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Solution of the Klein - Gordon Equations Using the Reduced Differential Polynomial and Elzaki Transform Method

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Abstract

In this work , the reduced differential polynomial is combined with the Elzaki transform to obtain the approximate solutions to the nonlinear Klein- Gordon equations which is expressed in series form. Four examples of the equations are considered to show the effectiveness and performance of the method. Graphs of the solution of the illustrated examples were also plotted to show shapes of the obtained solutions which are compared with those in the references and found to agree. The comparison between the reduced differential polynomial and the Elzaki transform method with other known methods showed the method is very effective and powerful in the solution of nonlinear equations.

Keywords: Klein-Gordon equations, reduced differential polynomial, Elzaki transform method.

1. INTRODUCTION

The Klein Gordon model is one of the most essential models in quantum mechanics. This model is also applicable in condensed matter physics, initial state recurrence, nonlinear wave equations and in the collision plasma for interaction of solution[24]. The Klein Gordon equation is also very important in mathematical physics such as chemical kinetics, fluid dynamics and solid state physics.[10],[13] ,[16]. Generally, the Klein Gordon equation is of the form;

$$u_{tt} - k^2 u_{xx} + g(u) = h(x,t) \tag{1}$$

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with the initial conditions

$$u(x,0) = f(x), u_t(x,0) = m(x) \tag{2}$$

where u is a function of x and t and h(x,t) a known analytic function [1]. Many developed methods are in existence to obtain the approximate solutions of the Klein-Gordon equations and nonlinear equations. This include the Homotopy perturbation method[9],[25] Variation iterational method [19],[29],[30], Adomian decomposition method [13],[16],[23],[42], exponential function method[22] and the homotopy analysis method [28] .

In this work, the solutions of the Klein Gordon nonlinear equations are gotten by the reduced differential polynomial and the Elzaki transform method. The method expresses the solution in series form which yields the analytical solution with few iterations. The paper is highlighted as follows: Section 2 displays the definitions and basic properties of the method proposed, section 3 has the theoretical approach of the method used on the considered equations and the reduced differential polynomial with the Elzaki transform method is applied to solve four equations so as to show its potency in section 4.

2. PROPERTIES OF ELZAKI TRANSFORM METHOD

The Elzaki Transform [5] [7] [8] [12] [32] defined for functions of exponential order and is given in a set R as;

$$R = \left\{ f(\phi) : \exists M, \sigma_1, \sigma_2 > 0, |f(\phi)| < Me^{\frac{|t|}{c_j}}, if\phi \in (-1)^j \times [0,\infty) \right\}$$

Where M is a constant that must be an infinite number and σ_1,σ_2 may be either finite or infinite. The Elzaki transform according to Tarig [6] is expressed as:

$$E\left[f(\phi)\right] = T(v) = v\int_0^\infty e^{\frac{-\phi}{v}} f(\phi)d\phi, \qquad \phi \geq 0, u \in (\sigma_1, \sigma_2) \tag{3}$$

where u is used to factor ϕ in the analysis of function f.

Let T(v) be the Elzaki transform of $f\phi$ such that $E[f(\phi)]$ =T(v). Then,

(i) $E[f'(\phi)]$=$\frac{1}{v}T(v) - vf(0)$

(ii) $E[f''(\phi)]$=$\frac{1}{v^2}T(v) - f(0) - vf'(0)$

(iii)$E[f^k(\phi)]$=$\frac{1}{v^k}T(v) - \sum_{n=0}^{k-1} v^{2-k+n} f^{(n)}(0)$

T(v) is the Elzaki transform of $f(\phi)$ and $f(\phi)$, the inverse Elzaki transform of T(v) :

$$f(\phi) = E^{-1}\left[T(v)\right]$$

Integration by part is applied on the definitions of the Elzaki transform to obtain the Elzaki transform of a partial derivative as[9];

(i) $E\left[\frac{\delta f(x,\phi)}{\delta\phi}\right]=\frac{1}{v}T(x,v) - vf(x,0),$

