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Pointwise Contact Problems for Composite Bodies with Non-Penetration Conditions on Separate Points of Rigid Stiffeners

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Abstract

A new type of non-classical 2D pointwise contact problems is proposed. Namely, we suppose that a composite body in its undeformed state touches a polygon shaped rigid obstacle at two separate contact points. Composite bodies under investigation consist of an elastic matrix and a thin rigid stiffener or a volume (bulk) inclusion. In this case, the displacements on the set corresponding to a rigid stiffener (inclusion) have a predetermined affine structure that describes possible parallel shifts and rotations of the stiffener. The rigid stiffener (inclusion) is located on the external boundary and has the form of the part of a rectangle. The presence of the rigid stiffener (inclusion) allows to impose a set of non-penetration conditions for certain geometrical configurations of the obstacle and the body near contact points. For both cases of a thin rigid stiffener, which is described by a three line segments and a volume rigid inclusion specified in a subdomain, the energy minimization problems are formulated. The unique solvability of the corresponding boundary value problems is proved. For initial variational problems equivalent differential formulations have been derived under an assumption of additional regularity of the solution.

Keywords: Signorini-type condition, non-penetration, pointwise contact, contact problem, rigid inclusion, rigid stiffener.

INTRODUCTION

As is known, approaches to solving contact problems initiated the development of directions in the calculus of variations. To date, contact problems occupy a significant place in applied mathematics, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. For an overview of contact problems we refer to [11, 12]. The classical Signorini type condition are imposed on a part of a body boundary that has non-zero measure. This connected with a circumstance of non-regularity of functions belonging to certain Sobolev spaces. When formulations of problems implies regular traces of functions on boundaries we can neglect the requirement concerning non-zero measure of contact sets. Our approach arises from an assumption of sufficient regularity of traces for sought functions. Namely, we impose non-penetration conditions on two separate points of boundary.

In [15] free boundary problems for elastic bodies with a rigid inclusion being in frictionless contact with another rigid non-deformable punch were proposed and investigated for the first time. Nonlinear crack problems with inequality type non-penetration conditions between opposite crack faces (see, for example [16, 17, 18, 19, 20, 21, 22, 23, 24, 25]) are a special subclass of the contact problems. Furthermore, contact problems with the Signorini type conditions can be obtained from a family of crack problems by passages to the limits when rigidity parameters go to infinity [26, 27, 28]. Some pointwise contact problems were considered in [29, 30]. In [30], authors investigated pointwise contact interaction between bodies with the help of a four-variable local contact problem, which solution is associated with general material points on contact surfaces, where contact mechanical action-reaction are represented. Special algorithms allowing a simplified description of contact between deformable body and rigid surfaces are developed based on the geometrically exact covariant description of contact [31].

In this work we pay attention to a variational problem describing a pointwise contact of composite body with a non-deformable obstacle. Namely, we study mathematical models describing equilibrium of elastic bodies containing a thin rigid stiffener (volume inclusion) in the form of a part of a rectangle. In contrast to the work [15], we consider a new nonlinear contact problem, where non-penetration conditions is imposed for two separate points located on the boundary of the stiffener (inclusion). Due to the presence of a rigid stiffener we can rewrite the non-penetration condition in the form of two sets of inequalities. We should reflect the difference of approaches of the present paper and the paper [33], where contact problems were considered for a sole point and non-penetration conditions were specified for displacements and rotations parameters of rigid inclusions. In contrast to that approach, we impose restrictions in the form of inequalities only for displacement vectors of two points in which there is a touch of the obstacle. Despite of this, we obtain, as a consequence of prescribed inequalities, restrictions for a rotation parameter of the whole rigid stiffener (inclusion) that depend on a uniform transfer vector of the whole stiffener (inclusion). We can emphasize that this approach can be considered as some branch of Signorini's approach for contact problems with a finite number of contacting points.

The contact problems are formulated as a minimization problem of an energy functional over a set of admissible displacements. The unique existence of weak solutions of corresponding variational problems has been proved. Under an assumption of additional regularity of the solutions, for the initial weak formulations, we have found equivalent differential statements.

1. THE CASE OF A THIN RIGID STIFFENER

For simplicity, we consider the case of two contact points that corresponds to a rigid stiffener with three line segments. In general, the same approach can be treated for a finite set of contact points which fits a polygon shaped stiffener and as well as a polygon shaped obstacle.

