) \to f(x,y)$, as $n \to \infty$, $\forall\, y \in [c,d]_{\mathbb{T}_2}$. Furthermore there exists $M > 0$ such that $|f(x_n,y)|$, $|f(x,y)| \le M$, $\forall\, y \in [c,d]_{\mathbb{T}_2}$. Hence by Lebesgue's bounded convergence theorem (see [8]), we get that

$$\lim_{n\to\infty} \int_c^d f(x_n,y)\,\Delta y = \int_c^d f(x,y)\,\Delta y, \tag{1.25}$$

and

$$\lim_{n\to\infty} \int_c^d f(x_n,y)\,\nabla y = \int_c^d f(x,y)\,\nabla y. \tag{1.26}$$

Combining (1.25) and (1.26), we obtain $g(x_n) \to g(x)$, as $n \to \infty$, proving the continuity of g.

Comment 1.9. In [6], we proved that if $\Phi : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}$ is convex and increasing, then Φ is continuous on $\mathbb{R}_+$. Furthermore for $x < 0$, we extend $\Phi(x) := \Phi(-x)$, to the symmetric branch of Φ. Both branches of Φ make a convex function on $(-\infty, \infty)$.

So now we can apply Jensen's inequality also on $\mathbb{R}$. Plus, it is well known that if $\Phi : (A, B) \subseteq \mathbb{R} \to \mathbb{R}$ is convex, then Φ is continuous on (A, B).

2 Main Results

We present inequalities on $\Diamond_\alpha$-integral operators.

Theorem 2.1. *Let $\mathbb{T}_1$, $\mathbb{T}_2$ be time scales, $a, b \in \mathbb{T}_1$; $c, d \in \mathbb{T}_2$; $k(x, y)$ is a kernel function with $x \in [a, b]_{\mathbb{T}_1}$, $y \in [c, d]_{\mathbb{T}_2}$; k is continuous function from $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ into $\mathbb{R}_+$. Consider*

$$K(x) := \int_c^d k(x, y) \Diamond_\alpha y, \quad \forall\, x \in [a, b]_{\mathbb{T}_1}. \tag{2.1}$$

We assume that $K(x) > 0$, $\forall\, x \in [a, b]_{\mathbb{T}_1}$. Consider $f : [c, d]_{\mathbb{T}_2} \to \mathbb{R}$ continuous, and the $\Diamond_\alpha$-integral operator function

$$g(x) := \int_c^d k(x, y) f(y) \Diamond_\alpha y, \tag{2.2}$$

$\forall\, x \in [a, b]_{\mathbb{T}_1}$. *Consider also the weight function*

$$u : [a, b]_{\mathbb{T}_1} \to \mathbb{R}_+, \tag{2.3}$$

which is continuous. Define further the function

$$v(y) := \int_a^b \frac{u(x) k(x, y)}{K(x)} \Diamond_\alpha x, \tag{2.4}$$

$\forall\, y \in [c, d]_{\mathbb{T}_2}$. *Let I denote any of $(0, \infty)$ or $[0, \infty)$, and $\Phi : I \to \mathbb{R}$ be a convex and increasing function. In particular, we assume that*

$$|f|\left([c, d]_{\mathbb{T}_2}\right) \subseteq I. \tag{2.5}$$

Then

$$\int_a^b u(x) \Phi\left(\frac{|g(x)|}{K(x)}\right) \Diamond_\alpha x \le \int_c^d v(y) \Phi(|f(y)|) \Diamond_\alpha y. \tag{2.6}$$

Proof. We see that

$$\int_a^b u(x)\,\Phi\left(\frac{|g(x)|}{K(x)}\right)\diamondsuit_\alpha x = \int_a^b u(x)\,\Phi\left(\frac{1}{K(x)}\left|\int_c^d k(x,y)\,f(y)\,\diamondsuit_\alpha y\right|\right)\diamondsuit_\alpha x$$

$$\leq \int_a^b u(x)\,\Phi\left(\frac{1}{K(x)}\int_c^d k(x,y)\,|f(y)|\,\diamondsuit_\alpha y\right)\diamondsuit_\alpha x$$

$$\leq \int_a^b \frac{u(x)}{K(x)}\left(\int_c^d k(x,y)\,\Phi(|f(y)|)\,\diamondsuit_\alpha y\right)\diamondsuit_\alpha x$$

$$= \int_a^b \left(\int_c^d \frac{u(x)\,k(x,y)}{K(x)}\Phi(|f(y)|)\,\diamondsuit_\alpha y\right)\diamondsuit_\alpha x$$

$$= \int_c^d \left(\int_a^b \frac{u(x)\,k(x,y)}{K(x)}\Phi(|f(y)|)\,\diamondsuit_\alpha x\right)\diamondsuit_\alpha y$$

$$= \int_c^d \Phi(|f(y)|)\left(\int_a^b \frac{u(x)\,k(x,y)}{K(x)}\diamondsuit_\alpha x\right)\diamondsuit_\alpha y$$

$$= \int_c^d v(y)\,\Phi(|f(y)|)\,\diamondsuit_\alpha y,$$

where we used the generalized Jensen's inequality, see Theorem 1.2 and Comment 1.9, and (1.24). This proves the claim. □

We continue with the next result.

Theorem 2.2. *All as in Theorem 2.1, however now* Φ *is not necessarily increasing and only from* $(0,\infty)$ *into* $\mathbb{R}$*. Additionally, we assume that* f *is of fixed strict sign. Then*

$$\int_a^b u(x)\,\Phi\left(\frac{|g(x)|}{K(x)}\right)\diamondsuit_\alpha x \leq \int_c^d v(y)\,\Phi(|f(y)|)\,\diamondsuit_\alpha y. \tag{2.7}$$

Proof. We notice that

$$|g(x)| = \left|\int_c^d k(x,y)\,f(y)\,\diamondsuit_\alpha y\right| = \int_c^d k(x,y)\,|f(y)|\,\diamondsuit_\alpha y. \tag{2.8}$$

Therefore we have

$$\int_a^b u(x)\,\Phi\left(\frac{|g(x)|}{K(x)}\right)\diamondsuit_\alpha x = \int_a^b u(x)\,\Phi\left(\frac{1}{K(x)}\left|\int_c^d k(x,y)\,f(y)\,\diamondsuit_\alpha y\right|\right)\diamondsuit_\alpha x$$

$$= \int_a^b u(x)\,\Phi\left(\frac{1}{K(x)}\int_c^d k(x,y)\,|f(y)|\,\diamondsuit_\alpha y\right)\diamondsuit_\alpha x. \tag{2.9}$$

The rest follows as in the proof of Theorem 2.1. □

The following is a corollary to Theorem 2.2.

Corollary 2.3. *It holds*

$$\int_a^b u(x) \ln \left(\frac{|g(x)|}{K(x)} \right) \diamondsuit_\alpha x \geq \int_c^d v(y) \ln (|f(y)|) \diamondsuit_\alpha y. \tag{2.10}$$

Proof. Apply (2.7) for $\Phi(x) = -\ln x$, which is convex with domain $(0, \infty)$. □

The following is a corollary to Theorem 2.1.

