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A New Family of Special Polynomials with Fractional Integrals

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Abstract

This paper introduces a new family of special polynomials like Bernoulli polynomials. We use the Mittag-Leffler function of two parameters in the generating function of these polynomials. We derived the recurrence relations of these special polynomials. From these polynomials, a new family of special numbers is introduced. The properties of new special polynomials and new special numbers are derived. The fractional integrals of new special polynomial functions are also discussed.

Keywords: Mittag Leffler functions, Bernoulli polynomials, Special Polynomials, Generating function, Fractional integrals

2020 Mathematics Subject Classifications: 11B75, 11B83, 12E10.

1. INTRODUCTION

Special polynomials have important applications in number theory, classical analysis, and recently fractional calculus. They appear in the integral representation of differentiable periodic functions. In this paper, we have introduced a new family of special polynomials like Bernoulli polynomials.

2. PRELIMINARIES

Definition 1. *The coefficients B_n of $\frac{t^n}{n!}$ in the expansion of the function*

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad 0 < |t| < 2\pi, \tag{1}$$

are called Bernoulli numbers [1, 8, 13], [15].

Thus, the function $\frac{t}{e^t-1}$ is a generating function for the Bernoulli numbers.

Definition 2. *The coefficients* $B_n(x)$ *of* $\frac{t^n}{n!}$ *in the expansion of the function*

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\qquad 0<|t|<2\pi, \tag{2}$$

are called Bernoulli polynomials [1, 8, 13], [15].

Thus, the function $\frac{te^{xt}}{e^t-1}$ is a generating function for the Bernoulli polynomials.

Definition 3. *A generalization of the Bernoulli polynomials is defined in a suitable neighborhood of* $t=0$ by Natalini and Bernardini in [11] *as the coefficients* $B_n^{[m-1]}(x), m\geq 1$ *of* $\frac{t^n}{n!}$ *in the expansion of the generating function,*

$$G^{[m-1]}(x,t)=\frac{t^m e^{xt}}{e^t-\sum_{h=0}^{m-1}\frac{t^h}{h!}}=\sum_{n=0}^{\infty}B_n^{[m-1]}(x)\frac{t^n}{n!}. \tag{3}$$

Definition 4. *The Riemann-Liouville fractional integral* [14] *of order* $\alpha\in\mathbb{R}$ *is defined by*

$$\left(I^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}dt. \tag{4}$$

3. A NEW FAMILY OF SPECIAL POLYNOMIALS

Definition 5. *A new family of special polynomials* $C_{(2,n)}(x)$ *is defined in a suitable neighborhood of* $t=0$ *through the generating function*

$$\frac{e^{xt}}{2E_{2,3}\left(t^2\right)}=\sum_{n=0}^{\infty}C_{(2,n)}(x)\frac{t^n}{n!}. \tag{5}$$

Let us derive some of the new special polynomials as below:
The equation (5) can be written as

$$\frac{e^{xt}}{2\sum_{j=0}^{\infty}\frac{t^{2j}}{(2j+2)!}}=\sum_{n=0}^{\infty}C_{(2,n)}(x)\frac{t^n}{n!}. \tag{6}$$

Since $e^{xt}=\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}$, equation (6) becomes

$$\Rightarrow\frac{1}{2}\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}=\sum_{j=0}^{\infty}\frac{t^{2j}}{(2j+2)!}\sum_{n=0}^{\infty}C_{(2,n)}(x)\frac{t^n}{n!}.$$

$$\Rightarrow\frac{1}{2}\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}\quad=\quad\sum_{j=0}^{\infty}\frac{t^{2j}}{(2j+2)!}\left[C_{(2,0)}(x)+C_{(2,1)}(x)\frac{t}{1!}\right.$$

