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Modified Definitions of SABC and SABR Fractional Derivatives and Applications

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Abstract

This paper introduces modified definitions of SABC and SABR fractional derivatives. We made some changes in the formulae of SABC and SABR fractional derivatives so that the new modified formulae are well defined. Many applications of modified definitions are also explained in this paper.

Keywords: SABC and SABR fractional derivatives, non-singular kernel, Mittag-Leffler function of two parameters, fractional differential equations.

2020 Mathematics Subject Classifications: 26A33, 34A08.

1. INTRODUCTION

Chinchole and Bhadane presented the SABC and SABR definitions of fractional derivatives with some properties and applications [8, 9]. The fractional derivatives introduced by Chinchole and Bhadane have been applied in various fields of Science with enormous success. However, these definitions have to be well defined. For that, we have made modifications to these SABC and SABR definitions of fractional derivatives.

2. PRELIMINARIES

Definition 1. *Let $f \in H^1(a,b), a < b, a \in [-\infty, t)$ and $\alpha \in [0,1]$ then, the Caputo fractional time derivative with non-singular kernel [6, 7] is defined by*

$$ {}_aD_t^{(\alpha)} f(t) = \frac{N(\alpha)}{1-\alpha} \int_a^t f^{'}(\tau) \exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau \tag{1} $$

where $N(\alpha)$ is a normalization function such that $N(0) = N(1) = 1$.

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Definition 2 (Generalized Mittag Leffler Function of two parameters)**.** *The generalized Mittag-Leffler function [11, 19] is defined as*

$$t^{\beta-1}E_{\alpha,\beta}(-t^{\alpha}) = t^{\beta-1}\sum_{k=0}^{\infty}\frac{(-t^{\alpha})^k}{\Gamma(\alpha k+\beta)} \tag{2}$$

Using a kernel that is non-local and non-singular, namely the Mittag-Leffler function, Atangana, and Baleanu have introduced the definition of fractional derivatives as below:

Definition 3. *Let $f \in H^1(a,b), a < b, \alpha \in [0,1]$ then, the definitions of the AB fractional derivative with non-local and non-singular kernel [2] are given by:*

$$^{ABC}{}_aD_t^{\alpha}f(t) = \frac{N(\alpha)}{1-\alpha}\int_a^t f'(\tau)E_{\alpha}\left[-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau \tag{3}$$

$$^{ABR}{}_aD_t^{\alpha}f(t) = \frac{N(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(\tau)E_{\alpha}\left[-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau \tag{4}$$

with $N(\alpha)$ being a normalisation function mentioned in Caputo fractional time derivative.

By changing the kernel $E_{\alpha}\left[-\frac{\alpha t^{\alpha}}{1-\alpha}\right]$ with the Mittag - Leffler function $t^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)t^{\alpha}}{2-\alpha-\beta}\right]$ of two parameter and $\frac{N(\alpha)}{1-\alpha}$ by $\frac{B(\alpha,\beta)}{2-\alpha-\beta}$, Chinchole and Bhadane proposed the new definitions of fractional derivatives as below::

Definition 4. *Let $f \in H^1(a,b), a < b, 0 \le \alpha,\beta \le 1$ then, the SABC fractional derivative [8] is defined by*

$$^{SABC}{}_aD_t^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_a^t f'(\tau)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau \tag{5}$$

where $B(\alpha,\beta)$ is a normalization function such that $B(\alpha,\beta) = N(\alpha+\beta-1)$.

Definition 5. *Let $f \in H^1(a,b), a < b, 0 \le \alpha,\beta \le 1$ then, the SABR fractional derivative [8] is defined by*

$$^{SABR}{}_aD_t^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_a^t f(\tau)(t-\tau)^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau \tag{6}$$

3. MODIFIED SABC AND SABR FRACTIONAL DERIVATIVES

The kernel $t^{\beta-1}E_{\alpha,\beta}\left[-\frac{(\alpha+\beta-1)t^{\alpha}}{2-\alpha-\beta}\right]$ in SABC fractional derivative (5) and SABR fractional derivative (6) of two parameters is replaced by the new kernel $t^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)t^{\alpha}}{2-\alpha-\beta}\right]$ to obtain the modified SABC and modified SABC fractional derivatives of two parameter as follows:

Definition 6. *Let $f \in H^1(a,b), a < b, \alpha,\beta \in [0,1], \alpha+\beta \neq 2$ then, the modified SABC fractional derivative ${}^{SABC*}{}_aD_t^{\alpha,\beta}f(t)$ is defined as below:*

$$ {}^{SABC*}{}_aD_t^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_a^t f'(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau. \tag{7} $$

Definition 7. *Let $f \in H^1(a,b), a < b, \alpha,\beta \in [0,1], \alpha+\beta \neq 2$ then, the modified SABR fractional derivative ${}^{SABR*}{}_aD_t^{\alpha,\beta}f(t)$ is defined as below:*

$$ {}^{SABR*}{}_aD_t^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_a^t f(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau \tag{8} $$

Obviously, equations (7) and (8) have a non-local, non-singular kernel.

Lemma 1. *Let $f \in H^1(a,b), a < b, \alpha,\beta \in [0,1], \alpha+\beta \neq 2$. For a given normalisation function $B(\alpha,\beta) = N(\alpha+\beta-1)$, the modified SABR derivative ${}^{SABR*}{}_aD_t^{\alpha,\beta}f(t)$ and the modified SABC derivative ${}^{SABC*}{}_aD_t^{\alpha,\beta}f(t)$ are well-defined for any function f if and only if the R. L. integral $D_a^{-(1-\beta)}f(t)$ is well-defined.*

Proof. From the equation (8) in the definition 7, we have

$$ {}^{SABR*}{}_aD_t^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_a^t f(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau $$

By differentiating the integral on the right side using fundamental theorem of calculus, we get

$$ \begin{aligned} {}^{SABR*}{}_aD_t^{\alpha,\beta}f(t) &= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\left[f(t)(t-t)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-t)^{\alpha}}{2-\alpha-\beta}\right]\right. \\ &\quad \left. + \int_a^t f(\tau)\frac{d}{dt}\left((t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]\right)d\tau\right] \\ &= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_a^t f(\tau)\frac{d}{dt}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\frac{(t-\tau)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)}d\tau \end{aligned} $$

$$\Rightarrow {}^{SABR*}{}_{a}D_{t}^{\alpha,\beta}f(t)=\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_{a}^{t}f(\tau)\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\frac{(t-\tau)^{\alpha n-\beta}}{\Gamma(\alpha n-\beta+1)}d\tau. \tag{9}$$

The Mittag-Leffler function $(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]$ and its t-derivative, considered as functions of τ, are holomorphic at all points in the interval $[a,t)$. Also, the interval of integration is finite, and so the only way that the integral could have the possibility of divergence is due to its behaviour near $\tau=t$. Therefore, the conditions for the modified SABR derivative to be well-defined are exactly that the integral in (9) should behave well as $\tau\to t$ from below.

When $\tau\to t$, we have $(t-\tau)^{\alpha n-\beta+1}\to 0$ and hence

$$\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\frac{(t-\tau)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)}\sim\frac{(t-\tau)^{1-\beta}}{\Gamma(2-\beta)}$$

giving us

$$\frac{d}{dt}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\frac{(t-\tau)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)}\sim\frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)}.$$

Therefore, the integral in (9) converges if and only if

$$\int_{a}^{t}f(\tau)(t-\tau)^{(1-\beta)-1}d\tau$$

converges, i.e., if and only if the R. L. integral $D_{a}^{-(1-\beta)}f(t)$ is well-defined.

Now, from the equation (7) in the definition 6 of modified SABC fractional derivative, we have

$${}^{SABC*}{}_{a}D_{t}^{\alpha,\beta}f(t)=\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_{a}^{t}f^{'}(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau$$

Here also, the function $(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]$ is holomorphic at all points in the interval $[a,t)$. Also, the interval of integration is finite, and hence, the conditions for the modified SABC derivative to be well-defined are precisely that the integral in (7) should behave well as $\tau\to t$ from below.

When $\tau\to t$, we have $(t-\tau)^{\alpha n-\beta+1}\to 0$ and hence

$$(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]\sim\frac{(t-\tau)^{1-\beta}}{\Gamma(2-\beta)}$$

Therefore, the integral in (7) converges if and only if

$$\int_{a}^{t}f^{'}(\tau)(t-\tau)^{1-\beta}d\tau=(t-a)^{1-\beta}f(a)+(1-\beta)\int_{a}^{t}f(\tau)(t-\tau)^{-\beta}d\tau$$

converges, i.e., if and only if the R. L. integral $D_{a}^{-(1-\beta)}f(t)$ is well-defined. □

Before we take the applications of this new derivative, we will introduce some of its basic mathematical properties.