Corollary 2.4. *It holds*

$$\int_a^b u(x) e^{\frac{|g(x)|}{K(x)}} \diamondsuit_\alpha x \leq \int_c^d v(y) e^{|f(y)|} \diamondsuit_\alpha y. \tag{2.11}$$

Proof. Apply (2.6) for $\Phi(x) = e^x$, $x \geq 0$. □

Notation 2.5. Let $\mathbb{T}_1$, $\mathbb{T}_2$ be time scales, $a, b \in \mathbb{T}_1$; $c, d \in \mathbb{T}_2$; $k_i(x, y)$ is a kernel function with $x \in [a, b]_{\mathbb{T}_1}$, $y \in [c, d]_{\mathbb{T}_2}$; k_i is continuous function from $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ into $\mathbb{R}_+$ for $i = 1, \ldots, m \in \mathbb{N}$. Consider

$$K_i(x) := \int_c^d k_i(x, y) \diamondsuit_\alpha y, \quad \forall\, x \in [a, b]_{\mathbb{T}_1}, \tag{2.12}$$

$i = 1, \ldots, m$. We assume that $K_i(x) > 0$, $\forall\, x \in [a, b]_{\mathbb{T}_1}$, $i = 1, \ldots, m$. Consider $f_i : [c, d]_{\mathbb{T}_2} \to \mathbb{R}$ continuous, $i = 1, \ldots, m$, and the $\diamondsuit_\alpha$-integral operator function

$$g_i(x) := \int_c^d k_i(x, y) f_i(y) \diamondsuit_\alpha y, \tag{2.13}$$

$\forall\, x \in [a, b]_{\mathbb{T}_1}$, $i = 1, \ldots, m$. Consider also the weight function

$$u : [a, b]_{\mathbb{T}_1} \to \mathbb{R}_+, \tag{2.14}$$

which is continuous. Define further the function

$$\lambda_m(y) := \int_a^b \frac{u(x) \prod_{i=1}^m k_i(x, y)}{\prod_{i=1}^m K_i(x)} \diamondsuit_\alpha x, \tag{2.15}$$

$\forall\, y \in [c, d]_{\mathbb{T}_2}$. Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, are convex and increasing functions.

We give the following result.

Theorem 2.6. *All as in Notation 2.5. Let $j \in \{1, \ldots, m\}$ be fixed. Then*

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{|g_i(x)|}{K_i(x)} \right) \diamondsuit_\alpha x \leq \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_c^d \Phi_i(|f_i(y)|) \diamondsuit_\alpha y \right) \left(\int_c^d \Phi_j(|f_j(y)|) \lambda_m(y) \diamondsuit_\alpha y \right). \tag{2.16}$$

Proof. We demonstrate the proof for $m = 3$. For general m it follows the same way. Here we use the extended Jensen's inequality, see Theorem 1.2 and Comment 1.9, $\diamondsuit_\alpha$-Fubini's theorem, see (1.24), and that Φ_i are increasing. We introduce the auxiliary function

$$\theta(x) := \frac{u(x)}{\prod_{i=1}^{3} K_i(x)}$$

and calculate

$$\begin{aligned}
&\int_a^b u(x) \prod_{i=1}^{3} \Phi_i \left(\frac{|g_i(x)|}{K_i(x)} \right) \diamondsuit_\alpha x \\
&= \int_a^b u(x) \prod_{i=1}^{3} \Phi_i \left(\left| \frac{1}{K_i(x)} \int_c^d k_i(x,y) f_i(y) \diamondsuit_\alpha y \right| \right) \diamondsuit_\alpha x \\
&\leq \int_a^b u(x) \prod_{i=1}^{3} \Phi_i \left(\frac{1}{K_i(x)} \int_c^d k_i(x,y) |f_i(y)| \diamondsuit_\alpha y \right) \diamondsuit_\alpha x \\
&\leq \int_a^b u(x) \prod_{i=1}^{3} \left(\frac{1}{K_i(x)} \int_c^d k_i(x,y) \Phi_i(|f_i(y)|) \diamondsuit_\alpha y \right) \diamondsuit_\alpha x \\
&= \int_a^b \left(\frac{u(x)}{\prod_{i=1}^{3} K_i(x)} \right) \left(\prod_{i=1}^{3} \int_c^d k_i(x,y) \Phi_i(|f_i(y)|) \diamondsuit_\alpha y \right) \diamondsuit_\alpha x \\
&= \int_a^b \theta(x) \left(\prod_{i=1}^{3} \int_c^d k_i(x,y) \Phi_i(|f_i(y)|) \diamondsuit_\alpha y \right) \diamondsuit_\alpha x \\
&= \int_a^b \theta(x) \left[\int_c^d \left(\prod_{i=1}^{2} \int_c^d k_i(x,y) \Phi_i(|f_i(y)|) \diamondsuit_\alpha y \right) \right. \\
&\quad \left. k_3(x,y) \Phi_3(|f_3(y)|) \diamondsuit_\alpha y \right] \diamondsuit_\alpha x \\
&= \int_a^b \left(\int_c^d \theta(x) \left(\prod_{i=1}^{2} \int_c^d k_i(x,y) \Phi_i(|f_i(y)|) \diamondsuit_\alpha y \right) \right. \\
&\quad \left. k_3(x,y) \Phi_3(|f_3(y)|) \diamondsuit_\alpha y \right) \diamondsuit_\alpha x \\
&= \int_c^d \left(\int_a^b \theta(x) \left(\prod_{i=1}^{2} \int_c^d k_i(x,y) \Phi_i(|f_i(y)|) \diamondsuit_\alpha y \right) \right. \\
&\quad \left. k_3(x,y) \Phi_3(|f_3(y)|) \diamondsuit_\alpha x \right) \diamondsuit_\alpha y \\
&= \int_c^d \Phi_3(|f_3(y)|) \left(\int_a^b \theta(x) k_3(x,y) \left(\prod_{i=1}^{2} \int_c^d k_i(x,y) \Phi_i(|f_i(y)|) \diamondsuit_\alpha y \right) \diamondsuit_\alpha x \right) \diamondsuit_\alpha y
\end{aligned}$$

$$= \int_c^d \Phi_3(|f_3(y)|)\left[\int_a^b \theta(x) k_3(x,y)\left(\int_c^d \left\{\int_c^d k_1(x,y)\Phi_1(|f_1(y)|)\diamondsuit_\alpha y\right\}\cdot\right.\right.$$
$$\left.\left. k_2(x,y)\Phi_2(|f_2(y)|)\diamondsuit_\alpha y\right)\diamondsuit_\alpha x\right]\diamondsuit_\alpha y$$
$$= \int_c^d \Phi_3(|f_3(y)|)\left[\int_a^b \left(\int_c^d \theta(x) k_2(x,y) k_3(x,y)\Phi_2(|f_2(y)|)\cdot\right.\right.$$
$$\left.\left.\left\{\int_c^d k_1(x,y)\Phi_1(|f_1(y)|)\diamondsuit_\alpha y\right\}\diamondsuit_\alpha y\right)\diamondsuit_\alpha x\right]\diamondsuit_\alpha y$$
$$= \left(\int_c^d \Phi_3(|f_3(y)|)\diamondsuit_\alpha y\right)\left[\int_a^b \left(\int_c^d \theta(x) k_2(x,y) k_3(x,y)\Phi_2(|f_2(y)|)\cdot\right.\right.$$
$$\left.\left.\left\{\int_c^d k_1(x,y)\Phi_1(|f_1(y)|)\diamondsuit_\alpha y\right\}\diamondsuit_\alpha y\right)\diamondsuit_\alpha x\right]$$
$$= \left(\int_c^d \Phi_3(|f_3(y)|)\diamondsuit_\alpha y\right)\left[\int_c^d \left(\int_a^b \theta(x) k_2(x,y) k_3(x,y)\Phi_2(|f_2(y)|)\cdot\right.\right.$$
$$\left.\left.\left\{\int_c^d k_1(x,y)\Phi_1(|f_1(y)|)\diamondsuit_\alpha y\right\}\diamondsuit_\alpha x\right)\diamondsuit_\alpha y\right]$$
$$= \left(\int_c^d \Phi_3(|f_3(y)|)\diamondsuit_\alpha y\right)\left[\int_c^d \Phi_2(|f_2(y)|)\left(\int_a^b \theta(x) k_2(x,y) k_3(x,y)\cdot\right.\right.$$
$$\left.\left.\left(\int_c^d k_1(x,y)\Phi_1(|f_1(y)|)\diamondsuit_\alpha y\right)\diamondsuit_\alpha x\right)\diamondsuit_\alpha y\right]$$
$$= \left(\int_c^d \Phi_3(|f_3(y)|)\diamondsuit_\alpha y\right)\left[\int_c^d \Phi_2(|f_2(y)|)\left\{\int_a^b \left(\int_c^d \theta(x)\prod_{i=1}^3 k_i(x,y)\cdot\right.\right.\right.$$
$$\left.\left.\left.\Phi_1(|f_1(y)|)\diamondsuit_\alpha y\right)\diamondsuit_\alpha x\right\}\diamondsuit_\alpha y\right]$$
$$= \left(\int_c^d \Phi_3(|f_3(y)|)\diamondsuit_\alpha y\right)\left(\int_c^d \Phi_2(|f_2(y)|)\diamondsuit_\alpha y\right)\cdot$$
$$\left(\int_a^b \left(\int_c^d \theta(x)\prod_{i=1}^3 k_i(x,y)\Phi_1(|f_1(y)|)\diamondsuit_\alpha y\right)\diamondsuit_\alpha x\right)$$
$$= \left(\prod_{i=2}^3 \int_c^d \Phi_i(|f_i(y)|)\diamondsuit_\alpha y\right)\cdot$$
$$\left(\int_c^d \left(\int_a^b \theta(x)\prod_{i=1}^3 k_i(x,y)\Phi_1(|f_1(y)|)\diamondsuit_\alpha x\right)\diamondsuit_\alpha y\right)$$
$$= \left(\prod_{i=2}^3 \int_c^d \Phi_i(|f_i(y)|)\diamondsuit_\alpha y\right)\cdot$$
$$\left(\int_c^d \Phi_1(|f_1(y)|)\left(\int_a^b \theta(x)\prod_{i=1}^3 k_i(x,y)\diamondsuit_\alpha x\right)\diamondsuit_\alpha y\right)$$