(ii) $E\left[\frac{\delta^2 f(x,\phi)}{\delta\phi^2}\right]=\frac{1}{v^2}T(x,v) - f(x,0) - v\frac{\delta f(x,0)}{\delta\phi}$

(iii)$E\left[\frac{\delta^2 f(x,\phi)}{\delta x^2}\right]=\frac{d^2}{dx^2}\left[T(x,v)\right]$

3. THEORETICAL APPROACH OF THE ELZAKI TRANSFORM AND THE REDUCED DIFFERENTIAL POLYNOMIAL

The focus of this paper is on the solution of the nonlinear partial differential equations of the Klein-Gordon type. The reduced differential polynomial is combined with the Elzaki transform method to obtain the approximate solutions of the aforementioned equations.
Considering [27] ,

$$\frac{\delta^r u(x,\phi)}{\delta\phi^r} + Ru(x,\phi) + Nu(x,\phi) = f(x,\phi) \tag{4}$$

where r=1,2,3 and the initial conditions as;

$$\frac{\delta^{r-1} u(x,\phi)}{\delta\phi^{r-1}} = g_{r-1}(x) \tag{5}$$

The partial derivatives of the function $u(x,\phi)$ of the r^{th} is given as $\frac{d^r(x,\phi)}{d\phi^r}$, where R indicates the linear differential equation, N is the nonlinear terms of the differential equations and $f(x,\phi)$ the source terms.
Applying the Elzaki transform into equation(5) then;

$$E\left[\frac{\delta^r u(x,\phi)}{\delta\phi^r}\right] + E\left[Ru(x,\phi)\right] + E\left[Nu(x,\phi)\right] = E\left[f(x,\phi)\right] \tag{6}$$

where,

$$E\left[\frac{\delta^r u(x,\phi)}{\delta\phi^r}\right] = \frac{E\left[u(x,\phi)\right]}{v^{(r)}} - \sum_{n=0}^{r-1} v^{2-r+n}\frac{\delta^n u(x,0)}{\delta\phi^n} \tag{7}$$

Substituting equation (7)into (6) we have;

$$\frac{E\left[u(x,\phi)\right]}{v^{(r)}}-\sum_{n=0}^{r-1}v^{2-r+n}\frac{\delta^{n}u(x,0)}{\delta\phi^{n}}+E\left[Ru(x,\phi)\right]+E\left[Nu(x,\phi)\right]=E\left[f(x,\phi)\right] \quad (8)$$

Thus,

$$\frac{E\left[u(x,\phi)\right]}{v^{(r)}}=E\left[f(x,\phi)\right]+\sum_{n=0}^{r-1}v^{2-r+n}\frac{\delta^{n}u(x,0)}{\delta\phi^{n}}-\left\{E\left[Ru(x,\phi)\right]+E\left[Nu(x,\phi)\right]\right\} \quad (9)$$

Thus, Simplifying equation(9) we obtain;

$$E\left[u(x,\phi)\right]=v^{r}E\left[f(x,\phi)\right]+\sum_{n=0}^{r-1}v^{2+k}\frac{\delta^{n}u(x,0)}{\delta\phi^{n}}-v^{r}\left\{E\left[Ru(x,\phi)\right]+E\left[Nu(x,\phi)\right]\right\} \quad (10)$$

Applying the inverse Elzaki transform on equation (10) gives ;

$$u(x,\phi)=E^{-1}\left[v^{r}E\left[f(x,\phi)\right]+\sum_{n=0}^{r-1}v^{2+n}\frac{\delta^{n}u(x,0)}{\delta\phi^{n}}\right]-E^{-1}\left[v^{r}\left\{E\left[Ru(x,\phi)\right]+E\left[Nu(x,\phi)\right]\right\}\right] \quad (11)$$

