

## Solution of Certain Fractional Differential Equations by Kamal Transform

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### Abstract

In this paper we investigated the solution of certain fractional differential equations and some applications involving Mittag-Leffler function and G-function. The solutions of these fractional differential equations are obtained by the method of Kamal transform. Kamal transform is a very powerful mathematical tool. Some known results are also obtained in special cases.

**Keywords:** Fractional differential equation, Mittag-Leffler function, Generalized G and R-function, Kamal transform, inverse Kamal transform.

### INTRODUCTION

The fractional kinetic equations play important role in the field of applied science and engineering. In this paper, we propose to investigate solution of a certain generalized fractional kinetic equation associated with the generalized Mittag-Leffler function [3]. The fundamental equation [1].

$$\frac{d}{dt}p(t) = -kp(0) \quad \dots \quad (1)$$

Consider an arbitrary reaction characterized by a time dependent quantity  $p(t)$ :  $p(t) = p(0) \exp\{-kt\}$ . The initial value of the quantity at  $t = 0$  is given by  $p(0)$ . Its fractional extension and generalized by integrating the standard kinetic equation.

$$\frac{d}{dt}N_i(t) = -C_i(t), \quad C_i > 0 \quad \dots \quad (2)$$

the solution of the equation (2) is easily seen to be given by [Haubold and mathi].

$$N_i(t) = N_o e^{-C_i(t)} \quad \dots \quad (3)$$

integration gives an alternative form of the equation (2) as follow

$$\mathcal{N}(t) - \mathcal{N}_0 = C {}_0D_t^{-1} \mathcal{N}(t) \dots \quad (4)$$

where  ${}_0D_t^{-1}$  is the standard integral operator and  $C$  is a constant

The fractional generalization of the equation (4) is given as in the following from [1], [2].

$$\mathcal{N}(t) - \mathcal{N}_0 = -C {}_0D_t^{-\nu} \mathcal{N}(t) \dots \quad (5)$$

where  ${}_0D_t^{-\nu}$  is the Riemann-Liouville fractional integral operator [4], [5].

$${}_0D_t^{-\nu} f(t) = \frac{1}{\sqrt[\nu]{\Gamma(\nu)}} \int_0^t (t-u)^{\nu-1} f(u) du ; \nu > 0, \text{ with } {}_0D_t^0 \mathcal{F}(t) = \mathcal{F}(t) \dots \quad (6)$$

The fractional generalized of the kinetic equation (5) and studied [8], [9] they considered the free term containing Mittag-Leffler function and derived the solution of the fractional kinetic equation is obtained in terms of Mittag-Leffler function.

The generalized Mittag-Leffler function (Prabhakar [10])

$$E_{\rho, \gamma}^a(\mathcal{Z}) = \sum_{n=0}^{\infty} \frac{(a)_n}{\Gamma(\rho n + \gamma)} \cdot \frac{\mathcal{Z}^n}{n!}, \quad (\rho, \gamma, a \in \mathbb{C}; \operatorname{Re}(\rho) > 0) \dots \quad (7)$$

where  $(a)_n$  is the Pochhammer's symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1) \dots (a+n-1), & n \in \mathbb{N}; a \neq 0 \dots \end{cases} \quad (8)$$

For  $a = 1$ , Erdelyi (1954) has studied of the generalized Mittag-Leffler function [13], [14].

$$E_{\rho, \gamma}^1(\mathcal{Z}) = E_{\rho, \gamma}(\mathcal{Z}) = \sum_{n=0}^{\infty} \frac{(1)_n}{\Gamma(\rho n + \gamma)} \cdot \frac{\mathcal{Z}^n}{n!} \quad (\rho, \gamma \in \mathbb{C}; \operatorname{Re}(\rho) > 0) \dots \quad (9)$$

and in particular for  $a = \gamma = 1$ , Generalized Mittag-Leffler function [12].

$$E_{\rho, 1}^1(\mathcal{Z}) = E_{\rho}(\mathcal{Z}) = \sum_{n=0}^{\infty} \frac{(1)_n}{\Gamma(\rho n + 1)} \cdot \frac{\mathcal{Z}^n}{n!}, \quad (\rho \in \mathbb{C}; \operatorname{Re}(\rho) > 0) \dots \quad (10)$$

The function  $E_{\rho, \gamma}(\mathcal{Z})$  and  $E_{\rho}(\mathcal{Z})$  are the Mittag-Leffler function [11].