$$+C_{(2,2)}(x)\frac{t^2}{2!}+C_{(2,3)}(x)\frac{t^3}{3!}+\cdots\Bigg]$$

$$\begin{aligned}
\Rightarrow \frac{1}{2}\sum_{n=0}^{\infty}x^n\frac{t^n}{n!} &= \frac{1}{2!}\left[C_{(2,0)}(x)+C_{(2,1)}(x)\frac{t}{1!}+C_{(2,2)}(x)\frac{t^2}{2!}+C_{(2,3)}(x)\frac{t^3}{3!}+...\right]\\
&+\frac{t^2}{4!}\left[C_{(2,0)}(x)+C_{(2,1)}(x)\frac{t}{1!}+C_{(2,2)}(x)\frac{t^2}{2!}+C_{(2,3)}(x)\frac{t^3}{3!}+\cdots\right]\\
&+\frac{t^4}{6!}\left[C_{(2,0)}(x)+C_{(2,1)}(x)\frac{t}{1!}+C_{(2,2)}(x)\frac{t^2}{2!}+C_{(2,3)}(x)\frac{t^3}{3!}+\cdots\right]\\
&+\frac{t^6}{8!}\left[C_{(2,0)}(x)+C_{(2,1)}(x)\frac{t}{1!}+C_{(2,2)}(x)\frac{t^2}{2!}+C_{(2,3)}(x)\frac{t^3}{3!}+\cdots\right]\\
&+\frac{t^8}{10!}\left[C_{(2,0)}(x)+C_{(2,1)}(x)\frac{t}{1!}+C_{(2,2)}(x)\frac{t^2}{2!}+C_{(2,3)}(x)\frac{t^3}{3!}+\cdots\right]\\
&+\frac{t^{10}}{12!}\left[C_{(2,0)}(x)+C_{(2,1)}(x)\frac{t}{1!}+C_{(2,2)}(x)\frac{t^2}{2!}+C_{(2,3)}(x)\frac{t^3}{3!}+\cdots\right]\\
&+\cdots.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{1}{2}\sum_{n=0}^{\infty}x^n\frac{t^n}{n!} &= \left[\frac{0!}{2!\cdot 0!}C_{(2,0)}(x)\right]\frac{t^0}{0!}+\left[\frac{1!}{2!\cdot 1!}C_{(2,1)}(x)\right]\frac{t^1}{1!}\\
&+\left[\frac{2!}{2!\cdot 2!}C_{(2,2)}(x)+\frac{2!}{4!\cdot 0!}C_{(2,0)}(x)\right]\frac{t^2}{2!}\\
&+\left[\frac{3!}{2!\cdot 3!}C_{(2,3)}(x)+\frac{3!}{4!\cdot 1!}C_{(2,1)}(x)\right]\frac{t^3}{3!}\\
&+\left[\frac{4!}{2!\cdot 4!}C_{(2,4)}(x)+\frac{4!}{4!\cdot 2!}C_{(2,2)}(x)+\frac{4!}{6!\cdot 0!}C_{(2,0)}(x)\right]\frac{t^4}{4!}\\
&+\left[\frac{5!}{2!\cdot 5!}C_{(2,5)}(x)+\frac{5!}{4!\cdot 3!}C_{(2,3)}(x)+\frac{5!}{6!\cdot 1!}C_{(2,1)}(x)\right]\frac{t^5}{5!}\\
&+\Bigg[\frac{6!}{2!\cdot 6!}C_{(2,6)}(x)+\frac{6!}{4!\cdot 4!}C_{(2,4)}(x)\\
&+\frac{6!}{6!\cdot 2!}C_{(2,2)}(x)+\frac{6!}{8!\cdot 0!}C_{(2,0)}(x)\Bigg]\frac{t^6}{6!}\\
&+\Bigg[\frac{7!}{2!\cdot 7!}C_{(2,7)}(x)+\frac{7!}{4!\cdot 5!}C_{(2,5)}(x)\\
&+\frac{7!}{6!\cdot 3!}C_{(2,3)}(x)+\frac{7!}{8!\cdot 1!}C_{(2,1)}(x)\Bigg]\frac{t^7}{7!}\\
&+\Bigg[\frac{8!}{2!\cdot 8!}C_{(2,8)}(x)+\frac{8!}{4!\cdot 6!}C_{(2,6)}(x)+\frac{8!}{6!\cdot 4!}C_{(2,4)}(x)\\
&+\frac{8!}{8!\cdot 2!}C_{(2,2)}(x)++\frac{8!}{10!\cdot 0!}C_{(2,0)}(x)\Bigg]\frac{t^8}{8!}\\
&+\cdots.
\end{aligned}\tag{7}$$