4. PROPERTIES OF THE NEW DERIVATIVES

It follows directly from the definitions 6 and 7 that, if $f(t) = C$, then

$$^{SABC*}{}_aD_t^{\alpha,\beta} f(t) = 0 \text{ and } ^{SABR*}{}_aD_t^{\alpha,\beta} f(t) = 0.$$

The new derivatives are the generalization of Atangana Baleanu fractional derivatives because these can be obtained from the new derivatives by appropriate substitution stated in the following property.

Theorem 1. *If $\beta = 1$, the derivatives $^{SABC*}{}_aD_t^{\alpha,\beta} f(t)$ are equal to $^{ABC}{}_aD_t^{\alpha} f(t)$ and $^{SABR*}{}_aD_t^{\alpha,\beta} f(t)$ are equal to $^{ABR}{}_aD_t^{\alpha} f(t)$.*

Proof. If $\beta = 1$, the equations (7) and (8) respectively reduces to

$$^{SABC*}{}_aD_t^{\alpha,1} f(t) = \frac{B(\alpha,1)}{1-\alpha} \int_a^t f'(\tau) E_{\alpha,1}\left[-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right] d\tau, \tag{10}$$

and

$$^{SABR*}{}_aD_t^{\alpha,1} f(t) = \frac{B(\alpha,1)}{1-\alpha} \frac{d}{dt} \int_a^t f(\tau) E_{\alpha,1}\left[-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right] d\tau. \tag{11}$$

Since $B(\alpha, 1) = N(\alpha)$, the formulas in the equations (10) and (11) are respectively equivalent to the formulas of $^{ABC}{}_aD_t^{\alpha} f(t)$ and $^{ABR}{}_aD_t^{\alpha} f(t)$ given in the equations (3) and (4).
Thus, we have obtained ABC and ABR derivatives respectively from modified SABC and modified SABR derivatives by substituting $\beta = 1$. □

Now, in the next theorem, we present the relation between both derivatives defined by equations (7) and (8) using Laplace transform.

Theorem 2. *Let $f \in H^1(a,b), a < b$ and $\alpha, \beta \in [0,1]$ then the following relation is obtained:*

$$^{SABC*}{}_0D_t^{\alpha,\beta} f(t) = {}^{SABR*}{}_0D_t^{\alpha,\beta} f(t) + H_0(t) \tag{12}$$

Proof. If we take the Laplace transform on both sides of the equation (7) using the formula for Laplace transform, we get

$$\mathcal{L}\left\{^{SABC*}{}_0D_t^{\alpha,\beta} f(t), p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta} \mathcal{L}\left\{\int_0^t f'(\tau)(t-\tau)^{1-\beta} E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^\alpha}{2-\alpha-\beta}\right] d\tau, p\right\}.$$

Hence by formula for Laplace transform of convolution of two functions, we get

$$\mathcal{L}\left\{{}^{SABC*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\mathcal{L}\left\{f'(t),p\right\}\mathcal{L}\left\{t^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)t^{\alpha}}{2-\alpha-\beta}\right],p\right\}.$$

Using the properties of Laplace transform, we get

$$\mathcal{L}\left\{{}^{SABC*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\left[p\mathcal{L}\left\{f(t),p\right\}-f(0)\right]\frac{p^{-(2-\beta)}}{1-\frac{-\frac{\alpha+\beta-1}{2-\alpha-\beta}}{p^{\alpha}}}, \left|\frac{-\frac{\alpha+\beta-1}{2-\alpha-\beta}}{p^{\alpha}}\right|<1.$$

$$\Rightarrow\mathcal{L}\left\{{}^{SABC*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-2}\left[p\mathcal{L}\left\{f(t),p\right\}-f(0)\right]}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}, \left|\frac{\alpha+\beta-1}{p^{\alpha}\left(2-\alpha-\beta\right)}\right|<1.$$

Therefore, we have

$$\mathcal{L}\left\{{}^{SABC*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-1}\mathcal{L}\left\{f(t),p\right\}}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}} - \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-2}f(0)}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}. \tag{13}$$

Similarly, if we take the Laplace transform on both sides of the equation (8) using the formula for Laplace transform, we get

$$\mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\mathcal{L}\left\{\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau,p\right\}.$$

Using the property of Laplace transform, we get

$$\mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}p\mathcal{L}\left\{\int_0^t f(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau,p\right\} - \left[\int_0^t f(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau\right]_{t=0}.$$

Hence by formulas for Laplace transform of convolution of two functions, we get

$$\mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}p\mathcal{L}\left\{f(t),p\right\}\mathcal{L}\left\{t^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)t^{\alpha}}{2-\alpha-\beta}\right],p\right\}-0.$$

Hence, using the property of Laplace transform, we get

$$\mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}p\mathcal{L}\left\{f(t),p\right\}\frac{p^{-(2-\beta)}}{1-\frac{-\frac{\alpha+\beta-1}{2-\alpha-\beta}}{p^{\alpha}}}.$$

$$\mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}p\frac{p^{\alpha+\beta-2}}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}\mathcal{L}\left\{f(t),p\right\}.$$

Therefore, we have

$$\mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-1}\mathcal{L}\left\{f(t),p\right\}}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}. \tag{14}$$

Using equation (13), we have

$$\mathcal{L}\left\{{}^{SABC*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\} - \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-2}f(0)}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}. \tag{15}$$

Applying the inverse Laplace transform on both sides of equation (15) we obtain

$${}^{SABC*}{}_0D_t^{\alpha,\beta}f(t) = {}^{SABR*}{}_0D_t^{\alpha,\beta}f(t) - \frac{B(\alpha,\beta)}{2-\alpha-\beta}f(0)t^{1-\beta}E_{\alpha,2-\beta}\left(-\frac{\alpha+\beta-1}{2-\alpha-\beta}t^{\alpha}\right). \tag{16}$$

This implies that

$${}^{SABC*}{}_0D_t^{\alpha,\beta}f(t) = {}^{SABR*}{}_0D_t^{\alpha,\beta}f(t) + H_0(t).$$

where

$$H_0(t) = -\frac{B(\alpha,\beta)}{2-\alpha-\beta}f(0)t^{1-\beta}E_{\alpha,2-\beta}\left(-\frac{\alpha+\beta-1}{2-\alpha-\beta}t^{\alpha}\right).$$

This completes the proof. □

Theorem 3. *Let* $f \in H^1(a,b), a < b$ *and* $\alpha,\beta \in [0,1]$. *Then the following inequality is obtained on* $[a,b]$*:*

$$\left\|{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t)\right\|_2 < \frac{B(\alpha,\beta)}{2-\alpha-\beta}K. \tag{17}$$

Proof. For $a < b$, $f \in H^1(a,b)$ implies that $f \in L^2(a,b)$ and $\partial f \in L^2(a,b)$. Hence, we have

$$\begin{aligned}
&\left\|{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t)\right\|_2 \\
&\quad = \left\|\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau\right\|_2 \\
&\quad = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\left\|\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau\right\|_2.
\end{aligned}$$

Using $\|h(t)\|_2 = \max_{a\le t\le b}|h(t)|$, and the lemma of behaviour of Mittag Leffler function [19], we have

$$\begin{aligned}&\left\|(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]\right\|_2\\ &\quad=\max_{a\le t-\tau\le b}\left|(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]\right|<K_0.\end{aligned}\tag{18}$$

where, K_0 is constant.
Therefore the above equation becomes

$$\begin{aligned}\left\|{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t)\right\|_2&<K_0\frac{B(\alpha,\beta)}{2-\alpha-\beta}\left\|\frac{d}{dt}\int_0^t f(\tau)d\tau\right\|_2\\ &=K_0\frac{B(\alpha,\beta)}{2-\alpha-\beta}\|f(t)\|_2.\end{aligned}$$

This implies that

$$\left\|{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t)\right\|_2<\frac{B(\alpha,\beta)}{2-\alpha-\beta}K.$$

where

$$K=K_0\|f(t)\|_2.$$

This completes the proof. □

Theorem 4. *The modified SAB derivative in Riemann and Caputo sense satisfies the Lipschitz condition. In other words, for a given functions f and h, the following inequalities are established*

$$\left\|{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t)-{}^{SABR*}{}_0D_t^{\alpha,\beta}h(t)\right\|_2\le H(t)\|f(t)-h(t)\|_2.\tag{19}$$

and

$$\left\|{}^{SABC*}{}_0D_t^{\alpha,\beta}f(t)-{}^{SABC*}{}_0D_t^{\alpha,\beta}h(t)\right\|_2\le H(t)\|f(t)-h(t)\|_2.\tag{20}$$