Equation (11) can then be written as;

$$u(x,\phi)=F(x,\phi)-E^{-1}\left[v^{r}\left\{E\left[Ru(x,\phi)\right]+E\left[Nu(x,\phi)\right]\right\}\right] \quad (12)$$

where $F(x,\phi)$ is the expression that arises from the initial conditions given and the source terms after it has been simplified. The solution will be expressed as:

$$u(x,\phi)=\sum_{m=0}^{\infty}u_{m}(x,\phi) \quad (13)$$

The nonlinear part is reduced as;

$$Nu(x,\phi)=\sum_{m=0}^{\infty}A_{m} \quad (14)$$

where A_m is define as the reduced differential polynomial which can be gotten by using the formula;

$$A_{m}=U_{m}(x)U_{n-m}(x),\quad m=0,1,...$$

Substituting equations(14)and (15)into equation(13) to obtain;

$$\sum_{m=0}^{\infty}u_{m}(x,\phi)=F(x,\phi)-E^{-1}\left[v^{r}\left\{E\left[R\sum_{m=0}^{\infty}u_{m}(x,\phi)\right]+E\left[\sum_{m=0}^{\infty}A_{m}\right]\right\}\right] \quad (15)$$

Then from equation(15) we have;

$$u_0(x,\phi) = F(x,\phi), \ \{m = 0\} \tag{16}$$

The recursive relation is expressed as;

$$u_{m+1} = -E^{-1}\left[v^r\{E\left[Ru_m(x,\phi)\right] + E\left[A_m\right]\}\right]$$

where r=1,2,3 and $m \geq 0$
The analytical solution $u(x,\phi)$ can be approximated by the truncated series;

$$u(x,\phi) = \lim_{N\to\infty} \sum_{m=0}^{N} u_m(x,\phi)$$

4. APPLICATIONS TO THE KLEIN-GORDON EQUATIONS

4.1 : Illustration One

Considering the nonhomogenous nonlinear Klein Gordon equation [2];

$$u_{tt} - u_{xx} + u^2 = x^2cos^2t - x\ cost \tag{17}$$

with the conditions:

$$u(x,0) = x, \quad u_t(x,0) = 0$$

Applying the Elzaki transform on Equation (17) we obtain;

$$\frac{1}{v^2}U(x,v) - u(x,0) - vu'(x,0) + E[u^2] = E\left[x^2cos^2t - x\ cost\right] - \frac{d^2}{dx^2}E[u] \tag{18}$$

Taking the inverse Elzaki transform of Equation (18) with the given conditions we have;

$$u(x,v) = E^{-1}\{v^2E\left[xt + x^2cos^2t - xcos\right] + v^2\frac{d^2}{dx^2}E[u] - v^2E[u^2] + xv^2\} \tag{19}$$

From equation(19);

$$u_0 = E^{-1}\{v^2E\left[x^2cos^2t - x\ cos\ t\right]\} \tag{20}$$

$$u_0 = xcost + \frac{x^2}{2}t^2 - \frac{x^2}{12}t^4$$

Therefore, the recursive relation is expressed as;

$$u_{m+1} = E^{-1}\left\{v^2\left(\frac{d^2}{dx^2}E\left[U_m\right] - E\left[A_m\right]\right)\right\} \tag{21}$$

when m=0 then equation (21) becomes;

$$u_1 = E^{-1}\left\{v^2\left(\frac{d^2}{dx^2}E\left[U_0\right] - E\left[A_0\right]\right)\right\} \tag{22}$$

Thus, simplifying equation (22) gives;

$$u_1 = -\frac{x^2}{2}t^2 + \frac{x^2}{12}t^4 \tag{23}$$

The series solution is;

$$u(x,t) = u_0 + u_1 + ... \tag{24}$$

$$u(x,t) = xcost + \frac{x^2}{2}t^2 - \frac{x^2}{12}t^4 - \frac{x^2}{2}t^2 + \frac{x^2}{12}t^4$$