$$= \left(\prod_{i=2}^{3} \int_c^d \Phi_i \left(|f_i(y)| \right) \diamondsuit_\alpha y \right) \left(\int_c^d \Phi_1 \left(|f_1(y)| \right) \lambda_3(y) \diamondsuit_\alpha y \right),$$

proving the claim. □

The following is a corollary to Theorem 2.6.

Corollary 2.7. *It holds*

$$\int_a^b u(x) e^{\sum_{i=1}^m \frac{|g_i(x)|}{K_i(x)}} \diamondsuit_\alpha x \leq \left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_c^d e^{|f_i(y)|} \diamondsuit_\alpha y \right) \left(\int_c^d e^{|f_j(y)|} \lambda_m(y) \diamondsuit_\alpha y \right).$$

Proof. Apply $\Phi_i(x) = e^x$, $x \geq 0$, for all $i = 1, \ldots, m$. □

We continue with the next result.

Theorem 2.8. *All as in Theorem 2.6, but now $\Phi_i : (0, \infty) \to \mathbb{R}_+$, $i = 1, \ldots, m$, are convex and not necessarily increasing. Furthermore all f_i, $i = 1, \ldots, m$, are of fixed strict sign. Then* (2.16) *is valid.*

Proof. Similar to Theorem 2.2, and Theorem 2.6. □

We give the following application.

Corollary 2.9. *All as in Theorem 2.8, with $\Phi_i(x) = -\ln x$, $i = 1, \ldots, m \in \mathbb{N}$. It holds*

$$(-1)^m \int_a^b u(x) \prod_{i=1}^{m} \ln \left(\frac{|g_i(x)|}{K_i(x)} \right) \diamondsuit_\alpha x$$
$$\leq (-1)^m \left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_c^d \ln \left(|f_i(y)| \right) \diamondsuit_\alpha y \right) \left(\int_c^d \ln \left(|f_j(y)| \right) \lambda_m(y) \diamondsuit_\alpha y \right).$$

Proof. By (2.16). □

We continue with the next result.

Theorem 2.10. *All as in Notation 2.5. Define*

$$u_i(y) := \int_a^b u(x) \frac{k_i(x, y)}{K_i(x)} \diamondsuit_\alpha x, \tag{2.17}$$

$\forall\, y \in [c, d]_{\mathbb{T}_2}$, $i = 1, \ldots, m \in \mathbb{N}$. *Let* $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. *Then*

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_i \left(\frac{|g_i(x)|}{K_i(x)} \right) \diamondsuit_\alpha x \leq \prod_{i=1}^{m} \left(\int_c^d u_i(y) \Phi_i \left(|f_i(y)| \right)^{p_i} \diamondsuit_\alpha y \right)^{\frac{1}{p_i}}. \tag{2.18}$$

Proof. Notice that Φ_i, $i = 1, \ldots, m$, are continuous functions. Here we use the generalized Hölder's inequality, see Theorem 1.4. We have

$$\begin{aligned}\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{|g_i(x)|}{K_i(x)} \right) \diamondsuit_\alpha x &= \int_a^b \prod_{i=1}^m \left(u(x)^{\frac{1}{p_i}} \Phi_i \left(\frac{|g_i(x)|}{K_i(x)} \right) \right) \diamondsuit_\alpha x \\ &\leq \prod_{i=1}^m \left(\int_a^b u(x) \Phi_i \left(\frac{|g_i(x)|}{K_i(x)} \right)^{p_i} \diamondsuit_\alpha x \right)^{\frac{1}{p_i}} \\ &\leq \prod_{i=1}^m \left(\int_c^d u_i(y) \Phi_i \left(|f_i(y)|\right)^{p_i} \diamondsuit_\alpha y \right)^{\frac{1}{p_i}}\end{aligned}$$

(notice here that $\Phi_i^{p_i}$, $i = 1, \ldots, m$, are convex, increasing and continuous, non-negative functions, and apply Theorem 2.1). This proves the claim. □

We also give the following result.

Theorem 2.11. *All as in Theorem 2.10, but now $\Phi_i : (0, \infty) \to \mathbb{R}_+$, $i = 1, \ldots, m$, are convex and not necessarily increasing. Furthermore all f_i, $i = 1, \ldots, m$, are of fixed strict sign. Then* (2.18) *is valid.*

Proof. Similar to Theorem 2.10, and by using Theorem 2.2. □

The following two results are corollaries to Theorem 2.10.

Corollary 2.12. *Let $\alpha_i \geq 1$, $i = 1, \ldots, m$. Then*

$$\int_a^b u(x) \prod_{i=1}^m \left(\frac{|g_i(x)|}{K_i(x)} \right)^{\alpha_i} \diamondsuit_\alpha x \leq \prod_{i=1}^m \left(\int_c^d u_i(y) \left(|f_i(y)|\right)^{\alpha_i p_i} \diamondsuit_\alpha y \right)^{\frac{1}{p_i}}.$$

Proof. Apply (2.18) for $\Phi_i(x) = x^{\alpha_i}$, $x \geq 0$, $i = 1, \ldots, m$. □

Corollary 2.13. *It holds*

$$\int_a^b u(x) e^{\sum_{i=1}^m \frac{|g_i(x)|}{K_i(x)}} \diamondsuit_\alpha x \leq \prod_{i=1}^m \left(\int_c^d u_i(y) e^{p_i |f_i(y)|} \diamondsuit_\alpha y \right)^{\frac{1}{p_i}}.$$

Proof. Apply (2.18) for $\Phi_i(x) = e^x$, $x \geq 0$, for all $i = 1, \ldots, m$. □

We need the following definition.

Definition 2.14 (See [3])**.** Let $\mathbb{T}$ be a time scale. Consider the coordinate wise rd-continuous functions $h_\alpha : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$, such that $h_0(t, s) = 1$,

$$h_{\alpha+1}(t, s) = \int_s^t h_\alpha(\tau, s) \Delta\tau, \tag{2.19}$$

$\forall\, s,t \in \mathbb{T}$.