The G-function and R-function are defined by ([15], p.15, eqn. (101); [16], p.1, eqn. (1.2))

$$G_{q, u, r}(\alpha, z) = z^{qr-u-1} \sum_{n=0}^{\infty} \frac{(r)_n (\alpha z^q)^n}{\Gamma(1+n) \Gamma(nq + rq - u)}, \quad \operatorname{Re}(qr - u) > 0 \dots \quad (11)$$

$$R_{q, u}(\alpha, c, z) = (z - c)^{q-u-1} \sum_{n=0}^{\infty} \frac{[\alpha(z - c)^q]^n}{\Gamma(nq + q - u)}, \quad c \geq 0, \operatorname{Re}(q - u) > 0 \dots \quad (12)$$

obviously at  $r = 1$  and  $z$  is replaced by  $(z - c)$  the G-function reduces to the R-function.

**Kamal transform:**

The Kamal transform is a new integral transform that are defined in the time domain  $t \geq 0$  and for function of exponential order. We consider the function set  $A$  defined by:

$$A = \left\{ \mathcal{F}: |\mathcal{F}(t)| < Qe^{\frac{|t|}{\eta}} \text{ if } t \in (-1)^j. [0, \infty], j = 1,2; (Q, \eta_1, \eta_2 > 0) \right\}$$

where  $Q$  is constant and  $\eta_1, \eta_2$  can be finite or infinite [6], [7]. The integral equation

$$\mathbb{K}(p) = K\mathcal{F}(t) = \int_0^\infty e^{\frac{-t}{p}} \mathcal{F}(t) dt, \quad p \in (\eta_1, \eta_2) \dots \tag{13}$$

The Kamal transform of the operator defined in (6) is given by [5].

$$K[{}_0D_t^{-\nu} \mathcal{F}(t); p] = p^\nu \mathbb{F}(p) \dots \tag{14}$$

The Kamal transform of  $G_{q,u,r}(\alpha, z)$  and  $R_{q,u}(\alpha, c, z)$  function are given by ([16] p.15, eqn. (97); p.4, eqn. (21)).

$$K[G_{q,u,r}(\alpha^q, z)] = p^{rq-u} \left[ 1 - \left( \frac{1}{p\alpha} \right)^{-q} \right]^{-r} = p^{rq-u} [1 - (p\alpha)^q]^{-r}$$

$$Re(rq - u) > 0, Re\left(\frac{1}{p}\right) > 0, |\alpha p| > 0 \dots \tag{15}$$

$$K[R_{q,u}(\alpha^q, c, z)] = \frac{e^{-cp} p^{-u}}{p^{-q} - \alpha^q}, \quad c \geq 0, Re(q - u) > 0, Re\left(\frac{1}{p}\right) > 0 \dots \tag{16}$$

The following Kamal transform are also required in the sequel:

$$K[(1 + c^\nu {}_0D_t^{-\nu})^n \mathcal{N}(t); p] = \sum_{r=0}^n \binom{n}{r} c^{vr} K[\{{}_0D_t^{-vr} \mathcal{N}(t)\}; p] \dots \tag{17}$$

where  $\binom{n}{r} = \frac{n!}{(n-r)!}$ , which in view of (14) gives

$$K[(1 + c^\nu {}_0D_t^{-\nu})^n \mathcal{N}(t); p] = [1 + c^\nu p^\nu]^n \mathbb{N}(p) \dots \tag{18}$$