Thus,

$$u(x,t) = xcost \tag{25}$$

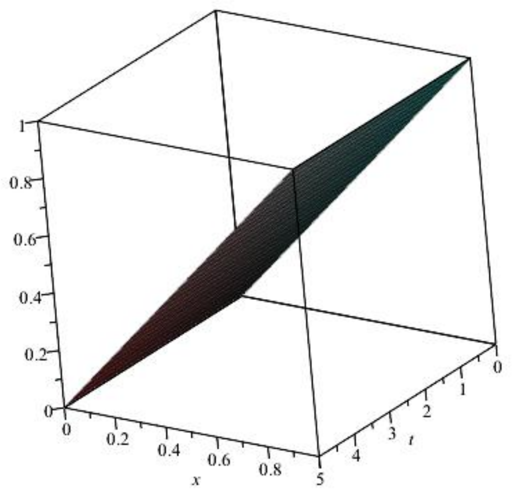

Figure 1: Numerical Solution for Application One

Illustration Two

Considering the nonlinear Klein Gordon equation [4];

$$u_{tt} - u_{xx} + u^2 = 6xt(x^2 - t^2) + x^6t^6 \tag{26}$$

with the conditions:

$$u(x,0) = 0, \quad u_t(x,0) = 0$$

Applying the Elzaki transform on Equation (26) gives;

$$\frac{1}{v^2}U(x,v) - u(x,0) - vu'(x,0) = E\left[6xt(x^2 - t^2) + x^6t^6 - u^2\right] + \frac{d^2}{dx^2}E[u] \tag{27}$$

where

$$E[u_{tt}] = \frac{1}{v^2}u(x,v) - u(x,0) - vu_t(x,0)$$

Taking the inverse Elzaki transform of Equation (27) with the given conditions we have;

$$u(x,v) = E^{-1}\{v^2 E\left[6xt(x^2-t^2)+x^6t^6\right] + v^2\frac{d^2}{dx^2}E[u] - v^2E[u^2]\} \tag{28}$$

From equation(28):

$$u_0 = E^{-1}\{v^2 E\left[6xt(x^2-t^2)+x^6t^6\right]\} \tag{29}$$

$$u_0 = x^3t^3 - \frac{3}{10}xt^5 + \frac{1}{56}x^6t^8$$

Therefore, the recursive relation is expressed as:

$$u_{m+1} = E^{-1}\left\{v^2\left(\frac{d^2}{dx^2}E\left[U_m\right] - E\left[A_m\right]\right)\right\} \tag{30}$$

when m=0 then equation (30) becomes:

$$u_1 = E^{-1}\left\{v^2\left(\frac{d^2}{dx^2}E\left[U_0\right] - E\left[A_0\right]\right)\right\} \tag{31}$$

Thus, simplifying equation (31) we obtain:

$$u_1 = \frac{3}{10}xt^5 - \frac{1}{56}x^6t^8 \tag{32}$$

The series solution is:

$$u(x,t) = u_0 + u_1 + ...$$

Hence,

$$u(x,t) = x^3t^3 \tag{33}$$

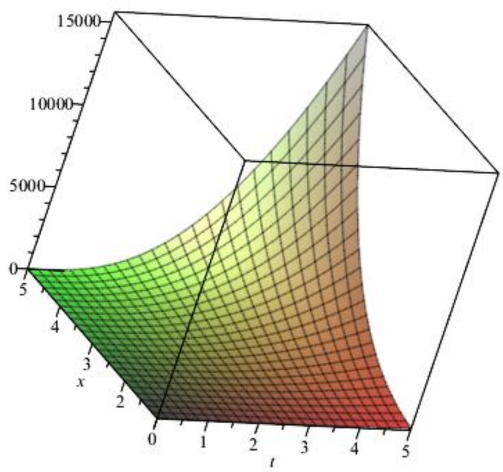

Figure 2: Numerical Solution for Application Two

Illustration Three

Considering the homogenous nonlinear Klein-Gordon equation [2];

$$u_{tt} + \alpha_{xx} + u + \gamma u^3 = 0 \tag{34}$$

with the conditions:

$$u(x,0) = B\tan(kx), \quad u_t(x,0) = bck\sec^2(kx)$$

where $B = \sqrt{\dfrac{\beta}{\gamma}}, \;\; K = \sqrt{\dfrac{-\beta}{2(\alpha + c^2)}}$

Applying the Elzaki transform on Equation (34) we obtain;

$$\frac{1}{v^2}U(x,v) - u(x,0) - vu'(x,0) = \left[\alpha\frac{d^2}{dx^2} - 1\right]E[u] - \gamma\; E[u^3] \tag{35}$$