Proof. From the definition (8) of modified SABR fractional derivative, we have

$$\begin{aligned}&\left\|{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t)-{}^{SABR*}{}_0D_t^{\alpha,\beta}h(t)\right\|_2\\ &=\left\|\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau\right.\\ &\quad\left.-\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_0^t h(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau\right\|_2.\end{aligned}$$

Using the inequality (18), we will get a positive constant ϕ_1 such that:

$$\left\| {}^{SABR*}{}_0D_t^{\alpha,\beta} f(t) - {}^{SABR*}{}_0D_t^{\alpha,\beta} h(t) \right\|_2 < \frac{B(\alpha,\beta)\phi_1}{2-\alpha-\beta} \left\| \int_0^t f(\tau)d\tau - \int_0^t h(\tau)d\tau \right\|_2 . \quad (21)$$

Using the Lipschitz condition of the first order derivative, the following result is obtained

$$\begin{aligned} \left\| {}^{SABR*}{}_0D_t^{\alpha,\beta} f(t) - {}^{SABR*}{}_0D_t^{\alpha,\beta} h(t) \right\|_2 & \\ &< \frac{B(\alpha,\beta)\phi_1}{2-\alpha-\beta} \left\| f(t) - h(t) \right\|_2 t \\ &= H(t) \left\| f(t) - h(t) \right\|_2 . \end{aligned}$$

where,

$$H(t) = \frac{B(\alpha,\beta)\phi_1}{2-\alpha-\beta} t.$$

which produces the requested result. The proof of (20) can be obtained similarly. □

Theorem 5. *For given any natural number $n \geq 1$, the modified SABC fractional derivative of bi-order (α, β), $\alpha, \beta \in (0,1)$ of t^n is given by*

$$\begin{aligned} {}^{SABC*}{}_aD_t^{\alpha,\beta} (t^n) = \frac{B(\alpha,\beta)}{2-\alpha-\beta} \sum_{k=1}^{n} \frac{n!}{(n-k)!} a^{n-k} (t-a)^{k+1-\beta} \\ E_{\alpha,k+2-\beta} \left(\frac{1-\alpha-\beta}{2-\alpha-\beta} (t-a)^{\alpha} \right). \end{aligned} \quad (22)$$

Proof. From the definition (7) of modified SABC fractional derivative, we have

$$\begin{aligned} {}^{SABC*}{}_aD_t^{\alpha,\beta} (t^n) &= \frac{B(\alpha,\beta)}{2-\alpha-\beta} \int_a^t n\tau^{n-1} (t-\tau)^{1-\beta} \\ & \quad E_{\alpha,2-\beta} \left(\frac{1-\alpha-\beta}{2-\alpha-\beta} (t-\tau)^{\alpha} \right) d\tau \\ &= \frac{B(\alpha,\beta)}{2-\alpha-\beta} \sum_{n=0}^{\infty} \left(\frac{1-\alpha-\beta}{2-\alpha-\beta} \right)^n \\ & \quad n \int_a^t \tau^{n-1} \frac{(t-\tau)^{\alpha n-\beta+1}}{\Gamma(\alpha n - \beta + 2)} d\tau \end{aligned}$$

Hence, using integration by parts rule, we get

$$
\begin{aligned}
{}^{SABC*}{}_{a}D_{t}^{\alpha,\beta}\left(t^{n}\right) &= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n} n\left\{\left[\tau^{n-1}\frac{(t-\tau)^{\alpha n-\beta+2}}{\Gamma(\alpha n-\beta+3)}\right]_{\tau=t}^{\tau=a}\right. \\
&\quad \left. +\int_{a}^{t}(n-1)\tau^{n-2}\frac{(t-\tau)^{\alpha n-\beta+2}}{\Gamma(\alpha n-\beta+3)}d\tau\right\} \\
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n} n\left\{a^{n-1}\frac{(t-a)^{\alpha n-\beta+2}}{\Gamma(\alpha n-\beta+3)}\right. \\
&\quad +(n-1)\left\{\left[\tau^{n-2}\frac{(t-\tau)^{\alpha n-\beta+3}}{\Gamma(\alpha n-\beta+4)}\right]_{\tau=t}^{\tau=a}\right. \\
&\quad \left.\left. +\int_{a}^{t}(n-2)\tau^{n-3}\frac{(t-\tau)^{\alpha n-\beta+3}}{\Gamma(\alpha n-\beta+4)}d\tau\right\}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow {}^{SABC*}{}_{a}D_{t}^{\alpha,\beta}\left(t^{n}\right) &= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\left\{na^{n-1}\frac{(t-a)^{\alpha n-\beta+2}}{\Gamma(\alpha n-\beta+3)}\right. \\
&\quad +n(n-1)a^{n-2}\frac{(t-a)^{\alpha n-\beta+3}}{\Gamma(\alpha n-\beta+4)} \\
&\quad +n(n-1)(n-2)\left\{\left[\tau^{n-3}\frac{(t-\tau)^{\alpha n-\beta+4}}{\Gamma(\alpha n-\beta+5)}\right]_{\tau=t}^{\tau=a}\right. \\
&\quad \left.\left. +\int_{a}^{t}(n-3)\tau^{n-4}\frac{(t-\tau)^{\alpha n-\beta+4}}{\Gamma(\alpha n-\beta+5)}d\tau\right\}\right\} \\
&= \cdots = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\left\{\frac{n!}{(n-1)!}a^{n-1}\right. \\
&\quad \frac{(t-a)^{\alpha n-\beta+2}}{\Gamma(\alpha n-\beta+3)}+\frac{n!}{(n-2)!}a^{n-2}\frac{(t-a)^{\alpha n-\beta+3}}{\Gamma(\alpha n-\beta+4)} \\
&\quad +\frac{n!}{(n-3)!}a^{n-3}\frac{(t-a)^{\alpha n-\beta+4}}{\Gamma(\alpha n-\beta+5)}+\cdots+\frac{n!}{1!} \\
&\quad \left.\left\{\left[\tau\frac{(t-\tau)^{\alpha n-\beta+n}}{\Gamma(\alpha n-\beta+n+1)}\right]_{\tau=t}^{\tau=a}+\int_{a}^{t}1\frac{(t-\tau)^{\alpha n-\beta+n}}{\Gamma(\alpha n-\beta+n+1)}d\tau\right\}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\left\{\frac{n!}{(n-1)!}a^{n-1}\right.\\
&\qquad \frac{(t-a)^{\alpha n-\beta+2}}{\Gamma(\alpha n-\beta+3)} + \frac{n!}{(n-2)!}a^{n-2}\frac{(t-a)^{\alpha n-\beta+3}}{\Gamma(\alpha n-\beta+4)}\\
&+\frac{n!}{(n-3)!}a^{n-3}\frac{(t-a)^{\alpha n-\beta+4}}{\Gamma(\alpha n-\beta+5)}+\cdots\\
&\left.+\frac{n!}{1!}a\frac{(t-a)^{\alpha n-\beta+n}}{\Gamma(\alpha n-\beta+n+1)}+\frac{n!}{0!}\frac{(t-a)^{\alpha n-\beta+n+1}}{\Gamma(\alpha n-\beta+n+2)}\right\}\\
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\\
&\left\{\frac{n!}{(n-1)!}a^{n-1}(t-a)^{2-\beta}E_{\alpha,3-\beta}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}(t-a)^{\alpha}\right)\right.\\
&+\frac{n!}{(n-2)!}a^{n-2}(t-a)^{3-\beta}E_{\alpha,4-\beta}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}(t-a)^{\alpha}\right)\\
&+\frac{n!}{(n-3)!}a^{n-3}(t-a)^{4-\beta}E_{\alpha,5-\beta}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}(t-a)^{\alpha}\right)+\cdots\\
&+\frac{n!}{1!}a(t-a)^{n-\beta}E_{\alpha,n+1-\beta}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}(t-a)^{\alpha}\right)\\
&\left.+\frac{n!}{0!}(t-a)^{n+1-\beta}E_{\alpha,n+2-\beta}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}(t-a)^{\alpha}\right)\right\}.
\end{aligned}
$$

This completes the proof. □

5. NEW MODIFIED FRACTIONAL INTEGRAL

Let for a natural number n, f be an n-times differentiable and $f^{(k)}(0) = 0$, for $k = 1, 2, 3, \ldots, n$, then by observation we get

$$
{}^{SABC*}{}_{0}D_{t}^{\alpha,\beta}\left[\frac{d^n}{dt^n}f(t)\right] = \frac{d^n}{dt^n}\left[{}^{SABR*}{}_{0}D_{t}^{\alpha,\beta}f(t)\right]. \tag{23}
$$