The Kamal transform in (18) is obtained with the help of binomial expansion and term by term integration in view of (14). The Kamal transform of power function is defined by:

$$K[t^{\mu-1}] = (\mu - 1)! p^\mu \dots \tag{19}$$

then the following inverse Kamal transform in (19)

$$K^{-1}[p^\mu] = \frac{t^{\mu-1}}{(\mu - 1)!}, \quad \Gamma\mu = (\mu - 1) \Gamma(\mu - 1) = (\mu - 1)! \dots \tag{20}$$

$$K^{-1}[e^{-bp} p^\mu] = \frac{(t - b)^{\mu-1}}{\Gamma\mu}$$

## 2. Main fractional differential equations.

The fractional differential equations and their solution are studied in the following theorems.

Theorem 1: If  $\mathfrak{b} > 0, \vartheta > 0, \delta > 0$  then the solution of the equation.

$$\mathcal{N}(t) - \mathcal{N}_0 G_{\vartheta, \delta, \lambda}(-\mathfrak{b}^\vartheta, t) = -\mathfrak{b}^\vartheta {}_0D_t^{-\vartheta} \mathcal{N}(t) \dots \quad (21)$$

are given by

$$\mathcal{N}(t) = \mathcal{N}_0 G_{\vartheta, \delta + \vartheta, \lambda + 1}(-\mathfrak{b}^\vartheta, t) \dots \quad (22)$$

and 
$$\mathcal{N}(t) = \frac{\mathcal{N}_0}{\lambda \vartheta} [G_{\vartheta, \delta - 1, \lambda}(-\mathfrak{b}^\vartheta, t) + (\mu + 1) G_{\vartheta, \delta, \lambda}(-\mathfrak{b}^\vartheta, t)] \dots \quad (23)$$

Proof: Taking Kamal transform of both sides of (22), we have

$$\begin{aligned} \mathcal{K}\{\mathcal{N}(t)\} - \mathcal{N}_0 \mathcal{K}\{G_{\vartheta, \delta, \lambda}(-\mathfrak{b}^\vartheta, t)\} &= -\mathfrak{b}^\vartheta \mathcal{K}\{{}_0D_t^{-\vartheta} \mathcal{N}(t)\} \\ \mathcal{N}(p) - \mathcal{N}_0 p^{-\delta + \lambda \vartheta} (1 + \mathfrak{b}^\vartheta p^\vartheta)^{-\lambda} &= -\mathfrak{b}^\vartheta p^\vartheta \mathcal{N}(p) \\ \mathcal{N}(p) \{(1 + \mathfrak{b}^\vartheta p^\vartheta)^{-\lambda}\} &= \mathcal{N}_0 p^{-\delta + \lambda \vartheta} (1 + \mathfrak{b}^\vartheta p^\vartheta)^{-\lambda} \\ \mathcal{N}(p) &= \mathcal{N}_0 p^{-\delta + \lambda \vartheta} (1 + \mathfrak{b}^\vartheta p^\vartheta)^{-(\lambda + 1)} \dots \end{aligned} \quad (24)$$

now taking inverse Kamal transform of both sides of (24), we have

$$\begin{aligned} \mathcal{N}(t) &= \mathcal{N}_0 \sum_{n=0}^{\infty} \frac{(\lambda + 1)_n}{n!} (-\mathfrak{b}^\vartheta)^n \mathcal{K}^{-1}\{p^{-\delta + \lambda \vartheta + n \vartheta}\} \\ \mathcal{N}(t) &= \mathcal{N}_0 \sum_{n=0}^{\infty} \frac{(\lambda + 1)_n}{n!} (-\mathfrak{b}^\vartheta)^n \frac{t^{\lambda \vartheta + n \vartheta - \delta - 1}}{(\lambda \vartheta + n \vartheta - \delta - 1)!} \\ \mathcal{N}(t) &= \mathcal{N}_0 t^{\lambda \vartheta - \delta - 1} \sum_{n=0}^{\infty} \frac{(\lambda + 1)_n}{n!} \frac{(-\mathfrak{b}^\vartheta t^\vartheta)^n}{(\lambda \vartheta + n \vartheta - \delta - 1)!} \\ \mathcal{N}(t) &= \mathcal{N}_0 t^{\lambda \vartheta - \delta - 1} \sum_{n=0}^{\infty} \frac{(\lambda + 1)_n}{n!} \frac{(-\mathfrak{b}^\vartheta t^\vartheta)^n}{\Gamma(\lambda \vartheta + n \vartheta - \delta)} \dots \end{aligned} \quad (25)$$