Taking the inverse Elzaki transform of Equation (35) with the given conditions we have;

$$u(x,v) = E^{-1}\{v^2B\;tan(kx) + v^3Bck\;sec^2(kx) + \alpha v^2\frac{d^2}{dx^2}E[u] - v^2E[u] - \gamma v^2E[u^3]\} \tag{36}$$

From equation(36);

$$u_0 = E^{-1}\{v^2B\;tan(kx) + v^3Bck\;sec^2(kx)\} \tag{37}$$

$$u_0 = B\;tan(kx) + tBck\;sec^2(kx)$$

Therefore, the recursive relation is given as:

$$u_{m+1} = E^{-1}\left\{v^2\left(\alpha\frac{d^2}{dx^2}E\left[U_m\right] - E[u_m] - \gamma E\left[A_m\right]\right)\right\} \tag{38}$$

when m=0 then equation (38) becomes:

$$u_1 = E^{-1}\left\{v^2\alpha\frac{d^2}{dx^2}E\left[U_0\right] - v^2E[u_0] - \beta v^2E\left[A_0\right]\right\} \tag{39}$$

Thus, simplifying equation (39) at α= -2.5 and γ=1.5 gives:

$$u_1 = -\frac{Bck}{12}t^3 \times \left[1 + 4.5B^2 - 20K^2 + cos(2Kx) + 10K^2cos(2Kx)\right] \times sec^4(kx)$$

$$-\frac{B}{4}t^2\left[1 + 1.5B^2 - 10K^2 + cos(2Kx) - 1.5B^2cos(2Kx)\right] \times sec^2(Kx)tan(Kx)$$

$$-\frac{1.5B^3c^3K^3}{20}t^5sec^5(Kx) - \frac{cosh^2x - 2}{8cosh^3x}t^2 \tag{40}$$

The series solution is:

$$u(x,t) = u_0 + u_1 + ...$$

Thus,

$$u(x,t) = Btan(K(x+ct)) \tag{41}$$

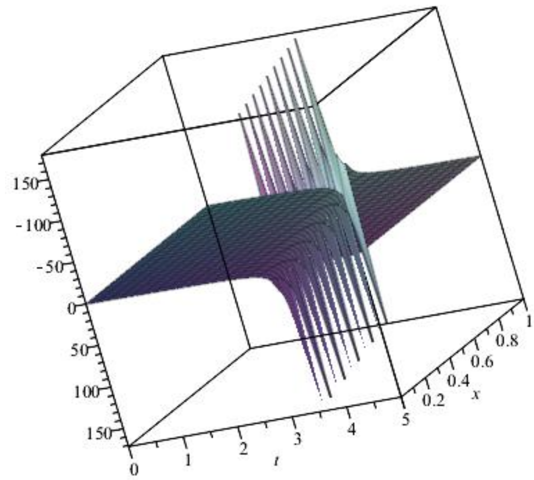

Figure 3: Numerical Solution for Application Three

Illustration Four

Considering the nonlinear Klein-Gordon equation [1];

$$u_{tt} - u_{xx} + \frac{3}{4}u - \frac{3}{2}u^3 = 0 \tag{42}$$

with the conditions:

$$u(x,0) = -sechx, \quad u_t(x,0) = \frac{1}{2}sechx\, tanhx$$

Applying the Elzaki transform on Equation (42) we obtain;

$$\frac{1}{v^2}U(x,v) - u(x,0) - vu'(x,0) = E[u]\left[\frac{d^2}{dx^2} - \frac{3}{4}\right] + \frac{3}{2}E[u^3] \tag{43}$$

Taking the inverse Elzaki transform of Equation (43) with the given conditions we have;

$$u(x,v) = E^{-1}\{\frac{v^3}{2}sechx\, tanhx - v^2 sechx + v^2\frac{d^2}{dx^2}E[u] - \frac{3v^2}{4}E[u^2] + \frac{3v^2}{2}E[u^3]\} \tag{44}$$