Let us formulate a contact problem for an elastic body containing a rigid stiffener on the external boundary. Such configuration may describe bodies covered by coatings. Consider a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ with the boundary $\Gamma \in C^{0,1}$, which consists of two continuous curves $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, meas$(\Gamma_1) > 0$. We suppose that the curve comprised of three line segments

$$\gamma = \bigcup_{i=1}^{3} l_i,$$

$$l_1 = [0,1] \times \{0\}, \quad l_2 = [0,1] \times \{-1\}, \quad l_3 = \{0\} \times [-1,0],$$

is a part of Γ_2, such that $\gamma \subset \text{int}(\Gamma_2)$ (see Fig.1). For the construction of rigid displacements in what follows, the domain Ω is delineated by the curve Γ containing γ. We assume that a thin rigid stiffener is given by γ, and a rigid obstacle is given by the rectilinear lines

$$O = \{(x_1, x_2) \,|\, -\infty < x_1 \le 0, \quad k_1 x_1 \le x_2 \text{ or } x_2 \le k_2 x_1 - 1\}, \quad -\infty < k_2 < 0 < k_1 < \infty.$$

Denote by $W = (w_1, w_2)$ the displacement vector. We suppose that the body is fixed on the part Γ_1 of the boundary, i.e.

$$W = (0,0) \quad \text{on} \quad \Gamma_1. \tag{1}$$

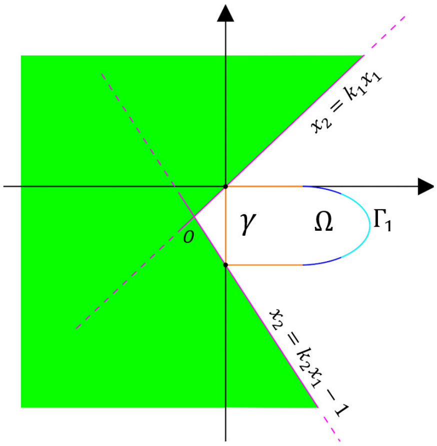

Figure 1: Geometry of the problem for composite body with the thin rigid stiffener

According to the last condition, we deal with the following Sobolev spaces

$$H^{1,0}_{\Gamma_1}(\Omega) = \{w \in H^1(\Omega) \mid w = 0 \quad \text{on} \quad \Gamma_1\}, \quad H(\Omega) = H^{1,0}_{\Gamma_1}(\Omega)^2.$$

We recall constitutive relations for the plane elasticity in the the framework of strain and stress tensors

$$\varepsilon_{ij}(W) = \frac{1}{2}(w_{i,j} + w_{j,i}), \quad \sigma_{ij}(W) = c_{ijkl}\varepsilon_{kl}(W), \; i,j = 1,2, \tag{2}$$

where the comma in the first formula in (2) denotes a corresponding derivative, and the Einstein summation convention is used. The tensor of elastic coefficients is given by entries c_{ijkl} assumed to be symmetric and positive definite:

$$c_{ijkl} = c_{klij} = c_{jikl}, \quad c_{ijkl} \in L^\infty(\Omega), \quad i,j,k,l = 1,2,$$

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2, \quad \forall\xi, \quad \xi_{ij} = \xi_{ji}, \quad i,j = 1,2, \quad c_0 = const > 0.$$

In order to provide a variational formulation describing the equilibrium state for the body with the rigid stiffener γ, we introduce the energy functional

$$\Pi(W) = \frac{1}{2}\int\limits_\Omega \sigma_{ij}(W)\varepsilon_{ij}(W) - \int\limits_\Omega FW,$$

where the vector $F = (f_1, f_2) \in L^2(\Omega)^2$ describes the external forces acting on the body, $FW = f_i w_i$. The coercivity of the functional $\Pi(W)$ provided by the well-known Korn inequality:

$$\int\limits_\Omega \sigma_{ij}(W)\varepsilon_{ij}(W) \geq c\|W\|^2_{H(\Omega)}, \;\; \forall W \in H(\Omega), \tag{3}$$

where the constant $c > 0$ is independent of W.

The space of infinitesimal rigid displacements $R(Z)$ consists of affine functions and prescibes a linear structure of displacements on Z [13]:

$$R(Z) = \{\rho = (\rho_1, \rho_2) \,|\, \rho(x) = b(x_2, -x_1) + (c_1, c_2);\, b, c_1, c_2 \in \mathbb{R},\ x \in Z\}, \tag{4}$$

where Z is some subset of $\overline{\Omega}$. In particular, for the curve γ we have the space $R(\gamma)$ of infinitesimal rigid displacements.