On comparing the coefficients of $\frac{t^n}{n!}$; $(n = 0, 1, 2, 3, \cdots)$ in equation (7), the recurrences for new special polynomials are obtained as below

$$\frac{1}{2} = \frac{0!}{2! \cdot 0!} C_{(2,0)}(x) \tag{8}$$

$$\frac{1}{2}x = \frac{1!}{2! \cdot 1!} C_{(2,1)}(x) \tag{9}$$

$$\frac{1}{2}x^2 = \frac{2!}{2! \cdot 2!} C_{(2,2)}(x) + \frac{2!}{4! \cdot 0!} C_{(2,0)}(x) \tag{10}$$

$$\frac{1}{2}x^3 = \frac{3!}{2! \cdot 3!} C_{(2,3)}(x) + \frac{3!}{4! \cdot 1!} C_{(2,1)}(x) \tag{11}$$

$$\frac{1}{2}x^4 = \frac{4!}{2! \cdot 4!} C_{(2,4)}(x) + \frac{4!}{4! \cdot 2!} C_{(2,2)}(x) + \frac{4!}{6! \cdot 0!} C_{(2,0)}(x) \tag{12}$$

$$\frac{1}{2}x^5 = \frac{5!}{2! \cdot 5!} C_{(2,5)}(x) + \frac{5!}{4! \cdot 3!} C_{(2,3)}(x) + \frac{5!}{6! \cdot 1!} C_{(2,1)}(x) \tag{13}$$

$$\frac{1}{2}x^6 = \frac{6!}{2! \cdot 6!} C_{(2,6)}(x) + \frac{6!}{4! \cdot 4!} C_{(2,4)}(x) + \frac{6!}{6! \cdot 2!} C_{(2,2)}(x) + \frac{6!}{8! \cdot 0!} C_{(2,0)}(x) \tag{14}$$

$$\frac{1}{2}x^7 = \frac{7!}{2! \cdot 7!} C_{(2,7)}(x) + \frac{7!}{4! \cdot 5!} C_{(2,5)}(x) + \frac{7!}{6! \cdot 3!} C_{(2,3)}(x) + \frac{7!}{8! \cdot 1!} C_{(2,1)}(x) \tag{15}$$

$$\begin{aligned}\frac{1}{2}x^8 = \frac{8!}{2! \cdot 8!} C_{(2,8)}(x) + \frac{8!}{4! \cdot 6!} C_{(2,6)}(x) + \frac{8!}{6! \cdot 4!} C_{(2,4)}(x) + \frac{8!}{8! \cdot 2!} C_{(2,2)}(x) \\ + \frac{8!}{10! \cdot 0!} C_{(2,0)}(x)\end{aligned} \tag{16}$$

From equations (8) to (16), we get the new special polynomials as below:

$$C_{(2,0)}(x) = 1 \tag{17}$$

$$C_{(2,1)}(x) = x \tag{18}$$

$$C_{(2,2)}(x) = x^2 - \frac{1}{6} \tag{19}$$

$$C_{(2,3)}(x) = x^3 - \frac{1}{2}x \tag{20}$$

$$C_{(2,4)}(x) = x^4 - x^2 + \frac{1}{10} \tag{21}$$

$$C_{(2,5)}(x) = x^5 - \frac{5}{3}x^3 + \frac{1}{2}x \tag{22}$$

$$C_{(2,6)}(x) = x^6 - \frac{5}{2}x^4 + \frac{3}{2}x^2 - \frac{5}{42} \tag{23}$$

$$C_{(2,7)}(x) = x^7 - \frac{7}{2}x^5 + \frac{7}{2}x^3 - \frac{5}{6}x \tag{24}$$

$$C_{(2,8)}(x) = x^8 - \frac{14}{3}x^6 + 7x^4 - \frac{10}{3}x^2 + \frac{7}{30} \tag{25}$$