From equation(44);

$$u_0 = E^{-1}\{\left[\frac{v^3}{2}sechx\, tanhx - v^2 sechx\right]\} \tag{45}$$

$$u_0 = \frac{t}{2} sechx \, tanhx - sechx$$

Therefore, the recursive relation is expressed as:

$$u_{m+1} = E^{-1}\left\{ v^2 \left(\frac{d^2}{dx^2} E\left[U_m\right] - \frac{3}{4} E[u_m] + \frac{3}{2} E\left[A_m\right] \right) \right\} \tag{46}$$

when m=0 then equation (46) becomes:

$$u_1 = E^{-1}\left\{ v^2 \frac{d^2}{dx^2} E\left[U_0\right] - \frac{3v^2}{4} E[u_0] + \frac{3v^2}{2} E\left[A_0\right] \right\} \tag{47}$$

Thus, simplifying equation (47) gives:

$$u_1 = \frac{sinhx(cosh^2x - 6)}{48cosh^4x} t^3 - \frac{cosh^2x - 2}{8cosh^3x} t^2 + ... \tag{48}$$

The series solution is:

$$u(x,t) = u_0 + u_1 + ... \tag{49}$$

$$u(x,t) = -\frac{1}{coshx} + \frac{sinhx}{2cosh^2x} t - \frac{cosh^2x - 2}{8cosh^3x} t^2 + \frac{sinhx(cosh^2x - 6)}{48cosh^4x} t^3 + ...$$

Thus,

$$u(x,t) = -sech(x + \frac{1}{2} t) \tag{50}$$

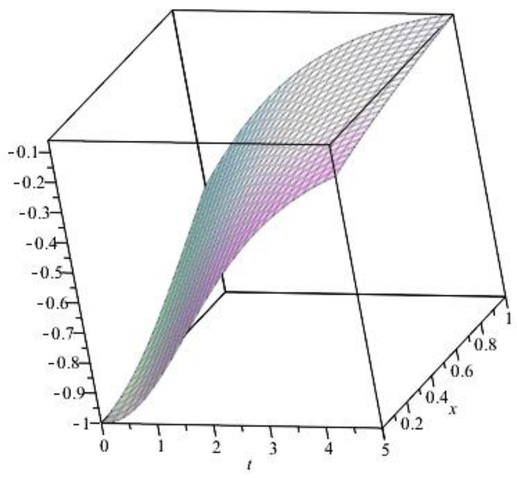

Figure 4: Numerical Solution for Application Four

5. DISCUSSION OF RESULTS

The Elzaki transform method combined with the reduced differential transform polynomials has been used to solve the nonlinear aspect of the partial differential equation illustrated. Four examples of the Klein-Gordon equations were solved with the scheme. The obtained results are in series solutions which are in concordance with the analytical solution and hence showed the efficiency of the scheme. The considered equations showed the method is effective in solving nonlinear partial differential equations as the result gotten agree with those in the references. Figures 1,2,3,4 also shows the three dimensional graphs of the examples considered to give detailed explanation of the behaviour and shape of the equations. The small computational size illustrated show that the proposed scheme is suitable and can be applied to other nonlinear partial differential equations.

6. CONCLUSION

In this work, the approximate solutions of the Klein-Gordon equations have been gotten with the reduced differential polynomial and Elzaki transform method. The obtained solution agree with the solutions gotten by the modified Adomian decomposition and novel expansion methods as given in the references. The problems considered showed that the reduced differential polynomial and Elzaki transform method is a very effective tool in the solution of Klein-Gordon equations. Therefore, the proposed method is powerful in solving nonlinear differential equations which provides the results in form of series.

CONFLICT OF INTEREST

The author(s) declare that there is no conflict of interest.

REFERENCES

[1] Agom ,E.U. Ogunfiditimi,F.O., 2018, ”Exact solution of nonlinear Klein-Gordon with quadratic nonlinearity by modified Adomian decomposition method,” Journal of Mathematical Computational Science, 8, 484-493.

[2] raza, N., rashid A.B and Jarid ,A., 2016, ”Approximate solution of Nonlinear Klein-Gordon equation using Sobolev Gradients. Hindawi Publishing Corporation,” Journal of Function Spaces.7, ID 1391594.

[3] Scheider,D., 2020, ”Brerther solutions of the cubic Klein-Gordon equation,” arXiv:2001, 04108v1.