Taking into account the given structure of displacements for points of the stiffener γ, we provide arguments justifying non-penetration conditions between the rigid stiffener and the rigid obstacle O in the case of their specific geometry configuration. Here we should note that our reasoning relies on assumption of the infinitesimal displacements within linearized elasticity.

The non-penetration of γ into obstacle O requires that

$$x + \rho(x) \notin int(O) \quad \text{for} \quad x \in \gamma. \tag{5}$$

In accordance to the geometry of the stiffener curve, we can see that for the initial state, the upper contact point A has coordinates $A(0, 0)$, and for an another contact point B touching the obstacle, we have coordinates $B(0, -1)$. Since displacements are expressed with the formula $\rho(x) = (c_1 + bx_2, c_2 - bx_1), x \in \gamma$, in an equilibrium state the coordinates are changed to $x + \rho(x)$, $x \in \gamma$. Therefore, for the points A, B we can exactly express their coordinates for an equilibrium state. For the point A we have:

$$x_A + \rho(x_A) = (0, 0) + (c_1, c_2) = (c_1, c_2).$$

And for the point B:

$$x_B + \rho(x_B) = (0, -1) + (c_1 - b, c_2) = (c_1 - b, c_2 - 1).$$

Further we can specify a condition for the point A in the sense of restrictions for displacements (in an equilibrium state), which reflects its final position with respect to the curve $x_2 = k_1 x_1$ delineating the obstacle

$$c_2 \leq k_1 c_1. \tag{6}$$

As can be easily seen, the point B should have final coordinates lying over the curve $x_2 = k_2 x_1 - 1$, so that

$$c_2 \geq k_2(c_1 - b)$$

or

$$c_2 - k_2 c_1 + k_2 b \geq 0. \tag{7}$$

Taking into account (6) and (7), let us write out the following convex and closed set in $H(\Omega)$, specifying admissible displacements

$$K = \{W \in H(\Omega) : k_2(c_1 - b) \leq c_2 \leq k_1 c_1, W|_\gamma = \rho, \rho(x) \in R(\gamma)\} \tag{8}$$

Here the term $c = (c_1, c_2)$ implies a uniform transfer vector (dilatation), whereas the term $b(x_2, -x_1)$ describes linearized rotation counterclock-wisely around the point $x+c$ at the right angle $\pi/2$ for $b < 0$, and $-\pi/2$ for $b > 0$.

Consider the minimization problem:

$$\text{find} \quad U \in K \quad \text{such that} \quad \Pi(U) = \inf_{W \in K} \Pi(W). \tag{9}$$

Due to coercivity and weak lower semicontinuity of $\Pi(W)$ on the Sobolev space $H(\Omega)$ implies that $\Pi(W)$ attains its minimums over K, at some function $U \in K$. Furthermore, strict convexity of the energy functional leads to uniqueness of the solution U. Let us reveal some relations characterizing qualitative properties of the problem under consideration. Under an assumption of additional regularity of the solution U, we can obtain from (9) an equivalent differential statement of the initial problem. The Gateaux differentiability of $\Pi(W)$ guarantees the equivalence of (9) to the following variational inequality

$$U \in K, \quad \int_\Omega \sigma_{ij}(U)\varepsilon_{ij}(W - U) \geq \int_\Omega F(W - U) \quad \forall W \in K. \tag{10}$$

By inserting the test functions of the following form $W = U + \phi$, $\phi \in C_0^\infty(\Omega)^2$ we obtain

$$\int_\Omega \sigma_{ij}(U)\varepsilon_{ij}(\phi) \geq \int_\Omega F\phi \quad \forall \phi \in C_0^\infty(\Omega)^2.$$

The substitution of test functions $W = U - \phi$, $\phi \in C_0^\infty(\Omega)^2$ gives us

$$\int_\Omega \sigma_{ij}(U)\varepsilon_{ij}(\phi) \leq \int_\Omega F\phi \quad \forall \phi \in C_0^\infty(\Omega)^2.$$

Therefore, we have in the sense of distributions

$$-\sigma_{ij,j}(U) = F_i \quad \text{in} \quad \Omega, \quad i = 1, 2. \tag{11}$$