When $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, we define

$$h_\alpha(t,s) := \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}, \quad \alpha \geq 0. \tag{2.20}$$

When $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1$, $t \in \mathbb{Z}$, and

$$h_k(t,s) = \frac{(t-s)^{(k)}}{k!}, \ \ \forall\, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{2.21}$$

$\forall\, t,s \in \mathbb{Z}$, where $t^{(0)} = 1$, $t^{(k)} = \prod_{i=0}^{k-1}(t-i)$ for $k \in \mathbb{N}$. Also it holds

$$\int_a^b f(t)\,\Delta t = \sum_{t=a}^{b-1} f(t)\,,\ \ a < b;\ a,b \in \mathbb{Z}. \tag{2.22}$$

We need the following definition.

Definition 2.15 (See [3])**.** For $\alpha \geq 1$, we define the time scale Δ-Riemann–Liouville type fractional integral ($a,b \in \mathbb{T}$)

$$K_a^\alpha f(t) = \int_a^t h_{\alpha-1}(t,\sigma(\tau))\,f(\tau)\,\Delta\tau, \tag{2.23}$$

(by [10] is an integral on $[a,t) \cap \mathbb{T}$)

$$K_a^0 f = f,$$

where $f \in L_1([a,b) \cap \mathbb{T})$ (Lebesgue Δ-integrable functions on $[a,b) \cap \mathbb{T}$, see [8,9,15]), $t \in [a,b] \cap \mathbb{T}$. Notice $K_a^1 f(t) = \int_a^t f(\tau)\,\Delta\tau$ is absolutely continuous in $t \in [a,b] \cap \mathbb{T}$, see [10].

Lemma 2.16 (See [3])**.** *Let* $\alpha > 1$, $f \in L_1([a,b) \cap \mathbb{T})$. *If additionally* $h_{\alpha-1}(s,\sigma(t))$ *is Lebesgue* Δ*-measurable on* $([a,b) \cap \mathbb{T})^2$, *then* $K_a^\alpha f \in L_1([a,b) \cap \mathbb{T})$.

We need the following definition.

Definition 2.17 (See [3])**.** Assume $\mathbb{T}$ time scale such that $\mathbb{T}^k = \mathbb{T}$. Let $\mu > 2 : m-1 < \mu < m \in \mathbb{N}$, i.e. $m = \lceil \mu \rceil$ (ceiling of the number), $\widetilde{\nu} = m - \mu$ ($0 < \widetilde{\nu} < 1$). Here we take $f \in C_{rd}^m([a,b] \cap \mathbb{T})$. Clearly here (see [15]) f^{Δ^m} is a Lebesgue Δ-integrable function. Assume $h_{\widetilde{\nu}}(s,\sigma(t))$ is continuous on $([a,b] \cap \mathbb{T})^2$. We define the delta fractional derivative on time scale $\mathbb{T}$ of order $\mu - 1$ as follows:

$$\Delta_{a*}^{\mu-1} f(t) = \left(K_a^{\widetilde{\nu}+1} f^{\Delta^m}\right)(t) = \int_a^t h_{\widetilde{\nu}}(t,\sigma(\tau))\,f^{\Delta^m}(\tau)\,\Delta\tau, \tag{2.24}$$

$\forall\ t \in [a,b] \cap \mathbb{T}$. Notice here that $\Delta_{a*}^{\mu-1} f \in C([a,b] \cap \mathbb{T})$ by a simple argument using dominated convergence theorem in Lebesgue Δ-sense. If $\mu = m$, then $\widetilde{\nu} = 0$ and by (2.24), we get

$$\Delta_{a*}^{m-1} f(t) = K_a^1 f^{\Delta^m}(t) = f^{\Delta^{m-1}}(t). \tag{2.25}$$

More generally, by [10], given that $f^{\Delta^{m-1}}$ is everywhere finite and absolutely continuous on $[a,b] \cap \mathbb{T}$, then f^{Δ^m} exists Δ-a.e. and is Lebesgue Δ-integrable on $[a,t) \cap \mathbb{T}$, $\forall$ $t \in [a,b] \cap \mathbb{T}$ and one can plug it into (2.24).

We need the following definition.

Definition 2.18 (See [4])**.** Consider the coordinate wise ld-continuous functions $\widehat{h}_\alpha$: $\mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$, such that $\widehat{h}_0(t,s) = 1$,

$$\widehat{h}_{\alpha+1}(t,s) = \int_s^t \widehat{h}_\alpha(\tau,s)\,\nabla\tau, \tag{2.26}$$

$\forall\ s,t \in \mathbb{T}$.

In the case of $\mathbb{T} = \mathbb{R}$; then $\rho(t) = t$, and $\widehat{h}_k(t,s) = \frac{(t-s)^k}{k!}$, $k \in \mathbb{N}_0$, and define

$$\widehat{h}_\alpha(t,s) := \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}, \quad \alpha \geq 0. \tag{2.27}$$

Let $T = \mathbb{Z}$, then $\rho(t) = t-1$, $t \in \mathbb{Z}$. Define $t^{\overline{0}} := 1$, $t^{\overline{k}} := t(t+1)\cdots(t+k-1)$, $k \in \mathbb{N}$, and by (2.26), we have $\widehat{h}_k(t,s) = \frac{(t-s)^{\overline{k}}}{k!}$, $s,t \in \mathbb{Z}$, $k \in \mathbb{N}_0$. Here $\int_{t_0}^t \nabla\tau = \sum_{t_0+1}^{t}$.

We need the following definition.

Definition 2.19 (See [4])**.** For $\alpha \geq 1$, we define the time scale ∇-Riemann–Liouville type fractional integral ($a,b \in \mathbb{T}$)

$$J_a^\alpha f(t) = \int_a^t \widehat{h}_{\alpha-1}(t,\rho(\tau))\,f(\tau)\,\nabla\tau, \tag{2.28}$$

(by [10] the last integral is on $(a,t] \cap \mathbb{T}$)

$$J_a^0 f(t) = f(t),$$

where $f \in L_1((a,b] \cap \mathbb{T})$ (Lebesgue ∇-integrable functions on $(a,b] \cap \mathbb{T}$, see [8,9,15]), $t \in [a,b] \cap \mathbb{T}$. Notice $J_a^1 f(t) = \int_a^t f(\tau)\,\nabla\tau$ is absolutely continuous in $t \in [a,b] \cap \mathbb{T}$, see [10].

Lemma 2.20 (See [4]). *Let* $\alpha > 1$, $f \in L_1((a,b] \cap \mathbb{T})$. *If additionally* $\widehat{h}_{\alpha-1}(s, \rho(t))$ *is Lebesgue* ∇*-measurable on* $((a,b] \cap \mathbb{T})^2$, *then* $J_a^\alpha f \in L_1((a,b] \cap \mathbb{T})$.

We also need the following definition.

Definition 2.21 (See [4]). Assume $\mathbb{T}_k = \mathbb{T}$. Let $\mu > 2$ such that $m - 1 < \mu < m \in \mathbb{N}$, i.e. $m = \lceil \mu \rceil$, $\widetilde{\nu} = m - \mu$ $(0 < \widetilde{\nu} < 1)$. Let $f \in C_{ld}^m([a,b] \cap \mathbb{T})$. Clearly here (see [15]) f^{∇^m} is a Lebesgue ∇-integrable function. Assume $\widehat{h}_{\widetilde{\nu}}(s, \rho(t))$ is continuous on $([a,b] \cap \mathbb{T})^2$. We define the nabla fractional derivative on time scale $\mathbb{T}$ of order $\mu - 1$ as follows:

$$\nabla_{a*}^{\mu-1} f(t) = \left(J_a^{\widetilde{\nu}+1} f^{\nabla^m}\right)(t) = \int_a^t \widehat{h}_{\widetilde{\nu}}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla\tau, \tag{2.29}$$

$\forall\ t \in [a,b] \cap \mathbb{T}$. Notice here that $\nabla_{a*}^{\mu-1} f \in C([a,b] \cap \mathbb{T})$ by a simple argument using dominated convergence theorem in Lebesgue ∇-sense.