On using the definition of G-function (11) we at once arrive at the solution in (22). To prove the assertion (23) of the theorem 1 we use in (22) the formulae for the Pochhammer symbol  $(Z)_n = \frac{\Gamma(Z+n)}{\Gamma(Z)}$  and for gamma function  $\Gamma(Z+1) = Z \Gamma(Z)$  we have

$$\mathcal{N}(t) = \mathcal{N}_0 t^{\lambda \vartheta - \delta - 1} \left[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{(-\mathfrak{b}^\vartheta t^\vartheta)^n \{\vartheta(\lambda + n) - \delta - 1 + \delta + 1\}}{\lambda \vartheta (\lambda \vartheta + n \vartheta - \delta - 1)!} \right]$$

$$\begin{aligned} \mathcal{N}(t) &= \frac{\mathcal{N}_0 t^{\lambda\vartheta - \delta - 1}}{\lambda\vartheta} \left[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{(-b^\vartheta t^\vartheta)^n \{\vartheta(\lambda + n) - \delta - 1\}}{(\lambda\vartheta + n\vartheta - \delta - 1)\{(\lambda\vartheta + n\vartheta - \delta - 1) - 1\}!} \right. \\ &\quad \left. + (\delta + 1) \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{(-b^\vartheta t^\vartheta)^n}{(\lambda\vartheta + n\vartheta - \delta - 1)!} \right] \\ \mathcal{N}(t) &= \frac{\mathcal{N}_0 t^{\lambda\vartheta - \delta - 1}}{\lambda\vartheta} \left[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{(-b^\vartheta t^\vartheta)^n}{\{(\lambda\vartheta + n\vartheta - \delta - 1) - 1\}!} + (\delta \right. \\ &\quad \left. + 1) \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{(-b^\vartheta t^\vartheta)^n}{(\lambda\vartheta + n\vartheta - \delta - 1)!} \right] \end{aligned}$$

Now on using the definition of G-function (11) we at once arrive at the solution in (23).

Theorem 2: If  $b > 0, \vartheta > 0, \delta > 0$  then the solution of the equation.

$$\mathcal{N}(t) - \mathcal{N}_0 = - \left\{ \sum_{r=1}^n \binom{n}{r} (b^\vartheta)^r {}_0D_t^{-\vartheta r} \right\} \mathcal{N}(t)$$

$$i. e. \quad (1 + b^\vartheta {}_0D_t^{-\vartheta})^n \mathcal{N}(t) = \mathcal{N}_0 \dots \tag{26}$$

is given by

$$\mathcal{N}(t) = \mathcal{N}_0 E_{\vartheta,1}^n(-b^\vartheta t^\vartheta) \dots \tag{27}$$

Proof: Taking Kamal transform of both sides of (26), we have

$$\begin{aligned} K\{(1 + b^\vartheta {}_0D_t^{-\vartheta})^n \mathcal{N}(t)\} &= \mathcal{N}_0 K\{1\} \\ \mathcal{N}(p)\{(1 + b^\vartheta p^\vartheta)^n\} &= p\mathcal{N}_0 \\ \mathcal{N}(p) &= \mathcal{N}_0 p(1 + b^\vartheta p^\vartheta)^{-n} \dots \end{aligned} \tag{28}$$

now taking inverse Kamal transform of both sides of (28), we have

$$\begin{aligned} \mathcal{N}(t) &= \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{\binom{n}{r}}{r!} (-b^\vartheta)^r K^{-1}\{p^{\vartheta r + 1}\} \\ \mathcal{N}(t) &= \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{\binom{n}{r}}{r!} (-b^\vartheta)^r \frac{t^{\vartheta r}}{(\vartheta r)!} \\ \mathcal{N}(t) &= \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{\binom{n}{r}}{r!} \frac{(-b^\vartheta t^\vartheta)^r}{(\vartheta r)!}, \quad \text{where } (\vartheta r)! = \Gamma(\vartheta r + 1) \end{aligned}$$