[4] Dehghan, M., Shokri, A., 2009, ”Numerical solution of the nonlinear Klein-Gordon equation using radial basis functions, J. Comput. Appl. Math, 230, 400-410.

[5] Elzaki, T.M., 2011 ”The new integral transform Elzaki Transform, Global Journal of Pure and Applied Mathematics, 7, 57-64.
[6] Elzaki, T.M., 2011 ”On the Elzaki transform and ordinary differential equation with variable coefficients,” Advances in Theoretical and Applied mathematics, 6, 13-18.
[7] Elzaki T.M and Alkhateeb, S.A., 2015, ”Modification of Sumudu transform ”Elzaki tranform” and Adomian decomposition method, Applied Mathematical Sciences, 9, 603-611.
[8] Elzaki,T.M On the connection between Laplace and Elzaki tranforms, Advances in Theoretical and Applied mathematics, 6(2011), 1-11.
[9] Elzaki T.M. and Hilal,E.M., 2012, ”Homotopy pertubation and Elzaki transform for solving nonlinear partial differential equations, Mathematical Theory and Modeling, 2, 33-42.
[10] Lakestani, M.and Dehghan,M., 2010 ”collocation methods for the solution of nonlinear Klein-Gordon equation,” Comput, Phys. Commun. 181(8) 1392-1401.
[11] Keskin, Y., Caglar, I. and Koc, A. B., 2011, ”Numerical solution of sine-gordon equation by reduced differential transform method, in: Proceedings of the World Congress on Engineering. 1, 6–8.
[12] Elzaki, T.M., 2011, ”Application of new transform Elzaki transform to partial differential equations, Global Journal of Pure and Applied Mathematics, 7, 1, 65-70.
[13] El-sayed,S.M., 2003, ”The decomposition method for studying the klein-Gordon equation, Chaos Solitons Fractals, 18,1025-1030.
[14] wazwaz, A.M., 2018, ”New travelling wave solutions to the Boussinesq and the Klein-Gordon equations,” Communications in Nonlinear Science and Numerical Simulation, 13, 889-908.
[15] Ibis, B., 2014, ”Application of reduced differential transformation method for solving Fourth-Order Parabolic Partial Differential Equations,” Journal of Mathematics and Computer Science, 12, 124-131.
[16] Basak, K.C. Ray P.C. and Bera, R.K., 2009, ”Solution of nonlinear Klein-Gordon equation with a quadratic nonlinear term by Adomian decomposition method,” Communications in Nonlinear Science and Numerical Simulation, 14, 3, 718-723.
[17] Liu, J.G., Yang, X.J., Feng, Y.Y and Cui, P., 2020, ”On the (N+1)-dimensional local fractional reduced differential transform method and its applications,” Mathematical Methods in the Applied Sciences. 43, 8856-8866.
[18] Keskin, Y. and Oturanç,G., 2009, ”Reduced differential transform method for partial differential equations,” Int J. Nonlinear Sci. Numerical Simulation. 10,741.
[19] Biazar, J. and Ghazvini, H., 2007, ”He’s variational iteration method for solving hyperbolic different equations,” International Journal of Nonlinear sciences and Numerical Simulation. 3,311-314.
[20] Yıldırım, K., İbiş , B. and Bayram, M., 2012, ”New solutions of the non linear Fisher type equations by the reduced differential transform, Nonlinear Science. Letters. 3, 29-36.