If $\mu = m$, then $\widetilde{\nu} = 0$ and by (2.29), we get

$$\nabla_{a*}^{m-1} f(t) = J_a^1 f^{\nabla^m}(t) = f^{\nabla^{m-1}}(t). \tag{2.30}$$

More generally, by [10], given that $f^{\nabla^{m-1}}$ is everywhere finite and absolutely continuous on $[a,b] \cap \mathbb{T}$, then f^{∇^m} exists ∇-a.e. and is Lebesgue ∇-integrable on $(a,t] \cap \mathbb{T}$, $\forall$ $t \in [a,b] \cap \mathbb{T}$, and one can plug it into (2.29).

We present the next result.

Theorem 2.22. *Let* $\mathbb{T}$ *be a time scale, and* $a, b \in \mathbb{T}$, $a < b$, *with* $\sigma(a) = a$. *Let* $\alpha \geq 1$, h_α *as in* (2.19), *and* $K_a^\alpha f$ *as in* (2.23), *where* $f \in L_1([a,b) \cap \mathbb{T})$. *Assume further that* $h_{\alpha-1}(s, \sigma(t))$ *is Lebesgue* Δ*-measurable on* $([a,b) \cap \mathbb{T})^2$. *Call*

$$K^*(x) := \int_a^b \chi_{[a,x)}(y) \left|h_{\alpha-1}(x, \sigma(y))\right| \Delta y \tag{2.31}$$

$$= \int_a^x \left|h_{\alpha-1}(x, \sigma(y))\right| \Delta y,$$

$\forall\ x \in ([a,b] \cap \mathbb{T})$, *where* $\chi_{[a,x)}(y)$ *is the characteristic function on* $[a,x) \cap \mathbb{T}$. *Assume that* $K^*(x) > 0$ *(delta) Lebesgue measure* Δ*-a.e. in* $x \in ([a,b] \cap \mathbb{T})$. *Consider also the weight function*

$$u : ([a,b) \cap \mathbb{T}) \to \mathbb{R}_+, \tag{2.32}$$

which is (delta) Lebesgue Δ*-measurable. Assume that the function*

$$x \to \frac{u(x)}{K^*(x)} \chi_{[a,x)}(y) \left|h_{\alpha-1}(x, \sigma(y))\right|$$

is Δ-integrable on $([a,b)\cap\mathbb{T})$ for each fixed $y\in([a,b)\cap\mathbb{T})$. Define v^* on $([a,b)\cap\mathbb{T})$ by

$$v^*(y) := \int_a^b \frac{u(x)}{K^*(x)}\chi_{[a,x)}(y)\left|h_{\alpha-1}(x,\sigma(y))\right|\Delta x < \infty. \tag{2.33}$$

Let $\Phi:\mathbb{R}_+\to\mathbb{R}$ be a convex and increasing function. Then

$$\int_a^b u(x)\,\Phi\left(\frac{|K_a^\alpha f(x)|}{K^*(x)}\right)\Delta x \le \int_a^b v^*(x)\,\Phi(|f(x)|)\,\Delta x, \tag{2.34}$$

under the further assumptions:

(i) $f,\Phi(|f|)$ are $\chi_{[a,x)}(y)|h_{\alpha-1}(x,\sigma(y))|\Delta y$-integrable, Δ-a.e. in $x\in([a,b)\cap\mathbb{T})$,

(ii) $v^*\Phi(|f|)$ is Δ-Lebesgue integrable.

Proof. By [6], Φ is continuous on $\mathbb{R}_+$. Here we use Jensen's inequality, Tonelli's theorem and Fubini's theorem, all on time scales setting. Also we use that Φ is convex and increasing. We extend $\Phi(x) := \Phi(-x)$, $x<0$, then both branches of Φ make a convex function on $\mathbb{R}$ and we can apply Jensen's inequality. Next we have

$$K_a^\alpha f(x) = \int_a^b \chi_{[a,x)}(y)\,h_{\alpha-1}(x,\sigma(y))\,f(y)\,\Delta y, \quad \forall\, x\in[a,b]\cap\mathbb{T},$$

and

$$|K_a^\alpha f(x)| \le \int_a^b \chi_{[a,x)}(y)\left|h_{\alpha-1}(x,\sigma(y))\right||f(y)|\,\Delta y.$$

By Jensen's inequality and Fubini's theorem, we see that

$$\begin{aligned}
&\int_a^b u(x)\,\Phi\left(\frac{|K_a^\alpha f(x)|}{K^*(x)}\right)\Delta x\\
\le\ & \int_a^b u(x)\,\Phi\left(\frac{1}{K_*(x)}\int_a^b \chi_{[a,x)}(y)\left|h_{\alpha-1}(x,\sigma(y))\right||f(y)|\,\Delta y\right)\Delta x\\
\le\ & \int_a^b \frac{u(x)}{K^*(x)}\left(\int_a^b \chi_{[a,x)}(y)\left|h_{\alpha-1}(x,\sigma(y))\right|\Phi(|f(y)|)\,\Delta y\right)\Delta x\\
=\ & \int_a^b \left(\int_a^b \frac{u(x)}{K^*(x)}\chi_{[a,x)}(y)\left|h_{\alpha-1}(x,\sigma(y))\right|\Phi(|f(y)|)\,\Delta y\right)\Delta x\\
=\ & \int_a^b \left(\int_a^b \frac{u(x)}{K^*(x)}\chi_{[a,x)}(y)\left|h_{\alpha-1}(x,\sigma(y))\right|\Phi(|f(y)|)\,\Delta x\right)\Delta y\\
=\ & \int_a^b \Phi(|f(y)|)\left(\int_a^b \frac{u(x)}{K^*(x)}\chi_{[a,x)}(y)\left|h_{\alpha-1}(x,\sigma(y))\right|\Delta x\right)\Delta y\\
=\ & \int_a^b \Phi(|f(y)|)\,v^*(y)\,\Delta y,
\end{aligned}$$

completing the proof of the theorem. □

The counterpart of the last result follows.

Theorem 2.23. *Let $\mathbb{T}$ be a time scale, and $a, b \in \mathbb{T} - \{\min \mathbb{T}\}$, $a < b$, with $\rho(a) = a$. Let $\alpha \geq 1$, $\widehat{h}_\alpha$ as in* (2.26)*, and $J_a^\alpha f$ as in* (2.28)*, where $f \in L_1((a, b] \cap \mathbb{T})$. Assume further that $\widehat{h}_{\alpha-1}(s, \rho(t))$ is Lebesgue ∇-measurable on $((a, b] \cap \mathbb{T})^2$. Call*

$$K_*(x) := \int_a^b \chi_{[a,x)}(y) \left| \widehat{h}_{\alpha-1}(x, \rho(y)) \right| \nabla y \tag{2.35}$$

$$= \int_a^x \left| \widehat{h}_{\alpha-1}(x, \rho(y)) \right| \nabla y,$$

$\forall\, x \in ([a, b] \cap \mathbb{T})$. Assume that $K_*(x) > 0$ (nabla) Lebesgue measure ∇-a.e. in $x \in ([a, b] \cap \mathbb{T})$. Consider also the weight function

$$w : ((a, b] \cap \mathbb{T}) \to \mathbb{R}_+, \tag{2.36}$$

which is (nabla) Lebesgue ∇-measurable. Assume that the function

$$x \to \frac{w(x)}{K_*(x)} \chi_{[a,x)}(y) \left| \widehat{h}_{\alpha-1}(x, \rho(y)) \right|$$

is ∇-integrable on $((a, b] \cap \mathbb{T})$ for each fixed $y \in ((a, b] \cap \mathbb{T})$. Define v_* on $((a, b] \cap \mathbb{T})$ by

$$v_*(y) := \int_a^b \frac{w(x)}{K_*(x)} \chi_{[a,x)}(y) \left| \widehat{h}_{\alpha-1}(x, \rho(y)) \right| \nabla x < \infty. \tag{2.37}$$

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be a convex and increasing function. Then

$$\int_a^b w(x) \Phi\left(\frac{|J_a^\alpha f(x)|}{K_*(x)} \right) \nabla x \leq \int_a^b v_*(x) \Phi(|f(x)|) \nabla x, \tag{2.38}$$

under the further assumptions:

(i) $f, \Phi(|f|)$ are $\chi_{[a,x)}(y) |\widehat{h}_{\alpha-1}(x, \rho(y))| \nabla y$-integrable, ∇-a.e. in $x \in ((a, b] \cap \mathbb{T})$,

(ii) $v_* \Phi(|f|)$ is ∇-Lebesgue integrable.