On using the definition of Mittag-Leffler function (7) we at once arrive at the solution in (27).

Theorem 3: If  $\mathfrak{b} > 0, \vartheta > 0, \delta > 0$  then the solution of the equation.

$$\mathcal{N}(t) - \mathcal{N}_0 t^{\delta-1} = - \left\{ \sum_{r=1}^n \binom{n}{r} (\mathfrak{b}^\vartheta)^r {}_0D_t^{-\vartheta r} \right\} \mathcal{N}(t)$$

$$i. e. \quad (1 + \mathfrak{b}^\vartheta {}_0D_t^{-\vartheta})^n \mathcal{N}(t) = \mathcal{N}_0 t^{\delta-1} \dots \quad (29)$$

is given by

$$\mathcal{N}(t) = \mathcal{N}_0 \Gamma(\delta) t^{\delta-1} E_{\vartheta, \delta}^n(-\mathfrak{b}^\vartheta t^\vartheta) \dots \quad (30)$$

The proof of theorem 3 is developed on following the similar lines as to prove the theorem 2.

Theorem 4: If  $\mathfrak{b} > 0, \vartheta > 0, \delta > 0$  then the solution of the equation.

$$\mathcal{N}(t) - \mathcal{N}_0 t^{\delta-1} E_{\vartheta, \delta}(-\mathfrak{b}^\vartheta t^\vartheta) = - \left\{ \sum_{r=1}^n \binom{n}{r} (\mathfrak{b}^\vartheta)^r {}_0D_t^{-\vartheta r} \right\} \mathcal{N}(t)$$

$$i. e. \quad (1 + \mathfrak{b}^\vartheta {}_0D_t^{-\vartheta})^n \mathcal{N}(t) = \mathcal{N}_0 t^{\delta-1} E_{\vartheta, \delta}(-\mathfrak{b}^\vartheta t^\vartheta) \dots \quad (31)$$

is given by

$$\mathcal{N}(t) = \mathcal{N}_0 t^{\delta-1} E_{\vartheta, \delta}^{1+n}(-\mathfrak{b}^\vartheta t^\vartheta) \dots \quad (32)$$

Proof: Taking Kamal transform of (31) both the sides and then in view of (18) and the result ([9], p. 282, equation (6)), we have

$$\mathcal{N}(p)(1 + \mathfrak{b}^\vartheta p^\vartheta)^n = \mathcal{N}_0 p^\delta (1 + \mathfrak{b}^\vartheta p^\vartheta)^{-1}$$

$$\mathcal{N}(p) = \mathcal{N}_0 p^\delta (1 + \mathfrak{b}^\vartheta p^\vartheta)^{-(n+1)} \dots \quad (33)$$

taking inverse Kamal transform of both sides of (33), we have

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{(n+1)_r}{r!} (-\mathfrak{b}^\vartheta)^r K^{-1}\{p^{\vartheta r + \delta}\}$$

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{(n+1)_r}{r!} (-\mathfrak{b}^\vartheta)^r \frac{t^{\vartheta r + \delta - 1}}{(\vartheta r + \delta - 1)!}$$

$$\mathcal{N}(t) = \mathcal{N}_0 t^{\delta-1} \sum_{r=0}^{\infty} \frac{(n+1)_r}{r!} \frac{(-\mathfrak{b}^\vartheta t^\vartheta)^r}{(\vartheta r + \delta - 1)!} ,$$

$$\text{where } (\vartheta r + \delta - 1)! = \Gamma(\vartheta r + \delta)$$

On using the definition of Mittag-Leffler function (7) we at once arrive at the solution in (30)