[21] Haghbin, A. and Hesam, S., 2012, ”Reduced Differential Transform Method For Solving Seventh Order Sawada Kotera Equations,” The Journal of Mathematics and Computer Science. 5, 53-59.
[22] Ebaid, A., 2009, ”Exact solutions for the generalized Klein-Gordon equation via a transformation and exp-function method and comparison with Adomian’s method,” Journal of computational and Applied Mathematics. 223, 1, 278-290.
[23] El-Sayed, S.M., 2003 ”The decomposition method for studying the Klein-Gordon equation,” Chaos, Solitions & Fractals. 18, 5, 1025-1030.
[24] Suleman, M. and Wu, Q., 2015, ”Transformed Homotopy Pertubation Method for Approximate Analytical Solution of Klein-Gordon and Sine-Gordon Equation.
[25] Odibat, Z. and Momani, S.,2007, ”A reliable treatment of homotopy pertubation method for solving Klein-Gordon equations, Physics Letters. 5, 351-357.
[26] He, J.H. and Wu, X. H., 2006, ”Construction of solitary solution and compacton–like solution by variational iteration method,” Chaos, Solitons and Fractals, 29, 108–113.
[27] Zaine, D. and Cherif, M.H., 2015, ”Resolution of nonlinear partial differential equations by Elzaki transform decomposition method,” J. Appro. Theo. Appl. Math, 5.
[28] Sun, Q., 2005, ”Solving the Klein-Gordon equation by means of the homotopy analysis method,” Applied Mathematics and Computation, 169, 355-365.
[29] Yusuf uglu, E., 2008,”The variational iteration method for studying the Klein-Gordon equation, Applied Mathematics Letters,” 21, 7, 669-674.
[30] Wang, Q. and Cheng, D., 2005, ”Numerical solution of damped nonlinear Klein-Gordon equations using variational method and finite element approach, Applied Mathematics and Computation,” 162 1,81-89.
[31] Keskin, Y. Servi, S. and Oturanc, G., 2011, ”Reduced differential transform method for solving Klein Gordon equations,” In: Proc. of the World Congress on Engineering.
[32] Alshikh, A.A. and Mahgoub, M. A., 2016, ” A Comparative Study Between Laplace Transform and Two New Integrals Elzaki Transform and Aboodh Transform,” Pure and Applied Mathematics Journal 5, 145-150.
[33] Zaine, D. and Cherif, M.H., 2015, ”Resolution of nonlinear partial differential equations by Elzaki transform decomposition method,” J. Appro. Theo. Appl. Math. 5, 17-30.
[34] Gupta, P.K., 2011, ”Approximate analytical solutions of fractional Benney–Lin equation by reduced differential transform method and the homotopy perturbation method,” Comput. Math. Appl. 9, 2829-2842.
[35] Bulbul, B. and Sezer, M., 2013, ”A new approach to numerical solution of nonlinear Klein-Gordon equation, ”Hindawi Publishing Corporation, Mathematical Problems in Engineering, 7, ID 869749.
[36] Song, M. Liu, Z., Zerrad E. and Biswas, A., 2013, ”Singular soliton solution and bifurcation analysis of klein-Gordon equation with power law nonlinearity,” Frontiers of mathematics in china, 8, 191-201.

[37] Biswas, A., Khalique, C.M. and Adem, A.R., 2012, ”Travelling wave solutions of nonlinear dispersive Klein-Gordon equations, Journal of King Saud University. 24(2012), 339-342.
[38] Song, M. Liu, Z., Zerrad, E. and Biswas, A., 2013, ”Singular soliton and bifurcation analysis of quadratic nonlinear klein-Gordon equation, Applied Mathematics and Information Sciences, 7, 1333-1340.
[39] Yin, F. Tian, T., Sing J. and Zhu, M., 2015, ”Spectral methods using Legendre wavelets for nonlinear klein-Gordon/Sine Gordon equations,” Journal of Computational and Applied Mathematics. 275, 321-334.
[40] Hafez, M.G., 2014, ”Exact travelling wave solutions to the Klein-Gordon equation using the Novel(G'/G)-Expansion Method,” Results in Physics, 4, 177-184.
[41] Venkatesh, S.G., 2016, ”An efficient approach for solving klein-Gordon equation arising in Quatum field theory using wavelets, Computerand Applied Mathematics, 37, 81-98.
[42] Agom, E.U., Ogunfiditimi, F.O., 2018, ”Exact solution of nonlinear Klein-Gordon with quadratic nonlinearity by modified Adomian decomposition method,” Journal of Mathematical Computational Science, 8, 484-493.
[43] Takei Y. and Iwata,Y. 2021, ”Space-time breather solution for nonlinear Klein-Gordon equations,” Journal of Physics: Conference series, 1730(2021), 012058.