Proof. Similar to the proof of Theorem 2.22. □

We continue with the next result.

Theorem 2.24. *Let $\mathbb{T}$ be a time scale, and $a, b \in \mathbb{T}$, $a < b$, with $\sigma(a) = a$. Let $\alpha \geq 1$, h_α as in* (2.19)*, and $K_a^\alpha f_i$ as in* (2.23)*, where $f_i \in L_1([a, b) \cap \mathbb{T})$, $i = 1, \ldots, m \in \mathbb{N}$. Assume further that $h_{\alpha-1}(s, \sigma(t))$ is Lebesgue Δ-measurable on $([a, b) \cap \mathbb{T})^2$. Call*

$$K^*(x) := \int_a^b \chi_{[a,x)}(y) \left| h_{\alpha-1}(x, \sigma(y)) \right| \Delta y \tag{2.39}$$

$\forall\, x \in ([a,b] \cap \mathbb{T})$. *Assume that* $K^*(x) > 0$, Δ*-a.e. in* $x \in ([a,b] \cap \mathbb{T})$. *Here the weight function*

$$u : ([a,b) \cap \mathbb{T}) \to \mathbb{R}_+,$$

is Lebesgue Δ*-measurable. Assume that the function*

$$x \to u(x)\, \chi_{[a,x)}(y) \left(\frac{|h_{\alpha-1}(x, \sigma(y))|}{K^*(x)} \right)^m$$

is Lebesgue Δ*-integrable on* $([a,b) \cap \mathbb{T})$ *for each fixed* $y \in ([a,b) \cap \mathbb{T})$. *Define* v_m^* *on* $([a,b) \cap \mathbb{T})$ *by*

$$v_m^*(y) := \int_a^b u(x)\, \chi_{[a,x)}(y) \left(\frac{|h_{\alpha-1}(x, \sigma(y))|}{K^*(x)} \right)^m \Delta x < \infty. \tag{2.40}$$

Let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, *are convex and increasing functions. Then*

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{|K_a^\alpha f_i(x)|}{K^*(x)} \right) \Delta x \\ \leq \left(\prod_{i=2}^m \int_a^b \Phi_i(|f_i(y)|)\, \Delta y \right) \left(\int_a^b \Phi_1(|f_1(y)|)\, v_m^*(y)\, \Delta y \right),$$

under the further assumptions:

(i) $f_i, \Phi_i(|f_i|)$ *are* $\chi_{[a,x)}(y)|h_{\alpha-1}(x,\sigma(y))|\Delta y$*-integrable,* Δ*-a.e. in* $x \in ([a,b) \cap \mathbb{T})$, $i = 1, \ldots, m$,

(ii) $v_m^* \Phi_1(|f_1|)$, $\Phi_2(|f_2|)$, $\Phi_3(|f_3|), \ldots, \Phi_m(|f_m|)$, *are all* Δ*-Lebesgue integrable.*

Proof. As in [7], and Theorem 2.6 here. □

The counterpart of the last result follows.

Theorem 2.25. *Let* $\mathbb{T}$ *be a time scale, and* $a, b \in \mathbb{T} - \{\min \mathbb{T}\}$, $a < b$, *with* $\rho(a) = a$. *Let* $\alpha \geq 1$, $\widehat{h}_\alpha$ *as in* (2.26)*, and* $J_a^\alpha f_i$ *as in (2.28), where* $f_i \in L_1((a,b] \cap \mathbb{T})$, $i = 1, \ldots, m \in \mathbb{N}$. *Assume further that* $\widehat{h}_{\alpha-1}(s, \rho(t))$ *is Lebesgue* ∇*-measurable on* $((a,b] \cap \mathbb{T})^2$. *Call*

$$K_*(x) := \int_a^b \chi_{[a,x)}(y) \left| \widehat{h}_{\alpha-1}(x, \rho(y)) \right| \nabla y \tag{2.41}$$

$\forall\, x \in ([a,b] \cap \mathbb{T})$. *Assume that* $K_*(x) > 0$, ∇*-a.e. in* $x \in ([a,b] \cap \mathbb{T})$. *Here the weight function*

$$w : ((a,b] \cap \mathbb{T}) \to \mathbb{R}_+,$$

is Lebesgue ∇-measurable. Assume that the function

$$x \to w(x)\,\chi_{[a,x)}(y)\left(\frac{\left|\widehat{h}_{\alpha-1}(x,\rho(y))\right|}{K_*(x)}\right)^m$$

is Lebesgue ∇-integrable on $((a,b]\cap\mathbb{T})$ for each fixed $y \in ((a,b]\cap\mathbb{T})$. Define v_{m*} on $((a,b]\cap\mathbb{T})$ by

$$v_{m*}(y) := \int_a^b w(x)\,\chi_{[a,x)}(y)\left(\frac{\left|\widehat{h}_{\alpha-1}(x,\rho(y))\right|}{K_*(x)}\right)^m \nabla x < \infty. \tag{2.42}$$

Let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,\ldots,m$, are convex and increasing functions. Then

$$\int_a^b w(x)\prod_{i=1}^m \Phi_i\left(\frac{\left|J_a^\alpha f_i(x)\right|}{K_*(x)}\right)\nabla x$$
$$\leq \left(\prod_{i=2}^m \int_a^b \Phi_i(|f_i(y)|)\,\nabla y\right)\left(\int_a^b \Phi_1(|f_1(y)|)\,v_{m*}(y)\,\nabla y\right),$$

under the further assumptions:

(i) $f_i, \Phi_i(|f_i|)$ *are* $\chi_{[a,x)}(y)|\widehat{h}_{\alpha-1}(x,\rho(y))|\nabla y$*-integrable,* ∇*-a.e. in* $x \in ((a,b]\cap\mathbb{T})$*,* $i = 1,\ldots,m$*,*

(ii) $v_{m*}\Phi_1(|f_1|)$*,* $\Phi_2(|f_2|)$*,* $\Phi_3(|f_3|),\ldots,\Phi_m(|f_m|)$*, are all* ∇*-Lebesgue integrable.*

Proof. As in [7], and Theorem 2.6 here. □

We continue with the following result.