Theorem 5: If  $\mathfrak{b} > 0, \vartheta > 0, \delta > 0$  then the solution of the equation.

$$\mathcal{N}(t) - \mathcal{N}_0 t^{\delta-1} E_{\vartheta, \delta}^{\lambda}(-\mathfrak{b}^{\vartheta} t^{\vartheta}) = - \left\{ \sum_{r=1}^n \binom{n}{r} (\mathfrak{b}^{\vartheta})^r {}_0D_t^{-\vartheta r} \right\} \mathcal{N}(t)$$

*i. e.*  $(1 + \mathfrak{b}^{\vartheta} {}_0D_t^{-\vartheta})^n \mathcal{N}(t) = \mathcal{N}_0 t^{\delta-1} E_{\vartheta, \delta}^{\lambda}(-\mathfrak{b}^{\vartheta} t^{\vartheta}) \dots$  (34)

is given by

$$\mathcal{N}(t) = \mathcal{N}_0 t^{\delta-1} E_{\vartheta, \delta}^{\lambda+n}(-\mathfrak{b}^{\vartheta} t^{\vartheta}) \dots$$
 (35)

Proof: Taking Kamal transform of (34) both the sides

$$\mathcal{N}(p)(1 + \mathfrak{b}^{\vartheta} p^{\vartheta})^n = \mathcal{N}_0 p^{\delta} (1 + \mathfrak{b}^{\vartheta} p^{\vartheta})^{-\lambda}$$

$$\mathcal{N}(p) = \mathcal{N}_0 p^{\delta} (1 + \mathfrak{b}^{\vartheta} p^{\vartheta})^{-(\lambda+n)} \dots$$
 (36)

taking inverse Kamal transform of (36) both the side

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{(\lambda+n)_r}{r!} (-\mathfrak{b}^{\vartheta})^r K^{-1}\{p^{\vartheta r + \delta}\}$$

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{(\lambda+n)_r}{r!} (-\mathfrak{b}^{\vartheta})^r \frac{t^{\vartheta r + \delta - 1}}{(\vartheta r + \delta - 1)!}$$

$$\mathcal{N}(t) = \mathcal{N}_0 t^{\delta-1} \sum_{r=0}^{\infty} \frac{(\lambda+n)_r}{r!} \frac{(-\mathfrak{b}^{\vartheta} t^{\vartheta})^r}{(\vartheta r + \delta - 1)!} ,$$

where  $(\vartheta r + \delta - 1)! = \Gamma(\vartheta r + \delta)$

On using the definition of Mittag-Leffler function (7) we at once arrive at the solution in (35).

Theorem 6: If  $\mathfrak{b} > 0, \vartheta > 0, \delta > 0$  then the solution of the equation.

$$\mathcal{N}(t) - \mathcal{N}_0 G_{\vartheta, \delta, \lambda}(-\mathfrak{b}^{\vartheta} t) = - \left\{ \sum_{r=1}^n \binom{n}{r} (\mathfrak{b}^{\vartheta})^r {}_0D_t^{-\vartheta r} \right\} \mathcal{N}(t)$$

*i. e.*  $(1 + \mathfrak{b}^{\vartheta} {}_0D_t^{-\vartheta})^n \mathcal{N}(t) = \mathcal{N}_0 G_{\vartheta, \delta, \lambda}(-\mathfrak{b}^{\vartheta} t) \dots$  (37)

is given by

$$\mathcal{N}(t) = \mathcal{N}_0 G_{\vartheta, \delta+n\vartheta, \lambda+n}(-\mathfrak{b}^{\vartheta} t) \dots$$
 (38)