In the following we will apply the following Green formula, which holds for sufficiently smooth functions V and $\overline{V} \in H(\Omega)$ [13, 14]

$$\int_{\Omega} \sigma_{ij}(V)\varepsilon_{ij}(\overline{V}) = -\int_{\Omega} \sigma_{ij,j}(V)\overline{v}_i + \int_{\Gamma} (\sigma_\nu(V)\overline{V}\nu + \sigma_\tau(V)\overline{V}\tau), \tag{12}$$

where $\nu = (\nu_1, \nu_2)$ is a unit normal vector to Γ,

$$\sigma_\nu(V) = \sigma_{ij}(V)\nu_i\nu_j, \quad \sigma_\tau(V) = (\sigma_\tau^1(V), \sigma_\tau^2(V)) = (\sigma_{1j}(V)\nu_j, \sigma_{2j}(V)\nu_j) - \sigma_\nu(V)\nu,$$
$$\overline{V}\nu = \overline{v}_i\nu_i, \quad \overline{V}\tau = (\overline{V}_\tau^1, \overline{V}_\tau^2), \quad \overline{v}_i = (\overline{V}\nu)\nu_i + \overline{V}_\tau^i, \quad i = 1, 2.$$

Taking into account the Green formula (12) and (11) we represent the variational inequality (10) in the following form

$$\int_{\Gamma_2} \Big(\sigma_\nu(U)(W - U)\nu + \sigma_\tau(U)(W - U)\tau\Big) \geq 0, \quad \forall W \in K. \tag{13}$$

Then, substituting into (13) functions $W = U + \tilde{W}$, with $\tilde{W} \in H(\Omega)$, $\tilde{W} = 0$ on Γ_1 and $\tilde{W} = U$ on γ, and applying (12) we infer

$$\int_{\Gamma_2 \setminus \gamma} (\sigma_\nu(U)\tilde{W}\nu + \sigma_\tau(U)\tilde{W}\tau) \geq 0. \tag{14}$$

From (14) it follows that

$$\sigma_\tau(U) = 0, \quad \sigma_\nu(U) = 0 \quad \text{on} \quad \Gamma_2 \setminus \gamma. \tag{15}$$

We insert the test functions $W = 0$ and $W = 2U$ into (13) and obtain

$$\int_{\gamma} (\sigma_\nu(U)\rho\nu + \sigma_\tau(U)\rho\tau) = 0, \tag{16}$$

where $\rho = U$ a.e. on γ. Finally, bearing in mind (15), (16) we have

$$\int_{\gamma} (\sigma_\nu(U)\rho\nu + \sigma_\tau(U)\rho\tau) \geq 0, \tag{17}$$

for all rigid displacements ρ described in K. The integral formulas (16), (17) imply a consequence of the principle of virtual displacements [19, 32], or equilibrium conditions for a rigid stiffener (inclusion) [34]. Following the same line of reasoning of [15], the converse can be proved, namely, that the differential setting consisting of (1), (2), (11), (15), (16), (17) leads to the variational formulation (10). The differential formulation of (10) can be obtained without additional regularity assumptions on the solution U. However, in this case, the expressions for the boundary conditions take the form of duality relations in the space of distributions.

2. THE CASE OF A VOLUME RIGID INCLUSION

We can consider a problem for a rigid volume inclusion, which may describe reinforcement of the body on a vicinity of its external boundary. Let us assume that a simply connected Lipschitz subdomain $\omega \subset \Omega$ has the boundary $\partial\omega$ satisfying $\partial\omega \cap \Gamma = \gamma$ (see Fig 2.).

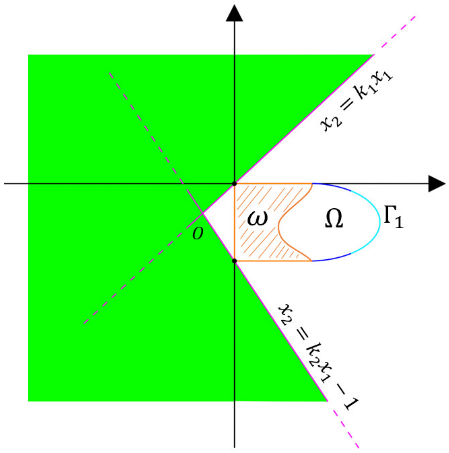

Figure 2: Geometry of the problem for composite body with the thin rigid inclusion