Theorem 2.26. *Let $\mathbb{T}$ be a time scale, and $a,b \in \mathbb{T}$, $a < b$, with $\sigma(a) = a$. Let $\alpha \geq 1$, h_α as in (2.19), and $K_a^\alpha f_i$ as in (2.23), where $f_i \in L_1([a,b)\cap\mathbb{T})$, $i = 1,\ldots,m \in \mathbb{N}$. Assume further that $h_{\alpha-1}(s,\sigma(t))$ is Lebesgue Δ-measurable on $([a,b)\cap\mathbb{T})^2$. Call*

$$K^*(x) := \int_a^b \chi_{[a,x)}(y)\left|h_{\alpha-1}(x,\sigma(y))\right|\Delta y \tag{2.43}$$

$\forall\, x \in ([a,b]\cap\mathbb{T})$. Assume that $K^*(x) > 0$, Δ-a.e. in $x \in ([a,b]\cap\mathbb{T})$. Here the weight function

$$u : ([a,b)\cap\mathbb{T}) \to \mathbb{R}_+,$$

is Lebesgue Δ-measurable. Assume that the function

$$x \to u(x)\,\chi_{[a,x)}(y)\left(\frac{\left|h_{\alpha-1}(x,\sigma(y))\right|}{K^*(x)}\right)$$

is Lebesgue Δ*-integrable on* $([a,b)\cap\mathbb{T})$ *for each fixed* $y\in([a,b)\cap\mathbb{T})$. *Define* v^* *on* $([a,b)\cap\mathbb{T})$ *by*

$$v^*(y) := \int_a^b \frac{u(x)}{K^*(x)}\chi_{[a,x)}(y)\left|h_{\alpha-1}(x,\sigma(y))\right|\Delta x < \infty. \tag{2.44}$$

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. *Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,\ldots,m$, *be convex and increasing. Then*

$$\int_a^b u(x)\prod_{i=1}^{m}\Phi_i\left(\frac{|K_a^{\alpha}f_i(x)|}{K^*(x)}\right)\Delta x \le \left(\prod_{i=1}^{m}\int_a^b v^*(y)\,\Phi_i(|f_i(y)|)^{p_i}\,\Delta y\right)^{\frac{1}{p_i}},$$

under the further assumptions:

(i) $f_i, \Phi_i(|f_i|)^{p_i}$ *are* $\chi_{[a,x)}(y)|h_{\alpha-1}(x,\sigma(y))|\Delta y$*-integrable,* Δ*-a.e. in* $x\in([a,b)\cap\mathbb{T})$, *for all* $i=1,\ldots,m$,

(ii) $v^*\Phi_i(|f_i|)^{p_i}$ *is* Δ*-Lebesgue integrable,* $i=1,\ldots,m$.

Proof. As in [5], and Theorem 2.10 here. □

The counterpart of the last result follows.

Theorem 2.27. *Let* $\mathbb{T}$ *be a time scale, and* $a,b\in\mathbb{T}-\{\min\mathbb{T}\}$, $a<b$, *with* $\rho(a)=a$. *Let* $\alpha\ge 1$, $\widehat{h}_{\alpha}$ *as in* (2.26)*, and* $J_a^{\alpha}f_i$ *as in (2.28), where* $f_i\in L_1((a,b]\cap\mathbb{T})$, $i=1,\ldots,m\in\mathbb{N}$. *Assume further that* $\widehat{h}_{\alpha-1}(s,\rho(t))$ *is Lebesgue* ∇*-measurable on* $((a,b]\cap\mathbb{T})^2$. *Call*

$$K_*(x) := \int_a^b \chi_{[a,x)}(y)\left|\widehat{h}_{\alpha-1}(x,\rho(y))\right|\nabla y, \tag{2.45}$$

$\forall\, x\in([a,b]\cap\mathbb{T})$. *Assume that* $K_*(x)>0$, ∇*-a.e. in* $x\in([a,b]\cap\mathbb{T})$. *Here the weight function*

$$w : ((a,b]\cap\mathbb{T})\to\mathbb{R}_+,$$

is Lebesgue ∇*-measurable. Assume that the function*

$$x\to w(x)\,\chi_{[a,x)}(y)\left(\frac{\left|\widehat{h}_{\alpha-1}(x,\rho(y))\right|}{K_*(x)}\right)$$

is Lebesgue ∇*-integrable on* $((a,b]\cap\mathbb{T})$ *for each fixed* $y\in((a,b]\cap\mathbb{T})$. *Define* v_* *on* $((a,b]\cap\mathbb{T})$ *by*

$$v_*(y) := \int_a^b \frac{w(x)}{K_*(x)}\chi_{[a,x)}(y)\left|\widehat{h}_{\alpha-1}(x,\rho(y))\right|\nabla x < \infty. \tag{2.46}$$

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. *Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, *be convex and increasing. Then*

$$\int_a^b w(x) \prod_{i=1}^{m} \Phi_i \left(\frac{|J_a^{\alpha} f_i(x)|}{K_*(x)} \right) \nabla x \le \left(\prod_{i=1}^{m} \int_a^b v_*(y) \Phi_i \left(|f_i(y)| \right)^{p_i} \nabla y \right)^{\frac{1}{p_i}},$$

under the further assumptions:

(i) $f_i, \Phi_i(|f_i|)^{p_i}$ *are* $\chi_{[a,x)}(y)|\widehat{h}_{\alpha-1}(x, \rho(y))|\nabla y$*-integrable,* ∇*-a.e. in* $x \in ((a, b] \cap \mathbb{T})$*, for all* $i = 1, \ldots, m$*,*

(ii) $v_* \Phi_i (|f_i|)^{p_i}$ *is* ∇*-Lebesgue integrable,* $i = 1, \ldots, m$.

Proof. As in [5], and Theorem 2.10 here. □

We give the following corollary to Theorem 2.24.

Corollary 2.28. *It holds*

$$\int_a^b u(x) e^{\sum_{i=1}^{m} \left(\frac{|K_a^{\alpha} f_i(x)|}{K^*(x)} \right)} \Delta x \le \left(\prod_{i=2}^{m} \int_a^b e^{|f_i(y)|} \Delta y \right) \left(\int_a^b e^{|f_1(y)|} v_m^*(y) \Delta y \right),$$

under the assumptions:

(i) $f_i, e^{|f_i|}$ *are* $\chi_{[a,x)}(y) |h_{\alpha-1}(x, \sigma(y))| \Delta y$*-integrable,* Δ*-a.e. in* $x \in ([a, b) \cap \mathbb{T})$*,* $i = 1, \ldots, m$*, and*

(ii) $v_m^* e^{|f_1|}$, $e^{|f_2|}$, $e^{|f_3|}, \ldots, e^{|f_m|}$*, are all* Δ*-integrable.*

We give the following corollary to Theorem 2.27.

Corollary 2.29. *It holds*

$$\int_a^b w(x) e^{\sum_{i=1}^{m} \left(\frac{|J_a^{\alpha} f_i(x)|}{K_*(x)} \right)} \nabla x \le \left(\prod_{i=1}^{m} \int_a^b v_*(y) e^{p_i |f_i(y)|} \nabla y \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $f_i, e^{p_i|f_i|}$ *are both* $\chi_{[a,x)}(y) \left| \widehat{h}_{\alpha-1}(x, \rho(y)) \right| \nabla y$*-integrable,* ∇*-a.e. in* $x \in ((a, b] \cap \mathbb{T})$*, for all* $i = 1, \ldots, m$*, and*

(ii) $v_* e^{p_i|f_i|}$ *is* ∇*-Lebesgue integrable,* $i = 1, \ldots, m$.

We continue with the following result.