Let

$$
{}^{SABR*}{}_{0}D_{t}^{\alpha,\beta}f(t) = u(t). \tag{24}
$$

$$
\Rightarrow \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau = u(t).
$$

$$
\begin{aligned}
\Rightarrow\mathcal{L}\Bigg\{&\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{1-\beta}\\
&E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau, p\Bigg\} = \mathcal{L}\left\{u(t),p\right\}.
\end{aligned}
$$

Using the convolution theorem, we get

$$\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-2}\left[p\mathcal{L}\left\{f(t),p\right\}\right]}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}=\mathcal{L}\left\{u(t),p\right\}.$$

$$\frac{B(\alpha,\beta)}{2-\alpha-\beta}p^{\alpha+\beta-1}\mathcal{L}\left\{f(t),p\right\}=p^{\alpha}\mathcal{L}\left\{u(t),p\right\}+\frac{\alpha+\beta-1}{2-\alpha-\beta}\mathcal{L}\left\{u(t),p\right\}.$$

$$\Rightarrow\mathcal{L}\left\{f(t),p\right\}=\frac{2-\alpha-\beta}{B(\alpha,\beta)}\frac{1}{p^{\beta-1}}\mathcal{L}\left\{u(t),p\right\}+\frac{\alpha+\beta-1}{B(\alpha,\beta)}\frac{1}{p^{\alpha+\beta-1}}\mathcal{L}\left\{u(t),p\right\}.$$

By taking the inverse Laplace transform, we get the unique solution

$$f(t)=\begin{cases}\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(\beta-1)}\int_0^t u(y)(t-y)^{\beta-2}dy+ & \\ \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha+\beta-1)}\int_0^t u(y)(t-y)^{\alpha+\beta-2}dy & \beta\neq 1\\ \frac{1-\alpha}{N(\alpha)}u(t)+\frac{\alpha}{N(\alpha)\Gamma(\alpha)}\int_0^t u(y)(t-y)^{\alpha-1}dy & \beta=1\end{cases}. \tag{25}$$

Definition 8. *The modified S fractional integral in two parameters corresponds to the new fractional derivative with non-local and non-singular kernel is defined as*

$${}^{S*}_{\ a}I_t^{\alpha,\beta}f(t)=\begin{cases}\frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(\beta-1)}\int_a^t f(y)(t-y)^{\beta-2}dy+ & \\ \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha+\beta-1)}\int_a^t f(y)(t-y)^{\alpha+\beta-2}dy & \beta\neq 1\\ \frac{1-\alpha}{N(\alpha)}f(t)+\frac{\alpha}{N(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy & \beta=1\end{cases}. \tag{26}$$

When $\alpha=0,\beta=1$ we recover the initial function, and if $\alpha=1,\beta=1$, we obtain the ordinary integral.

Remark 1. *The formula of ${}^{S*}_{\ a}I_t^{\alpha,\beta}f(t)$ can be expressed as*

$${}^{S*}_{\ a}I_t^{\alpha,\beta}f(t)=\begin{cases}\frac{2-\alpha-\beta}{B(\alpha,\beta)}{}^{RL}_{\ a}I_t^{\beta-1}f(t)+\frac{\alpha+\beta-1}{B(\alpha,\beta)}{}^{RL}_{\ a}I_t^{\alpha+\beta-1}f(t) & \beta\neq 1\\ \frac{1-\alpha}{N(\alpha)}f(t)+\frac{\alpha}{N(\alpha)}{}^{RL}_{\ a}I_t^{\alpha}f(t) & \beta=1\end{cases}. \tag{27}$$

Theorem 6. *The modified SABR fractional derivative can be expressed as*

$${}^{SABR*}{}_aD_t^{\alpha,\beta}f(t)=\frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\left({}^{RL}_{\ a}I_t^{\alpha n-\beta+1}f(t)\right). \tag{28}$$

Proof. As two parametric Mittag-Leffler function $E_{\alpha,\beta}(x)$ is an entire function of x, the modified SABR fractional derivative will be rewritten as follows:

$${}^{SABR*}{}_aD_t^{\alpha,\beta}f(t)=\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_a^t f(\tau)\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\frac{(t-\tau)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)}d\tau.$$

$$\Rightarrow {}^{SABR*}{}_{a}D_{t}^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\frac{d}{dt}\left(\frac{1}{\Gamma(\alpha n-\beta+2)}\int_{a}^{t}f(\tau)(t-\tau)^{\alpha n-\beta+1}d\tau\right).$$

$$\Rightarrow {}^{SABR*}{}_{a}D_{t}^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\frac{d}{dt}\left({}_{a}^{RL}I_{t}^{\alpha n-\beta+2}f(t)\right),$$

This implies the relation

$${}^{SABR*}{}_{a}D_{t}^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n-\beta+1}f(t)\right).$$

where the integral ${}_{a}^{RL}I_{t}$ is the standard Riemann–Liouville fractional integral defined in (11). □

Theorem 7. *The modified SABC fractional derivative can be expressed as*

$${}^{SABC*}{}_{a}D_{t}^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n-\beta+2}f'(t)\right). \quad (29)$$

Proof. As two parametric Mittag-Leffler function $E_{\alpha,\beta}(x)$ is an entire function of x, the modigied SABC fractional derivative will be rewritten as follows:

$${}^{SABC*}{}_{a}D_{t}^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_{a}^{t}f'(\tau)\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\frac{(t-\tau)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)}d\tau.$$

$$\Rightarrow {}^{SABC*}{}_{a}D_{t}^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\left(\frac{1}{\Gamma(\alpha n-\beta+2)}\int_{a}^{t}f'(\tau)(t-\tau)^{\alpha n-\beta+1}d\tau\right).$$

$$\Rightarrow {}^{SABR*}{}_{a}D_{t}^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n-\beta+2}f'(t)\right),$$

This implies the relation

$${}^{SABR*}{}_{a}D_{t}^{\alpha,\beta}f(t) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n-\beta+2}f'(t)\right).$$

where the integral ${}_{a}^{RL}I_{t}$ is the standard Riemann–Liouville fractional integral defined in (11). □

Theorem 6 allows us to derive the formula for Laplace transforms of modified SABR fractional derivative in a way different from that used in theorem 2. The following identity is equation (14) in theorem 2:

$$\mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-1}\mathcal{L}\{f(t),p\}}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}.$$

Now, we will prove this by taking Laplace transforms of modified SABR fractional derivative using the series in (28) as follows:

$$\begin{aligned}&\mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\}\\&=\frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\mathcal{L}\left\{\left({}_0^{RL}I_t^{\alpha n-\beta+1}f(t)\right),p\right\}.\end{aligned}$$

Using the formula for the Laplace transform of the Riemann Liouville fractional integral, we get

$$\begin{aligned}\mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\} =&\frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\\&\left(p^{-\alpha n+\beta-1}\mathcal{L}\{f(t),p\}\right).\end{aligned}\tag{30}$$

Hence (30) reduces to

$$\mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{p^{\alpha}(2-\alpha-\beta)}\right)^n p^{\beta-1}\mathcal{L}\{f(t),p\}.$$

$$\Rightarrow \mathcal{L}\left\{{}^{SABR*}{}_0D_t^{\alpha,\beta}f(t),p\right\} = \frac{B(\alpha,\beta)}{2-\alpha-\beta}p^{\beta-1}\mathcal{L}\{f(t),p\}\frac{1}{1-\frac{1-\alpha-\beta}{p^{\alpha}(2-\alpha-\beta)}}.$$

This implies the identity in equation (14) as required.

Theorem 8. *The modified S. fractional integral operator* ${}_a^{S*}I_t^{\alpha,\beta}$ *defined by* (25) *is an inverse to the modified SABR fractional differential operator* ${}^{SABR*}{}_aD_t^{\alpha,\beta}$ *defined by* (8).
In other words,

$${}^{SABR*}{}_aD_t^{\alpha,\beta}\left({}_a^{S*}I_t^{\alpha,\beta}f(t)\right) = f(t).\tag{31}$$

and

$${}_a^{S*}I_t^{\alpha,\beta}\left({}^{SABR*}{}_aD_t^{\alpha,\beta}f(t)\right) = f(t).\tag{32}$$

Proof. We will proceed the proof as follows:
Case (1): Suppose $\beta \neq 1$. Then from the remark 1 and definition 7 we have

$$\begin{aligned}{}^{SABR*}{}_aD_t^{\alpha,\beta}\left({}_a^{S*}I_t^{\alpha,\beta}f(t)\right) = {}^{SABR*}{}_aD_t^{\alpha,\beta}&\left(\frac{2-\alpha-\beta}{B(\alpha,\beta)}{}_a^{RL}I_t^{\beta-1}\right.\\&\left.+\frac{\alpha+\beta-1}{B(\alpha,\beta)}{}_a^{RL}I_t^{\alpha+\beta-1}\right)f(t).\end{aligned}$$