The space of infinitesimal rigid displacements $R(\omega)$ is defined according to (4). Now we introduce the following sets of possible displacements

$$K^\omega = \{W \in H(\Omega) \,:\, W|_\omega = \rho, \quad \rho(x) \in R(\omega),$$
$$k_2(c_1 - b) \le c_2 \le k_1 c_1\},$$

Consider the minimization problem:

$$\text{find} \quad U^\omega \in K^\omega \quad \text{such that} \quad \Pi(U^\omega) = \inf_{W \in K^\omega} \Pi(W). \tag{18}$$

As in the previous case, the set K^ω is convex and closed [20]. If the boundary of the domain $\Omega \setminus \overline{\omega}$ belongs to the class $C^{1,1}$, then variational problem (18) is equivalent to differential statement provided that solutions U^ω are sufficiently smooth in $\Omega \setminus \overline{\omega}$. For example, in the case of the set K^ω we have

$$-\sigma_{ij,j}(U^\omega) = F_i \quad \text{in} \quad \Omega \setminus \overline{\omega}, \quad i = 1, 2,$$

$$\sigma_{ij}(U^\omega) = c_{ijkl}\varepsilon_{kl}(U^\omega) \quad \text{in} \quad \Omega, \quad i, j = 1, 2,$$

$$U^\omega = (0, 0) \quad \text{on} \quad \Gamma_1,$$

$$\int_{\partial\omega\setminus\gamma} (\sigma_\nu(U^\omega)^- \rho_\omega \nu + \sigma_\tau(U^\omega)^- \rho_\omega \tau) = \int_\omega F\rho_\omega, \quad \text{where} \quad \rho_\omega = U^\omega \quad \text{in} \quad \omega, \tag{19}$$

$$\sigma_\tau(U^\omega)^- = 0, \quad \sigma_\nu(U^\omega)^- = 0 \quad \text{on} \quad \Gamma_2 \setminus \gamma, \tag{20}$$

$$\int_{\partial\omega\setminus\gamma} (\sigma_\nu(U^\omega)^- \rho\nu + \sigma_\tau(U^\omega)^- \rho\tau) \geq \int_\omega F\rho, \tag{21}$$

for all ρ from K^ω. Here, the relations (19)–(21) are written for the unit external normal vector ν to the boundary of the domain $\Omega \setminus \overline{\omega}$. The traces $\sigma_\nu(U^\omega)^-$ and $\sigma_\tau(U^\omega)^-$ are defined on the negative side $(\partial\omega\setminus\gamma)^-$. The negative $(\partial\omega\setminus\gamma)^-$ and positive side $(\partial\omega\setminus\gamma)^+$ of the curve $\partial\omega \setminus \gamma$ are selected with respect to the normal ν such that the positive side is a part of the boundary of inclusion ω. Then $\sigma_\nu(U^\omega)^-$ and $\sigma_\tau(U^\omega)^-$ are defined on the boundary of the deformed body $\Omega \setminus \overline{\omega}$. In addition, we note that the values $\sigma_\nu(U^\omega)^-$, $\sigma_\tau(U^\omega)^-$ can be nonzero on $(\partial\omega \setminus \gamma)^-$, despite of $\sigma_\nu(U^\omega)^+ = 0$, $\sigma_\tau(U^\omega)^+ = 0$ on $(\partial\omega \setminus \gamma)^+$ due to $\varepsilon_{ij}(U^\omega) = 0$ in the rigid inclusion ω, $i, j = 1, 2$.This case arises when the jumps of functions $\sigma_{ij}(U^\omega)$, $i, j = 1, 2$, are not equal to zero on $(\partial\omega \setminus \gamma)^-$ provided that $U^\omega \in H(\Omega)$, but $U^\omega \notin H^2(\Omega)^2$.

3. CONCLUSION

The obtained results justify the new class of pointwise contact problems. In contrast to the well-known Signorini condition, the proposed non-penetration conditions are imposed for possible displacements of two separate points located on the boundary of the rigid stiffener (inclusion). It is obvious that this approach can be applied for other geometrical configurations prescribed by polygons, and moreover, for a finite set of separate points. Namely, a finite set of points with non-penetration conditions can be considered for bodies with polygon shaped stiffeners contacting with polygon shaped obstacles. The obtained rigorous mathematical results are subject to research from the point of view of applications of the solid mechanics in the framework of contact problems for reinforced composite bodies. It should be noted that the results require numerical simulations and their subsequent comparison with experimental data. As the possible directions for further research we can mention the following issues: approximation by a family of equilibrium problems for elastic bodies, inverse problems, etc.

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