Theorem 2.30. *Let $\mathbb{T}$ be a time scale, and $a, b \in \mathbb{T}$, $a < b$, with $\sigma(a) = a$. Let all as in Definition 2.17, with $f_i \in C_{rd}^m([a,b] \cap \mathbb{T})$, $i = 1, \dots, m_* \in \mathbb{N}$; $h_{\widetilde{\nu}}$ as in* (2.19). *Call*

$$K_1(x) := \int_a^b \chi_{[a,x)}(y) \left| h_{\widetilde{\nu}}(x, \sigma(y)) \right| \Delta y \tag{2.47}$$

$\forall\, x \in ([a,b] \cap \mathbb{T})$. *Assume that $K_1(x) > 0$, Δ-a.e. in $x \in ([a,b] \cap \mathbb{T})$. Here the weight function*

$$u : ([a,b) \cap \mathbb{T}) \to \mathbb{R}_+,$$

is Lebesgue Δ-measurable. Assume that the function

$$x \to u(x)\, \chi_{[a,x)}(y) \left(\frac{\left| h_{\widetilde{\nu}}(x, \sigma(y)) \right|}{K_1(x)} \right)^{m_*}$$

is Lebesgue Δ-integrable on $([a,b) \cap \mathbb{T})$ for each fixed $y \in ([a,b) \cap \mathbb{T})$. Define φ_{m_*} on $([a,b) \cap \mathbb{T})$ by

$$\varphi_{m_*}(y) := \int_a^b u(x)\, \chi_{[a,x)}(y) \left(\frac{\left| h_{\widetilde{\nu}}(x, \sigma(y)) \right|}{K_1(x)} \right)^{m_*} \Delta x < \infty. \tag{2.48}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \dots, m_*$, are convex and increasing functions. Then

$$\int_a^b u(x) \prod_{i=1}^{m_*} \Phi_i \left(\frac{\left| \Delta_{a*}^{\mu-1} f_i(x) \right|}{K_1(x)} \right) \Delta x$$
$$\leq \left(\prod_{i=2}^{m_*} \int_a^b \Phi_i \left(\left| f_i^{\Delta^m}(y) \right| \right) \Delta y \right) \left(\int_a^b \Phi_1 \left(\left| f_1^{\Delta^m}(y) \right| \right) \varphi_{m_*}(y)\, \Delta y \right),$$

under the assumption that φ_{m_*} is Δ-Lebesgue integrable on $([a,b) \cap \mathbb{T})$.

Proof. By Theorem 2.24. □

We also derive the next result.

Theorem 2.31. *Let $\mathbb{T}$ be a time scale, and $a, b \in \mathbb{T} - \{\min \mathbb{T}\}$, $a < b$, with $\rho(a) = a$. Let all as in Definition 2.21, with $f_i \in C_{ld}^m([a,b] \cap \mathbb{T})$, $i = 1, \dots, m_* \in \mathbb{N}$; $h_{\widetilde{\nu}}$ as in (2.26). Call*

$$K_2(x) := \int_a^b \chi_{[a,x)}(y) \left| \widehat{h}_{\widetilde{\nu}}(x, \rho(y)) \right| \nabla y, \tag{2.49}$$

$\forall\, x \in ([a,b] \cap \mathbb{T})$. *Assume that $K_2(x) > 0$, ∇-a.e. in $x \in ([a,b] \cap \mathbb{T})$. Here the weight function*

$$w : ((a,b] \cap \mathbb{T}) \to \mathbb{R}_+,$$

is Lebesgue ∇-measurable. Assume that the function

$$x \to w(x)\,\chi_{[a,x)}(y)\left(\frac{\left|\widehat{h}_{\widetilde{\nu}}(x,\rho(y))\right|}{K_2(x)}\right)$$

is Lebesgue ∇-integrable on $((a,b]\cap\mathbb{T})$ for each fixed $y \in ((a,b]\cap\mathbb{T})$. Define ψ on $((a,b]\cap\mathbb{T})$ by

$$\psi(y) := \int_a^b \frac{w(x)}{K_2(x)}\chi_{[a,x)}(y)\left|\widehat{h}_{\widetilde{\nu}}(x,\rho(y))\right|\nabla x < \infty. \tag{2.50}$$

Let $p_i > 1 : \sum_{i=1}^{m_*}\frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,\ldots,m_*$, be convex and increasing. Then

$$\int_a^b w(x)\prod_{i=1}^m \Phi_i\left(\frac{\left|\nabla_a^{\mu-1}f_i(x)\right|}{K_2(x)}\right)\nabla x \leq \left(\prod_{i=1}^m \int_a^b \psi(y)\,\Phi_i\left(\left|f_i^{\nabla^m}(y)\right|\right)^{p_i}\nabla y\right)^{\frac{1}{p_i}},$$

under the assumption that ψ is ∇-Lebesgue integrable on $((a,b]\cap\mathbb{T})$.

Proof. By Theorem 2.27. □

We finish with the following remark concerning Theorem 2.1.

Remark 2.32. (i) Let $\mathbb{T}_1 = \mathbb{R}$, $\mathbb{T}_2 = \mathbb{Z}$; $0 \leq \alpha \leq 1$. Then

$$\int_a^b \cdot\Diamond_\alpha x = \int_a^b \cdot dx, \tag{2.51}$$

and

$$\int_c^d \cdot\Diamond_\alpha y = \alpha\sum_{y=c}^{d-1}\cdot + (1-\alpha)\sum_{y=c+1}^{d}\cdot. \tag{2.52}$$

Assume $k : [a,b]\times[c,d]_{\mathbb{Z}} \to \mathbb{R}_+$, a continuous function. So here

$$K(x) = \alpha\sum_{y=c}^{d-1}k(x,y) + (1-\alpha)\sum_{y=c+1}^{d}k(x,y) > 0, \tag{2.53}$$

$\forall\, x \in [a,b]$, and $f : [c,d]_{\mathbb{Z}} \to \mathbb{R}$, with

$$g(x) = \alpha\sum_{y=c}^{d-1}k(x,y)\,f(y) + (1-\alpha)\sum_{y=c+1}^{d}k(x,y)\,f(y), \tag{2.54}$$

$\forall\, x \in [a,b]$. Here $u : [a,b] \to \mathbb{R}_+$ continuous, and

$$v(y) = \int_a^b \frac{u(x)\,k(x,y)}{K(x)} dx, \tag{2.55}$$

$\forall\, y \in [c,d]_{\mathbb{Z}}$. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ convex and increasing function. Then, by (2.6), we obtain

$$\int_a^b u(x)\,\Phi\left(\frac{|g(x)|}{K(x)}\right) dx \\ \leq \alpha \sum_{y=c}^{d-1} v(y)\,\Phi(|f(y)|) + (1-\alpha) \sum_{y=c+1}^{d} v(y)\,\Phi(|f(y)|).$$

(ii) Let $\mathbb{T}_1 = \mathbb{Z}$, $\mathbb{T}_2 = \mathbb{R}$. Then

$$\int_a^b \cdot \diamondsuit_\alpha x = \alpha \sum_{x=a}^{b-1} \cdot + (1-\alpha) \sum_{x=a+1}^{b} \cdot, \tag{2.56}$$

and

$$\int_c^d \cdot \diamondsuit_\alpha y = \int_c^d \cdot dy. \tag{2.57}$$

Assume $k : [a,b]_{\mathbb{Z}} \times [c,d] \to \mathbb{R}_+$, a continuous function. So here

$$K(x) = \int_c^d k(x,y)\,dy > 0, \ \ \forall x \in [a,b]_{\mathbb{Z}}. \tag{2.58}$$

Consider $f : [c,d] \to \mathbb{R}$ continuous and

$$g(x) = \int_c^d k(x,y)\,f(y)\,dy, \ \ \forall x \in [a,b]_{\mathbb{Z}}. \tag{2.59}$$

Let $u : [a,b]_{\mathbb{Z}} \to \mathbb{R}_+$. Here it is

$$v(y) = \alpha \sum_{x=a}^{b-1} \frac{u(x)\,k(x,y)}{K(x)} + (1-\alpha) \sum_{x=a+1}^{b} \frac{u(x)\,k(x,y)}{K(x)}, \tag{2.60}$$

$\forall\, y \in [c,d]$. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ convex and increasing function. Then, by (2.6), we derive

$$\alpha \sum_{x=a}^{b-1} u(x)\,\Phi\left(\frac{|g(x)|}{K(x)}\right) + (1-\alpha) \sum_{x=a+1}^{b} u(x)\,\Phi\left(\frac{|g(x)|}{K(x)}\right) \\ \leq \int_c^d v(y)\,\Phi(|f(y)|)\,dy.$$

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