Hence using equation (28), we get

$$ {}^{SABR*}{}_{a}D_t^{\alpha,\beta}\left({}_{a}^{S*}I_t^{\alpha,\beta}f(t)\right) = \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\left({}_{a}^{RL}I_t^{\alpha n-\beta+1}\right) \left(\frac{2-\alpha-\beta}{B(\alpha,\beta)}{}_{a}^{RL}I_t^{\beta-1}+\frac{\alpha+\beta-1}{B(\alpha,\beta)}{}_{a}^{RL}I_t^{\alpha+\beta-1}\right)f(t). $$

$$ {}^{SABR*}{}_{a}D_t^{\alpha,\beta}\left({}_{a}^{S*}I_t^{\alpha,\beta}f(t)\right) = \sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\left({}_{a}^{RL}I_t^{\alpha n-\beta+1}\right)\left({}_{a}^{RL}I_t^{\beta-1}\right)f(t) + \frac{\alpha+\beta-1}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\left({}_{a}^{RL}I_t^{\alpha n-\beta+1}\right)\left({}_{a}^{RL}I_t^{\alpha+\beta-1}\right)f(t). $$

$$ \Rightarrow {}^{SABR*}{}_{a}D_t^{\alpha,\beta}\left({}_{a}^{S*}I_t^{\alpha,\beta}f(t)\right) = \sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\left({}_{a}^{RL}I_t^{\alpha n}\right)f(t) - \sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n+1}\left({}_{a}^{RL}I_t^{\alpha(n+1)}\right)f(t). $$

$$ \Rightarrow {}^{SABR*}{}_{a}D_t^{\alpha,\beta}\left({}_{a}^{S*}I_t^{\alpha,\beta}f(t)\right) = \sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\left({}_{a}^{RL}I_t^{\alpha n}\right)f(t) - \sum_{n=1}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\left({}_{a}^{RL}I_t^{\alpha(n)}\right)f(t). $$

$$ \Rightarrow {}^{SABR*}{}_{a}D_t^{\alpha,\beta}\left({}_{a}^{S*}I_t^{\alpha,\beta}f(t)\right) = \left[\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\left({}_{a}^{RL}I_t^{\alpha n}\right)f(t)\right]_{n=0}. $$

This proves the identity (31).

Also, by using equation (28), we get

$$ {}_{a}^{S*}I_t^{\alpha,\beta}\left({}^{SABR*}{}_{a}D_t^{\alpha,\beta}f(t)\right) = \left(\frac{2-\alpha-\beta}{B(\alpha,\beta)}{}_{a}^{RL}I_t^{\beta-1}+\frac{\alpha+\beta-1}{B(\alpha,\beta)}{}_{a}^{RL}I_t^{\alpha+\beta-1}\right) \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\left({}_{a}^{RL}I_t^{\alpha n-\beta+1}\right)f(t). $$

$$ \Rightarrow {}_{a}^{S*}I_t^{\alpha,\beta}\left({}^{SABR*}{}_{a}D_t^{\alpha,\beta}f(t)\right) = \sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\left({}_{a}^{RL}I_t^{\beta-1}\right)\left({}_{a}^{RL}I_t^{\alpha n-\beta+1}\right)f(t) + \frac{\alpha+\beta-1}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\left({}_{a}^{RL}I_t^{\alpha+\beta-1}\right)\left({}_{a}^{RL}I_t^{\alpha n-\beta+1}\right)f(t). $$

$$\Rightarrow {}_{a}^{S*}I_{t}^{\alpha,\beta}\left({}^{SABR*}{}_{a}D_{t}^{\alpha,\beta}f(t)\right) = \sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n}\right)f(t)$$
$$-\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n+1}\left({}_{a}^{RL}I_{t}^{\alpha(n+1)}\right)f(t)$$
$$= f(t).$$

This proves the identity (32).

Case (2): Suppose $\beta = 1$. Then from the remark 1 and definition 7 we have

$$^{SABR*}{}_{a}D_{t}^{\alpha,\beta}\left({}_{a}^{S*}I_{t}^{\alpha,\beta}f(t)\right) = {}^{SABR*}{}_{a}D_{t}^{\alpha,1}\left(\frac{1-\alpha}{N(\alpha)} + \frac{\alpha}{N(\alpha)}{}_{a}^{RL}I_{t}^{\alpha}\right)f(t).$$

Hence using equation (28), we get

$$^{SABR*}{}_{a}D_{t}^{\alpha,\beta}\left({}_{a}^{S*}I_{t}^{\alpha,\beta}f(t)\right) = \frac{B(\alpha,1)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n}\right)\left(\frac{1-\alpha}{N(\alpha)} + \frac{\alpha}{N(\alpha)}{}_{a}^{RL}I_{t}^{\alpha}\right)f(t).$$

Since $B(\alpha, 1) = N(\alpha)$, we get

$$^{SABR*}{}_{a}D_{t}^{\alpha,\beta}\left({}_{a}^{S*}I_{t}^{\alpha,\beta}f(t)\right) = \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n}\right)f(t)$$
$$+\frac{\alpha}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n}\right)\left({}_{a}^{RL}I_{t}^{\alpha}\right)f(t).$$

$$\Rightarrow {}^{SABR*}{}_{a}D_{t}^{\alpha,\beta}\left({}_{a}^{S*}I_{t}^{\alpha,\beta}f(t)\right) = \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n}\right)f(t)$$
$$-\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n+1}\left({}_{a}^{RL}I_{t}^{\alpha(n+1)}\right)f(t).$$

$$\Rightarrow {}^{SABR*}{}_{a}D_{t}^{\alpha,\beta}\left({}_{a}^{S*}I_{t}^{\alpha,\beta}f(t)\right) = \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n}\right)f(t)$$
$$-\sum_{n=1}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n}\right)f(t).$$

$$\Rightarrow {}^{SABR*}{}_{a}D_{t}^{\alpha,\beta}\left({}_{a}^{S*}I_{t}^{\alpha,\beta}f(t)\right) = \left[\left(\frac{-\alpha}{1-\alpha}\right)^{n}\left({}_{a}^{RL}I_{t}^{\alpha n}\right)f(t)\right]_{n=0}.$$

This proves the identity (31).

Also, by using equation (28), we get

$$
{}^{S*}_{\ a}I_t^{\alpha,\beta}\left({}^{SABR*}{}_aD_t^{\alpha,\beta}f(t)\right) = \left(\frac{1-\alpha}{N(\alpha)} + \frac{\alpha}{N(\alpha)}{}^{RL}_{\ a}I_t^{\alpha}\right)
\frac{B(\alpha,1)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n\left({}^{RL}_{\ a}I_t^{\alpha n}\right)f(t).
$$

Since $B(\alpha,1) = N(\alpha)$, we get

$$
\Rightarrow {}^{S*}_{\ a}I_t^{\alpha,\beta}\left({}^{SABR*}{}_aD_t^{\alpha,\beta}f(t)\right) = \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n\left({}^{RL}_{\ a}I_t^{\alpha n}\right)f(t)
+ \frac{\alpha}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n\left({}^{RL}_{\ a}I_t^{\alpha n}\right)\left({}^{RL}_{\ a}I_t^{\alpha}\right)f(t).
$$

$$
\begin{aligned}
\Rightarrow {}^{S*}_{\ a}I_t^{\alpha,\beta}\left({}^{SABR*}{}_aD_t^{\alpha,\beta}f(t)\right) &= \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n\left({}^{RL}_{\ a}I_t^{\alpha n}\right)f(t) \\
&\quad - \sum_{n=1}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n\left({}^{RL}_{\ a}I_t^{\alpha n}\right)f(t) \\
&= f(t).
\end{aligned}
$$

This proves the identity (32). □

Example 1. *In this example, we will verify the result of theorem 8 for the function* $f(t) = 1$. *Let us calculate* ${}^{SABR*}{}_aD_t^{\alpha,\beta}(1)$ *as below:*

$$
\begin{aligned}
{}^{SABR*}{}_aD_t^{\alpha,\beta}(1) &= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_a^t (t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau \\
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\int_a^t\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\frac{(t-\tau)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)}d\tau \\
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\int_a^t\frac{(t-\tau)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)}d\tau \\
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{d}{dt}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\frac{(t-a)^{\alpha n-\beta+2}}{\Gamma(\alpha n-\beta+3)} \\
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\frac{(t-a)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)} \\
&= \frac{B(\alpha,\beta)}{2-\alpha-\beta}(t-a)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-a)^{\alpha}}{2-\alpha-\beta}\right].
\end{aligned}
$$