Proof: Taking Kamal transform of (37) both the sides

$$\mathcal{N}(p) = \mathcal{N}_0 p^{\lambda\vartheta - \delta} (1 + \mathfrak{b}^{\vartheta} p^{\vartheta})^{-(\lambda+n)} \dots$$
 (39)

now taking inverse Kamal transform of (39) both sides, we have

$$K^{-1}\{\mathcal{N}(p)\} = \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{(\lambda+n)_r}{r!} (-b^\vartheta)^n K^{-1}\{p^{\lambda\vartheta+\vartheta r-\delta}\}$$

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{(\lambda+n)_r}{r!} (-b^\vartheta)^n \frac{t^{\lambda\vartheta+\vartheta r-\delta-1}}{(\lambda\vartheta+\vartheta r-\delta-1)!}$$

$$\mathcal{N}(t) = \mathcal{N}_0 t^{\lambda\vartheta-\delta-1} \sum_{r=0}^{\infty} \frac{(\lambda+n)_r}{r!} \frac{(-b^\vartheta t^\vartheta)^r}{(\lambda\vartheta+\vartheta r-\delta-1)!},$$

$$\text{Where } (\lambda\vartheta+\vartheta r-\delta-1)! = \Gamma(\lambda\vartheta+\vartheta r-\delta)$$

on using the definition of G-function (11) we at once arrive at the solution in (38).

Theorem 7: If  $b > 0, \vartheta > 0, d > 0, \delta > 0$  then the solution of the equation.

$$\mathcal{N}(t) - \mathcal{N}_0 \mathcal{R}_{\vartheta,\delta}(-b^\vartheta, d, t) = - \left\{ \sum_{r=1}^n \binom{n}{r} (b^\vartheta)^r {}_0D_t^{-\vartheta r} \right\} \mathcal{N}(t)$$

$$i.e. \quad (1 + b^\vartheta {}_0D_t^{-\vartheta})^n \mathcal{N}(t) = \mathcal{N}_0 \mathcal{R}_{\vartheta,\delta}(-b^\vartheta, d, t) \dots \quad (40)$$

is given by

$$\mathcal{N}(t) = \mathcal{N}_0 \mathbb{G}_{\vartheta,\delta+n\vartheta,n+1}(-b^\vartheta, t, -d) \dots \quad (41)$$

and

$$\mathcal{N}(t) = \frac{\mathcal{N}_0}{n\vartheta} [(t-d) \mathbb{G}_{\vartheta,\vartheta-\delta-1,n}(-b^\vartheta, t, -d) + \{\delta+1-\vartheta(1-n)\} \mathbb{G}_{\vartheta,\vartheta-\delta,n}(-b^\vartheta, t, -d)] \dots \quad (42)$$

Proof: Taking Kamal transform of (40) both side and then in view of (18) and (20) we have

$$\mathcal{N}(p)(1 + b^\vartheta p^\vartheta)^n = \mathcal{N}_0 \left( \frac{e^{-dp} p^{-\delta}}{p^{-\vartheta} + b^\vartheta} \right),$$

$$\mathcal{N}(p)(1 + b^\vartheta p^\vartheta)^n = \mathcal{N}_0 \left( \frac{e^{-dp} p^{-\delta+\vartheta}}{1 + b^\vartheta p^\vartheta} \right)$$

$$\mathcal{N}(p) = \mathcal{N}_0 \left\{ e^{-dp} p^{-\delta+\vartheta} (1 + b^\vartheta p^\vartheta)^{-(n+1)} \right\} \dots \quad (43)$$

taking inverse Kamal transform of (43)

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{(n+1)_r}{r!} (-b^\vartheta)^r K^{-1}\{e^{-dp} p^{\vartheta r+\vartheta-\delta}\}$$

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{(n+1)_r}{r!} (-b^\vartheta)^r \frac{(t-d)^{\vartheta r+\vartheta-\delta-1}}{(\vartheta r+\vartheta-\delta-1)!}$$

$$\mathcal{N}(t) = \mathcal{N}_0(t-d)^{\vartheta-\delta-1} \sum_{r=0}^{\infty} \frac{(n+1)_r}{r!} \frac{\{-b^\vartheta(t-d)^\vartheta\}^r}{(\vartheta r + \vartheta - \delta - 1)!}$$