4. A NEW FAMILY OF SPECIAL NUMBERS

A new special numbers $C_{(2,n)}, (n = 0, 1, 2, \cdots)$ can be obtained from equations (17) to (25) by formula

$$C_{(2,n)} = C_{(2,n)}(0) \tag{26}$$

as below:

$$\begin{aligned} C_{(2,0)} = 1, C_{(2,1)} = 0, C_{(2,2)} = -\frac{1}{6}, C_{(2,3)} = 0, C_{(2,4)} = \frac{1}{10}, \\ C_{(2,5)} = 0, C_{(2,6)} = -\frac{5}{42}, C_{(2,7)} = 0, C_{(2,8)} = \frac{7}{30}, C_{(2,9)} = 0, \\ C_{(2,10)} = -\frac{15}{22}, C_{(2,11)} = 0, C_{(2,12)} = \frac{7601}{2730}, C_{(2,13)} = 0, \\ C_{(2,14)} = -\frac{91}{6}, C_{(2,15)} = 0, C_{(2,16)} = \frac{3617}{34}, C_{(2,17)} = 0, \cdots. \end{aligned}$$

A few of the patterns we see are among the following:

1. $C_{(2,n)}$ is rational.
2. $C_{(2,2n+1)} = 0$ for $n \geq 0$.
3. $C_{(2,2n)}$ alternates sign: $C_{(2,4n)} > 0$ and $C_{(2,4n+2)} < 0$ for $n \geq 0$.

5. PROPERTIES OF NEW SPECIAL POLYNOMIALS AND NUMBERS

Theorem 1. *The new special numbers satisfy the relation*

$$\sum_{k=0}^{2n} \binom{2n+2}{k} C_{(2,k)} = 0, \quad n \geq 1. \tag{27}$$

Proof. The new special numbers can be obtained by

$$\frac{1}{2\sum_{j=0}^{\infty} \frac{t^{2j}}{(2j+2)!}} = \sum_{n=0}^{\infty} C_{(2,n)} \frac{t^n}{n!}. \tag{28}$$

$$\Rightarrow 1 = 2\sum_{j=0}^{\infty} \frac{t^{2j}}{(2j+2)!} \sum_{n=0}^{\infty} C_{(2,n)} \frac{t^n}{n!}.$$

From equation (7), we have

$$\begin{aligned} 1 \;=\; & \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{2(2n)!}{(2n+2-2k)!(2k)!} C_{(2,2k)} \frac{t^{2n}}{(2n)!} \\ & + \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{2(2n+1)!}{(2n+2-2k)!(2k+1)!} C_{(2,2k+1)} \frac{t^{2n+1}}{(2n+1)!}. \end{aligned} \tag{29}$$

Since $C_{(2,2k+1)} = 0, \forall k \geq 0$, equation (29) can be written as

$$1 = \sum_{n=0}^{\infty}\sum_{k=0}^{2n} \frac{2(2n)!}{(2n+2-k)!(k)!} C_{(2,k)} \frac{t^{2n}}{(2n)!}$$

$$\Rightarrow 1 = \sum_{n=0}^{\infty}\sum_{k=0}^{2n} \frac{2(2n+2)!}{(2n+2-k)!(k)!} C_{(2,k)} \frac{t^{2n}}{(2n+2)!}$$

$$\Rightarrow 1 = \sum_{n=0}^{\infty}\sum_{k=0}^{2n} 2 \binom{2n+2}{k} C_{(2,k)} \frac{t^{2n}}{(2n+2)!} \tag{30}$$

On the left hand side of equation (30) we have only 1 and the coefficient of $t^0 = 1$ on the right hand side is $C_{(2,0)}$, and the coefficient of every other power of t is 0 on left side. Thus, the desired relation results:
$C_{(2,0)} = 1$ and

$$\sum_{k=0}^{2n} \binom{2n+2}{k} C_{(2,k)} = 0 \text{ for} \, n \geq 1.$$

◄

Theorem 2. *For* $k = 1, 2, 3, ...$, *we have*

$$C^{'}_{(2,k)}(x) = kC_{(2,k-1)}(x). \tag{31}$$

Proof. From the equation (7), we have

$$x^{2n} = \sum_{k=0}^{n} \frac{2(2n)!}{(2n+2-2k)!(2k)!} C_{(2,2k)}(x) \tag{32}$$

$$x^{2n+1} = \sum_{k=0}^{n} \frac{2(2n+1)!}{(2n+2-2k)!(2k+1)!} C_{(2,2k+1)}(x) \tag{33}$$

Differentiate equation (33) with respect to 'x', we get

$$(2n+1)x^{2n} = \sum_{k=0}^{n} \frac{2(2n+1)!}{(2n+2-2k)!(2k+1)!} C^{'}_{(2,2k+1)}(x).$$