Hence, by using equation (27)*, we get*

$$
\begin{aligned}
{}_{a}^{S*}I_t^{\alpha,\beta}\left({}^{SABR*}{}_{a}D_t^{\alpha,\beta}\right)(1) = {}_{a}^{RL}I_t^{\beta-1}(t-a)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-a)^{\alpha}}{2-\alpha-\beta}\right] \\
+\frac{\alpha+\beta-1}{2-\alpha-\beta}{}_{a}^{RL}I_t^{\alpha+\beta-1}(t-a)^{1-\beta} \\
E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-a)^{\alpha}}{2-\alpha-\beta}\right]
\end{aligned}
$$

$$
\begin{aligned}
{}_{a}^{S*}I_t^{\alpha,\beta}\left({}^{SABR*}{}_{a}D_t^{\alpha,\beta}\right)(1) = {}_{a}^{RL}I_t^{\beta-1}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\frac{(t-a)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)} \\
+\frac{\alpha+\beta-1}{2-\alpha-\beta}{}_{a}^{RL}I_t^{\alpha+\beta-1} \\
\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\frac{(t-a)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)}
\end{aligned}
$$

Using the definition of Riemann-Liouville fractional integral [6], we get

$$
\begin{aligned}
{}_{a}^{S*}I_t^{\alpha,\beta}\left({}^{SABR*}{}_{a}D_t^{\alpha,\beta}\right)(1) = \frac{1}{\Gamma(\beta-1)}\int_a^t(\tau-a)^{\beta-2}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n \\
\frac{(\tau-a)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)}d\tau+\frac{\alpha+\beta-1}{2-\alpha-\beta}\frac{1}{\Gamma(\alpha+\beta-1)} \\
\int_a^t(\tau-a)^{\alpha+\beta-2}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\frac{(\tau-a)^{\alpha n-\beta+1}}{\Gamma(\alpha n-\beta+2)}d\tau
\end{aligned}
$$

Using the definition and properties of the gamma function and the beta function, we will get the conclusion: ${}_{a}^{S*}I_t^{\alpha,\beta}\left({}^{SABR*}{}_{a}D_t^{\alpha,\beta}\right)(1)=1.$

Theorem 9. *The modified S. fractional integral operator* ${}_{a}^{S*}I_t^{\alpha,\beta}$ *defined by* (26) *and the modified S A BC fractional differential operator* ${}^{SABR*}{}_{a}D_t^{\alpha,\beta}$ *defined by* (7) *satisfy the Newton–Leibniz formula. In other words,*

$$
{}_{a}^{S*}I_t^{\alpha,\beta}\left({}^{SABC*}{}_{a}D_t^{\alpha,\beta}f(t)\right)=f(t)-f(a). \tag{33}
$$

Proof. We will proceed the proof as follows:
Case (1): Suppose $\beta\neq 1$. Then from the remark 1 and identity (29), we have

$$
\begin{aligned}
{}_{a}^{S*}I_t^{\alpha,\beta}\left({}^{SABC*}{}_{a}D_t^{\alpha,\beta}f(t)\right) = \left(\frac{2-\alpha-\beta}{B(\alpha,\beta)}{}_{a}^{RL}I_t^{\beta-1}+\frac{\alpha+\beta-1}{B(\alpha,\beta)}{}_{a}^{RL}I_t^{\alpha+\beta-1}\right) \\
\frac{B(\alpha,\beta)}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n\left({}_{a}^{RL}I_t^{\alpha n-\beta+2}\right)f'(t).
\end{aligned}
$$

$$\Rightarrow {}^{S*}_{a}I_t^{\alpha,\beta}\left({}^{SABC*}{}_{a}D_t^{\alpha,\beta}f(t)\right) = \sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n \left({}^{RL}_{a}I_t^{\beta-1}\right)\left({}^{RL}_{a}I_t^{\alpha n-\beta+2}\right) f'(t)$$
$$+\frac{\alpha+\beta-1}{2-\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n \left({}^{RL}_{a}I_t^{\alpha+\beta-1}\right)\left({}^{RL}_{a}I_t^{\alpha n-\beta+2}\right) f'(t).$$

$$\begin{aligned}\Rightarrow {}^{S*}_{a}I_t^{\alpha,\beta}\left({}^{SABC*}{}_{a}D_t^{\alpha,\beta}f(t)\right) &= \sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^n \left({}^{RL}_{a}I_t^{\alpha n+1}\right) f'(t) \\ &\quad -\sum_{n=0}^{\infty}\left(\frac{1-\alpha-\beta}{2-\alpha-\beta}\right)^{n+1} \left({}^{RL}_{a}I_t^{\alpha(n+1)+1}\right) f'(t) \\ &= \left({}^{RL}_{a}I_t^{1}\right) f'(t) \\ &= f(t)-f(a).\end{aligned}$$

This proves the identity (33).

Case (2): Suppose $\beta = 1$. Then from the remark 1 and identity (29), we have

$$\begin{aligned}{}^{S*}_{a}I_t^{\alpha,\beta}\left({}^{SABC*}{}_{a}D_t^{\alpha,\beta}f(t)\right) &= \left(\frac{1-\alpha}{N(\alpha)}+\frac{\alpha}{N(\alpha)}{}^{RL}_{a}I_t^{\alpha}\right) \\ &\quad \frac{B(\alpha,1)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n \left({}^{RL}_{a}I_t^{\alpha n+1}\right) f'(t).\end{aligned}$$

Since $B(\alpha, 1) = N(\alpha)$, we get

$$\Rightarrow {}^{S*}_{a}I_t^{\alpha,\beta}\left({}^{SABR*}{}_{a}D_t^{\alpha,\beta}f(t)\right) = \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n \left({}^{RL}_{a}I_t^{\alpha n+1}\right) f'(t)$$
$$+\frac{\alpha}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n \left({}^{RL}_{a}I_t^{\alpha n+1}\right)\left({}^{RL}_{a}I_t^{\alpha}\right) f'(t).$$

$$\begin{aligned}\Rightarrow {}^{S*}_{a}I_t^{\alpha,\beta}\left({}^{SABR*}{}_{a}D_t^{\alpha,\beta}f(t)\right) &= \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n \left({}^{RL}_{a}I_t^{\alpha n+1}\right) f'(t) \\ &\quad -\sum_{n=1}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n \left({}^{RL}_{a}I_t^{\alpha n+1}\right) f'(t) \\ &= \left({}^{RL}_{a}I_t^{1}\right) f'(t) \\ &= f(t)-f(a).\end{aligned}$$

This proves the identity (33). □

6. SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, we study some simple but useful fractional differential equations.

Lemma 2. *Let* $0 < \alpha, \beta < 1$ *and let* f *be a solution of the following fractional differential equation,*

$$^{SABC*}{}_0D_t^{\alpha,\beta} f(t) = 0, \;\; t \geq 0.$$

Then, the function f *is a constant function.*

Proof. Taking Laplace transform on both sides of differential equation, we get

$$\mathcal{L}\left\{^{SABC*}{}_0D_t^{\alpha,\beta} f(t), p\right\} = 0, \;\; p > 0. \tag{34}$$

Hence from equation (13), we have

$$\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-1}\mathcal{L}\{f(t),p\}}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}} - \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-2}f(0)}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}} = 0, \;\; p > 0.$$

$$\Rightarrow \mathcal{L}\{f(t),p\} = \frac{1}{p}f(0).$$

After taking inverse Laplace transform on both sides, we get

$$f(t) = f(0) \quad \forall t \geq 0.$$

This proves that f is a constant function. □

Theorem 10. *Let* $0 < \alpha, \beta < 1$. *Then, the unique solution of the initial value problem:*

$$^{SABC*}{}_0D_t^{\alpha,\beta} f(t) = \sigma(t), \;\; t \geq 0. \tag{35}$$

$$f(0) = f_0; \tag{36}$$

is given by

$$f(t) = f_0 + P_{\alpha,\beta}\int_0^t \sigma(y)(t-y)^{\beta-2}dy + Q_{\alpha,\beta}\int_0^t \sigma(y)(t-y)^{\alpha+\beta-2}dy \tag{37}$$

where

$$P_{\alpha,\beta} = \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(\beta-1)}, Q_{\alpha,\beta} = \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha+\beta-1)}. \tag{38}$$

Proof. Suppose that the initial value problem defined in the equations (35)) and (36) has two solutions, $f_1(t)$ and $f_2(t)$. Then, we have

$$^{SABC*}{}_0D_t^{\alpha,\beta} f_1(t) - {}^{SABC*}{}_0D_t^{\alpha,\beta} f_2(t) = \sigma(t) - \sigma(t) = 0, \;\; t \geq 0.$$

and

$$f_1(0) - f_2(0) = f_0 - f_0 = 0.$$

This means that

$$
\begin{aligned}
&{}^{SABC*}{}_0D_t^{\alpha,\beta}\left(f_1-f_2\right)f(t)=0,\ \ t\geq 0\\
&\text{and}\\
&\left(f_1-f_2\right)f(0)=0.
\end{aligned}
$$