Where  $(\vartheta r + \vartheta - \delta - 1)! = \Gamma(\vartheta r + \vartheta - \delta)$

On using the definition of G-function (11) we at once arrive at the solution in (41). To prove the assertion (42) of the theorem 7 we use in (41) the formulae for the Pochhammer symbol  $(Z)_n = \frac{\Gamma(Z+n)}{\Gamma(Z)}$  and for gamma function  $\Gamma(Z+1) = Z \Gamma Z$  we have:

$$\begin{aligned} \mathcal{N}(t) &= \mathcal{N}_0(t-d)^{(n+1)\vartheta-(\delta+n\vartheta)-1} \sum_{r=0}^{\infty} \frac{(n+1)_r}{r!} \frac{\{-b^\vartheta(t-d)^\vartheta\}^r}{\{\vartheta r + (n+1)\vartheta - (\delta + n\vartheta) - 1\}!} \\ \mathcal{N}(t) &= \mathcal{N}_0(t-d)^{(n+1)\vartheta-(\delta+n\vartheta)-1} \left[ \sum_{r=0}^{\infty} \frac{(n)_r}{r!} \{-b^\vartheta(t-d)^\vartheta\}^r \cdot \frac{\{\vartheta(n+r) + \vartheta(1-n) - \delta - 1 + \delta + 1 - \vartheta(1-n)\}}{n\vartheta\{\vartheta r + (n+1)\vartheta - (\delta + n\vartheta) - 1\}!} \right] \\ \mathcal{N}(t) &= \frac{\mathcal{N}_0(t-d)^{(n+1)\vartheta-(\delta+n\vartheta)-1}}{n\vartheta} \left[ \sum_{r=0}^{\infty} \frac{(n)_r \{-b^\vartheta(t-d)^\vartheta\}^r}{r! [\{\vartheta r + (n+1)\vartheta - (\delta + n\vartheta) - 1\} - 1]} \right. \\ &\quad \left. + \{\delta + 1 - \vartheta(1-n)\} \sum_{r=0}^{\infty} \frac{(n)_r}{r!} \frac{\{-b^\vartheta(t-d)^\vartheta\}^r}{\{\vartheta r + (n+1)\vartheta - (\delta + n\vartheta) - 1\}!} \right] \end{aligned}$$

where  $\{\vartheta r + (n+1)\vartheta - (\delta + n\vartheta) - 1\}! = \Gamma\{\vartheta r + (n+1)\vartheta - (\delta + n\vartheta)\}$

On interpreting the resulting series with the help of (11) we at once arrive at the solution in (42).

**Special Cases:-**

- (i) If we take  $\delta = \beta, \lambda = \gamma, \vartheta = \alpha, n = 1$  and  $b = c$  in equations (34) and (35) we get the known result of ([18], eqns. (2.5)-(2.6), p. 42). If we consider  $q = 1$  in equations (2.5) and (2.6),
- (ii) If we take  $n = 1, \beta = \mu$  then these reduce to known result [19, eqns.(27)-(28), p.661] which in turn at  $\gamma = 1$  provide the known results [19, eqns. (33)-(34), p.662] respectively.
- (iii) If we replace  $\delta = 1$  in theorem-3 then we get the same result of theorem-2.
- (iv) If we take  $n = 1, v = \vartheta, \lambda = \gamma, b = c$  and  $\delta = \mu - 1$  in equations (34) and (35) we get the known result of ([17], eqns. (30), p.326).

## CONCLUSION

In this work we have studied of certain fractional differential equation and some applications involving Mittag-Leffler function and G-function. The solutions of this certain fractional differential equation are obtained by Kamal transform. We observed that Kamal transform solves fractional differential equation with a few applications and derived solution for same. We can easily obtain various known and new fractional differential equations. We noted that results of Laplace transform see in [1] and Kamal transform are closely connected to each other.

## REFERENCE

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