Substituting x^{2n} from equation (32), we get

$$(2n+1)\sum_{k=0}^{n} \frac{2(2n)!C_{(2,2k)}(x)}{(2n+2-2k)!(2k)!} = \sum_{k=0}^{n} \frac{2(2n+1)!C^{'}_{(2,2k+1)}(x)}{(2n+2-2k)!(2k+1)!}.$$

$$\Rightarrow \sum_{k=0}^{n} \frac{2(2n+1)!C_{(2,2k)}(x)}{(2n+2-2k)!(2k)!} = \sum_{k=0}^{n} \frac{2(2n+1)!C^{'}_{(2,2k+1)}(x)}{(2k+1)\,(2n+2-2k)!(2k)!}. \tag{34}$$

On comparing both sides of equation (34), we have

$$C^{'}_{(2,2k+1)}(x) = (2k+1)C_{(2,2k)}(x), \forall k \geq 0. \tag{35}$$

Similarly by differentiating equation (32) and substituting x^{2n-1} from equation (33), we get

$$C^{'}_{(2,2k)}(x) = (2k)C_{(2,2k-1)}(x), \forall k \geq 1. \tag{36}$$

The results in equations (35) and (36) proves that

$$C^{'}_{(2,k)}(x) = kC_{(2,k-1)}(x), \forall k \geq 1.$$

◀

Theorem 3. *The new special polynomials satisfy the following property for* $k \geq 1$

$$\int_0^1 C_{(2,2k+1)}(x)dx = 0. \tag{37}$$

Proof. From the result in theorem (2), we have

$$C^{'}_{(2,2k+2)}(x) = (2k+2)C_{(2,2k+1)}(x).$$

Hence, we have

$$\int_0^1 C_{(2,2k+1)}(x)dx = \int_0^1 \frac{1}{(2k+2)} C^{'}_{(2,2k+2)}(x)dx$$

$$\int_0^1 C_{(2,2k+1)}(x)dx = \frac{1}{(2k+2)} \left[C_{(2,2k+2)}(x)\right]_0^1 \tag{38}$$

But, for every $k \geq 2$, we have $C_{(2,2k)}(1) = C_{(2,2k)}(0)$. Hence an equation (38) gives us the desired result

$$\int_0^1 C_{(2,2k+1)}(x)dx = 0.$$

◀

6. THE FRACTIONAL INTEGRATIONS OF NEW SPECIAL POLYNOMIAL FUNCTIONS

The Riemann-Liouville fractional integrals of new special polynomial functions are given by

$$\left(I^{\alpha}f\right)\left(C_{(2,0)}(x)\right) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{1}{(x-t)^{1-\alpha}}dt = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \tag{39}$$

$$\left(I^{\alpha}f\right)\left(C_{(2,1)}(x)\right) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{x}{(x-t)^{1-\alpha}}dt = \frac{x^{1+\alpha}}{\Gamma(2+\alpha)} \tag{40}$$

$$\left(I^{\alpha}f\right)\left(C_{(2,2)}(x)\right) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{x^2-\frac{1}{6}}{(x-t)^{1-\alpha}}dt = \frac{2x^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{x^{\alpha}}{6\Gamma(1+\alpha)} \tag{41}$$

$$(I^{\alpha}f)\left(C_{(2,3)}(x)\right)=\frac{6x^{3+\alpha}}{\Gamma(4+\alpha)}-\frac{x^{1+\alpha}}{2\Gamma(2+\alpha)} \tag{42}$$

$$(I^{\alpha}f)\left(C_{(2,4)}(x)\right)=\frac{24x^{4+\alpha}}{\Gamma(5+\alpha)}-\frac{2x^{2+\alpha}}{\Gamma(3+\alpha)}+\frac{x^{\alpha}}{10\Gamma(1+\alpha)}, \tag{43}$$

etc.

7. CONCLUSION

Many mathematicians have explored degenerate versions of many special numbers and polynomials in the last few years. This paper aimed to introduce a new family of special polynomials and numbers. To achieve this goal, we make use of the base of Bernoulli polynomials. We derive the different properties of these special polynomials. The research on this new special polynomials and numbers will be continued for applications to physics, science, and engineering

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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