Hence, by Lemma 2, we get the conclusion that $f_1-f_2=0$. Therefore, $f_1(t)=f_2(t)$ for all $t\geq 0$. From equation (13), by taking Laplace transform on both sides of differential equation (35), we get

$$
\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-1}\mathcal{L}\left\{f(t),p\right\}}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}-\frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-2}f(0)}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}=\mathcal{L}\left\{\sigma(t),p\right\},\ \ p>0.
$$

$$
\begin{aligned}
\mathcal{L}\left\{f(t),p\right\}=\frac{1}{p}f(0) &+ \frac{2-\alpha-\beta}{B(\alpha,\beta)}\frac{1}{p^{\beta-1}}\mathcal{L}\left\{\sigma(t),p\right\}\\
&+ \frac{\alpha+\beta-1}{B(\alpha,\beta)}\frac{1}{p^{\alpha+\beta-1}}\mathcal{L}\left\{\sigma(t),p\right\}.
\end{aligned}
$$

After taking inverse Laplace transform on both sides, we get

$$
\begin{aligned}
f(t)=f_0 &+ \frac{2-\alpha-\beta}{B(\alpha,\beta)\Gamma(\beta-1)}\int_0^t\sigma(y)(t-y)^{\beta-2}dy\\
&+ \frac{\alpha+\beta-1}{B(\alpha,\beta)\Gamma(\alpha+\beta-1)}\int_0^t\sigma(y)(t-y)^{\alpha+\beta-2}dy.
\end{aligned}
$$

□

Remark 2. *For* $\beta=1$, *we have*

$$
f(t)=f_0+\frac{1-\alpha}{N(\alpha)}\sigma(t)+\frac{\alpha}{N(\alpha)\Gamma(\alpha)}\int_0^t\sigma(y)(t-y)^{\alpha-1}dy. \tag{39}
$$

Remark 3. *For* $\alpha=\beta=1$, *we have the solution of* (35) *is the usual antiderivative of* $\sigma(t)$.

Corollary 1. *Let* $0<\alpha,\beta<1$. *Then, the unique solution of the initial value problem:*

$$
\begin{aligned}
&{}^{SABC*}{}_0D_t^{\alpha,\beta}f(t)=\lambda f(t)+u(t),\ \ t\geq 0 \qquad (40)\\
&f(0)=f_0.
\end{aligned}
$$

where $\lambda\in\mathbb{R},\lambda\neq 0$ *(*$\lambda=0$ *corresponds to the case in theorem 10 is given by*

$$
\begin{aligned}
f(t)- &\ \lambda\ \ P_{\alpha,\beta}\int_0^t f(y)(t-y)^{\beta-2}dy-\lambda Q_{\alpha,\beta}\int_0^t f(y)(t-y)^{\alpha+\beta-2}dy\\
=f_0 &+ P_{\alpha,\beta}\int_0^t u(y)(t-y)^{\beta-2}dy+Q_{\alpha,\beta}\int_0^t u(y)(t-y)^{\alpha+\beta-2}dy. \qquad (41)
\end{aligned}
$$

Proof. From Theorem 10, it is clear that solving equation (40) is equivalent to find a function f such that

$$f(t) = f_0 + P_{\alpha,\beta} \int_0^t (\lambda f + u)(y)(t-y)^{\beta-2} dy + Q_{\alpha,\beta} \int_0^t (\lambda f + u)(y)(t-y)^{\alpha+\beta-2} dy, \ \ t \geq 0.$$

This gives the unique solution as given in the equation (41). □

7. APPLICATIONS

7.1 Thermal Science

Fractional time Fourier's law equation The Fourier's law [10, 12, 15] is defined by the classical parabolic equation as

$$\chi \frac{\partial^2 T(x,t)}{\partial x^2} - \frac{\partial T(x,t)}{\partial x} = 0 \tag{42}$$

where

$$\chi = \frac{k}{\rho C_p}$$

is the thermal diffusivity, with the thermal conductivity k, density ρ, and the specific heat capacity C_p. T is the temperature conduction in a planar medium with constant properties.

Assuming that time derivative is fractional and space derivative is ordinary. Then, taking into consideration equation (42), the temporal fractional equation will be as follows:

$$^{SABC*}{}_0D_t^{\alpha,\beta} T(x,t) - \chi \frac{\partial^2 T(x,t)}{\partial x^2} = 0 \tag{43}$$

A particular solution to the equation (43) will be in the following form

$$T(x,t) = T_0 e^{-i\bar{k}x} u(t) \tag{44}$$

Substituting the equation (44) into the equation (43), we get

$$T_0 e^{-i\bar{k}x SABC*}{}_0D_t^{\alpha,\beta} u(t) - \chi(-i\bar{k})^2 T_0 e^{-i\bar{k}x} u(t) = 0$$

$$\Rightarrow T_0 e^{-i\bar{k}x} \left[^{SABC*}{}_0D_t^{\alpha,\beta} u(t) + \chi \bar{k}^2 u(t) \right] = 0$$

$$^{SABC*}{}_0D_t^{\alpha,\beta} u(t) + \omega u(t) = 0 \tag{45}$$

where $\omega = \chi \bar{k}^2 \sigma_t^{2-\alpha-\beta}$ is the angular frequency.
The numerical approximation to the equation (45) is given by

$$\frac{B(\alpha,\beta)}{2-\alpha-\beta}\int_0^t u'(\tau)(t-\tau)^{1-\beta}E_{\alpha,2-\beta}\left[-\frac{(\alpha+\beta-1)(t-\tau)^{\alpha}}{2-\alpha-\beta}\right]d\tau+\omega u(t)=0$$

By taking Laplace transform on both sides, we get

$$\mathcal{L}\left\{{}^{SABC*}{}_0D_t^{\alpha,\beta}u(t),p\right\}+\omega\mathcal{L}\left\{u(t),p\right\}=0.$$

$$\Rightarrow \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{p^{\alpha+\beta-2}\left[p\mathcal{L}\left\{u(t),p\right\}-u(0)\right]}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}+\omega\mathcal{L}\left\{u(t),p\right\}=0.$$

$$\Rightarrow \frac{B(\alpha,\beta)}{2-\alpha-\beta}\frac{\left[p^{\alpha+\beta-1}\mathcal{L}\left\{u(t),p\right\}-u(0)p^{\alpha+\beta-2}\right]}{p^{\alpha}+\frac{\alpha+\beta-1}{2-\alpha-\beta}}+\omega\mathcal{L}\left\{u(t),p\right\}=0.$$

$$\Rightarrow \frac{B(\alpha,\beta)p^{\alpha+\beta-1}}{(2-\alpha-\beta)p^{\alpha}+\alpha+\beta-1}\mathcal{L}\left\{u(t),p\right\}-\frac{B(\alpha,\beta)p^{\alpha+\beta-2}u(0)}{(2-\alpha-\beta)p^{\alpha}+\alpha+\beta-1}+\omega\mathcal{L}\left\{u(t),p\right\}=0.$$

$$\Rightarrow \left[\frac{B(\alpha,\beta)p^{\alpha+\beta-1}}{(2-\alpha-\beta)p^{\alpha}+\alpha+\beta-1}+\omega\right]\mathcal{L}\left\{u(t),p\right\}-\frac{B(\alpha,\beta)p^{\alpha+\beta-2}u(0)}{(2-\alpha-\beta)p^{\alpha}+\alpha+\beta-1}=0.$$

$$\Rightarrow \left[\frac{B(\alpha,\beta)p^{\alpha+\beta-1}+\left((2-\alpha-\beta)p^{\alpha}+\alpha+\beta-1\right)\omega}{(2-\alpha-\beta)p^{\alpha}+\alpha+\beta-1}\right]\mathcal{L}\left\{u(t),p\right\}-\frac{B(\alpha,\beta)p^{\alpha+\beta-2}u(0)}{(2-\alpha-\beta)p^{\alpha}+\alpha+\beta-1}=0.$$

$$\Rightarrow \left[B(\alpha,\beta)p^{\alpha+\beta-1}+\left((2-\alpha-\beta)p^{\alpha}+\alpha+\beta-1\right)\omega\right]\mathcal{L}\left\{u(t),p\right\}=B(\alpha,\beta)p^{\alpha+\beta-2}u(0).$$

$$\Rightarrow \mathcal{L}\left\{u(t),p\right\}=\frac{B(\alpha,\beta)p^{\alpha+\beta-2}u(0)}{B(\alpha,\beta)p^{\alpha+\beta-1}+\left((2-\alpha-\beta)p^{\alpha}+\alpha+\beta-1\right)\omega}.$$

$$\Rightarrow \mathcal{L}\left\{u(t),p\right\}=\frac{u(0)}{p}\frac{1}{1+\frac{\left((2-\alpha-\beta)p^{2-\beta}+(\alpha+\beta-1)p^{2-\alpha-\beta}\right)\omega}{B(\alpha,\beta)}}.$$

$$\Rightarrow \mathcal{L}\left\{u(t),p\right\}=\frac{u(0)}{p}\sum_{k=0}^{\infty}\left[-\frac{\left((2-\alpha-\beta)p^{2-\beta}+(\alpha+\beta-1)p^{2-\alpha-\beta}\right)\omega}{B(\alpha,\beta)}\right]^k.$$

$$\Rightarrow \mathcal{L}\left\{u(t),p\right\}=u(0)\sum_{k=0}^{\infty}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}\right]^k\left(1+\frac{\alpha+\beta-1}{2-\alpha-\beta}p^{-\alpha}\right)^k p^{2k-\beta k-1}.$$

$$\Rightarrow u(t) = u(0)\sum_{k=0}^{\infty}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}\right]^k\int_0^t\left[\delta(\tau)+\frac{k(\alpha+\beta-1)}{\Gamma(\alpha)(2-\alpha-\beta)}\tau^{\alpha-1}\right]$$
$$\frac{(t-\tau)^{(\beta-2)k}}{\Gamma(\beta k-2k+1)}d\tau.$$
$$\Rightarrow u(t) = u(0)\sum_{k=0}^{\infty}\frac{1}{\Gamma(\beta k-2k+1)}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}\right]^k$$
$$\left\{\left(t^{\beta k-2k+1}-t^{\beta k-2k}\right)+\frac{k(\alpha+\beta-1)}{\Gamma(\alpha)(2-\alpha-\beta)}\frac{\Gamma(\alpha)\Gamma(\beta k-2k+1)}{\Gamma(\alpha-2k+\beta k+1)}t^{\alpha-2k+\beta k+1}\right\}.$$
$$\Rightarrow u(t) = u(0)\sum_{k=0}^{\infty}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}\right]^k$$
$$\left\{\frac{(t-1)\,t^{\beta k-2k}}{\Gamma(1+\beta k-2k)}+\frac{k(\alpha+\beta-1)}{(2-\alpha-\beta)}\frac{t^{\alpha+\beta k-2k+1}}{\Gamma(\alpha+\beta k-2k+1)}\right\}.$$
$$\Rightarrow u(t) = u(0)\left\{(t-1)E_{\beta-2,1}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}t^{\beta-2}\right]\right.$$
$$\left.+\frac{\alpha+\beta-1}{2-\alpha-\beta}t^{\alpha+1}E_{\beta-2,\alpha+1}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}t^{\beta-2}\right]\right\}.$$

Therefore, the relation (44) gives us

$$T(x,t) = T_0e^{-i\bar{k}x}u(0)\left\{(t-1)E_{\beta-2,1}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}t^{\beta-2}\right]\right.$$
$$\left.+\frac{\alpha+\beta-1}{2-\alpha-\beta}t^{\alpha+1}E_{\beta-2,\alpha+1}\left[-\omega\frac{(2-\alpha-\beta)}{B(\alpha,\beta)}t^{\beta-2}\right]\right\}. \quad (46)$$

This is an alternative representation of the fractional-time Fourier's law equation using the concept of derivative with two fractional orders α and β.
For $\alpha = 1$, we have

$$T(x,t) = T_0e^{-i\bar{k}x}u(0)\left\{(t-1)E_{\beta-2,1}\left[-\omega\frac{(1-\beta)}{B(1,\beta)}t^{\beta-2}\right]\right.$$
$$\left.+\frac{\beta}{1-\beta}t^2E_{\beta-2,2}\left[-\omega\frac{(1-\beta)}{B(1,\beta)}t^{\beta-2}\right]\right\}. \quad (47)$$

Now, by substituting $\omega = \chi\bar{k}^2\sigma_t^{\beta-1} = \frac{\bar{k}^2}{T_\beta}$, where χ is a time constant or thermal diffusion coefficient, we get

$$T(x,t) = T_0e^{-i\bar{k}x}u(0)\left\{(t-1)E_{\beta-2,1}\left[-\bar{k}^2\frac{(1-\beta)}{T_\beta B(1,\beta)}t^{\beta-2}\right]\right.$$
$$\left.+\frac{\beta}{1-\beta}t^2E_{\beta-2,2}\left[-\bar{k}^2\frac{(1-\beta)}{T_\beta B(1,\beta)}t^{\beta-2}\right]\right\}. \quad (48)$$

This equation represents the fractional-time Fourier's law equation using the concept of the derivative with only one fractional order β.

8. CONCLUSION

This paper aimed to suggest a modified SABC and SABR fractional derivative with a Mittag-Leffler kernel of two parameters. We made some changes in the formulae of SABC and SABR fractional derivatives so that the new modified formulae are well defined. Many applications of modified definitions are also explained in this paper. We derive many results and gave some examples in this paper.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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REFERENCES

[1] Abdon Atangana, On the new fractional derivative and application to nonlinear Fisher's reaction–diffusion equation, *Applied Mathematics and Computation* **273** (2016), 948–956.

[2] Abdon Atangana and Dumitru Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, *arXiv preprint arXiv:1602.03408* (2016), 8 – 15.

[3] Abdon Atangana and Juan Jose Nieto, Numerical solution for the model of rlc circuit via the fractional derivative without singular kernel, *Advances in Mechanical Engineering* **7(10)** (2015), 1 – 7.

[4] Dumitru Baleanu and Arran Fernandez, On some new properties of fractional derivatives with Mittag-Leffler kernel, *Communications in Nonlinear Science and Numerical Simulation* **59** (2018), 444–462.

[5] David A Benson, Stephen W Wheatcraft, and Mark M Meerschaert, Application of a fractional advection-dispersion equation, *Water resources research* **36(6)** (2000), 1403–1412.

[6] Michele Caputo and Mauro Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl* **1(2)** (2015), 1–13.

[7] Michele Caputo and Mauro Fabrizio, Applications of new time and spatial fractional derivatives with exponential kernels, *Progr. Fract. Differ. Appl* **2(2)** (2016), 1–11.

[8] Sanjay Madhukar Chinchole and Ashok Bhadane, A new definition of fractional derivative with Mittag-Leffler kernel of two parameters, *Communications in Mathematics and Applications* **13(1)** (2022), 1–11.

[9] Sanjay Madhukar Chinchole and Ashok Parashram Bhadane, On Properties of New Fractional Derivative with Mittag-Leffler Kernel of Two Parameters, *International Journal of Difference Equations (IJDE)* **17(1)** (2022), 47–52.

[10] JF Gomez-Aguilar, Abdon Atangana, and RF Escobar-Jimenez, Numerical solutions of fourier's law involving fractional derivatives with bi-order, *Scientia Iranica* **25(4)** (2018), 2175–2185.

[11] Rudolf Gorenflo, Francesco Mainardi and Serge Rogosin, Mittag-Leffler function: properties and applications, *Basic Theory* (2019), 269–296.

[12] Ji-Huan He, A new fractal derivation, *Thermal Science* **15(suppl. 1)** (2011), 145–147.

[13] Jordan Hristov, Approximate solutions to time-fractional models by integral-balance approach, *Fractional dynamics* (2016), 78–109.

[14] Jordan Hristov, Double integral-balance method to the fractional subdiffusion equation: approximate solutions, optimization problems to be resolved and numerical simulations, *Journal of Vibration and Control* **23(17)** (2017), 2795–2818.

[15] Xiaoyun Jiang and Mingyu Xu, The time fractional heat conduction equation in the general orthogonal curvilinear coordinate and the cylindrical coordinate systems, *Physica A: Statistical Mechanics and its Applications* **389(17)** (2010), 3368–3374.

[16] Anatoli Aleksandrovich Kilbas, Hari M Srivastava, and Juan J Trujillo, Theory and applications of fractional differential equations, *elsevier* **204** (2006), 3368–3374.

[17] Fu-Juan Liu, Zheng-Biao Li, Sheng Zhang, and Hong-Yan Liu, He's fractional derivative for heat conduction in a fractal medium arising in silkworm cocoon hierarchy, *Thermal Science* **19(4)** (2015), 1155–1159.

[18] Jorge Losada and Juan J Nieto, Properties of a new fractional derivative without singular kernel, *Progr. Fract. Differ. Appl* **1(2)** (2015), 87–92.

[19] Sergei Rogosin, The role of the Mittag-Leffler function in fractional modeling, *Progr. Fract. Differ. Appl* **3(2)** (2015), 368–381.