

# Partially Invariant Solutions of the Navier-Stokes Equations by Using the Polar Coordinate System

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## Abstract

The purpose of this paper is to find the partially invariant solutions of the Navier-Stokes equations. These solutions are constructed based on the 4-dimensional algebra that admitted the Navier-Stokes equations. The transformation from the Cartesian coordinate system to the polar coordinate system is presented and some partially invariant solutions are also found.

**Keywords:** Admitted Lie group, invariant solutions, partially invariant solutions, Navier-Stokes equations, Polar coordinate system

## 1. INTRODUCTION

Many physical phenomena are described by differential equations. Physical and mathematical modeling are very closely related. Physical phenomena are used to construct mathematical models, and on the other hand, the solutions of mathematical models are used to explain the physical phenomena. Almost all fundamental equations of physics are nonlinear equations, and in general, are very difficult to solve explicitly. Group analysis is a method for constructing exact solutions of differential equations. This method uses symmetry properties for constructing exact solutions. There are two types of solutions, the class of invariant solutions and partially invariant solutions which can be obtained by group analysis. Constructing of invariant and partially invariant solutions consists of some steps: choosing a subgroup of the admitted group, finding a representation of solution, substituting the representation into the studied system of equations and the study of compatibility of the obtained reduced system of equations. A survey of this method can be found in [11, 19].

In this manuscript, we construct partially invariant solutions of the Navier-Stokes equations by using the polar coordinate system. These solutions are constructed based on the 4-dimensional algebra that admitted the Navier-Stokes equations. Example of some partially invariant solutions are also found. They can return to new solutions of the Navier-Stokes equations.

## 2. INVARIANT AND PARTIALLY INVARIANT SOLUTIONS

Group analysis is one of the methods for constructing exact solutions of partial differential equations. This method uses the symmetry properties of differential equations. Symmetry means that any solutions of a given system of partial differential equations is transformed by a Lie group of transformations to a solution of the same system. Moreover, a symmetry allows finding new solutions of the system. There are two types of solutions, the class of invariant solutions and partially invariant solutions which can be obtained by Group analysis. The notion of invariant solution was introduced by Sophus Lie in 1895.

The notion of a partially invariant solution was introduced by Ovsiannikov [20]. This notion of partially invariant solutions generalized the notion of an invariant solution and extends the scope of applications of group analysis for constructing exact solutions of partial differential equations. The algorithm of finding invariant and partially invariant solutions consist of the following steps.

Let  $L^r$  be a Lie algebra with the basis  $X_1, \dots, X_r$ . The universal invariant  $J$  consists of  $s = m + n - r_*$  functionally independent invariants

$$J = (J^1(x, u), J^2(x, u), \dots, J^{m+n-r_*}(x, u)), \quad (1)$$

where  $n, m$  are the numbers of independent and dependent variables respectively and  $r_*$  is the total rank of the matrix composed by the coefficients of the generators  $X_i, (i = 1, 2, \dots, r)$ . If the rank of the Jacobi matrix  $\frac{\partial(J^1, \dots, J^{m+n-r_*})}{\partial(u^1, \dots, u^m)}$  is equal to  $q$ , then one can choose the first  $q \leq m$

invariants  $J^1, \dots, J^q$  such that the rank of the Jacobi matrix  $\frac{\partial(J^1, \dots, J^q)}{\partial(u_1, \dots, u_m)}$  is equal to  $q$ . A

partially invariant solution is characterized by two integers:  $\sigma \geq 0$  and  $\delta \geq 0$ . These solutions are also called  $H(\sigma, \delta)$ -solutions. The number  $\sigma$  is called the rank of a partially invariant solution. This number gives the number of the independent variables in the representation of the partially invariant solution. The number  $\delta$  is called the defect of a partially invariant solution. The defect is the number of the dependent functions which can not be found from the representation of partially invariant solution. The rank  $\sigma$  and the defect  $\delta$  must satisfy the conditions

$$\begin{aligned} \sigma &= \delta + n - r_* \geq 0, \delta \geq 0, \\ \rho \leq \sigma < n, \max\{r_* - n, m - q, 0\} \leq \delta \leq \min\{r_* - 1, m - 1\}, \end{aligned}$$

where  $\rho$  is the maximum number of invariants which depends only on the independent variables. Note that for the invariant solutions,  $\delta = 0$  and  $q = m$ .

For constructing a representation of a  $H(\sigma, \delta)$ -solution one needs to choose  $l = m - \delta$  invariants and separates the universal invariant in two parts:

$$\bar{J} = (J^1, \dots, J^l), \quad \bar{\bar{J}} = (J^{l+1}, J^{l+2}, \dots, J^{m+n-r_*}).$$

The number  $l$  satisfies the inequality  $1 \leq l \leq q \leq m$ . The representation of the  $H(\sigma, \delta)$ -

solution is obtained by assuming the first  $l$  coordinates  $\bar{J}$  of the universal invariant are functions of the invariants  $\bar{J}$ :

$$\bar{J} = W(\bar{J}). \quad (2)$$

Equation (2) is a form of the invariant part of the representation of a solution. The next assumption about a partially invariant solution is that Equation (2) can be solved for the first  $l$  dependent functions, for example,

$$u^i = \phi^i(u^{l+1}, u^{l+2}, \dots, u^m, x), (i=1, \dots, l). \quad (3)$$

It is important to note that the function  $W^i, (i=1, \dots, l)$  is involved in the expression for the functions  $\phi^i, (i=1, \dots, l)$ . The functions  $u^{l+1}, u^{l+2}, \dots, u^m$  are called superfluous. The rank and the defect of the  $H(\sigma, \delta)$ -solution are  $\delta = m - l$  and  $\sigma = m + n - r_* - l = \delta + n - r_*$  respectively.

Note that if  $\delta = 0$ , then the above algorithm is the algorithm for finding a representation of an invariant solution. If  $\delta \neq 0$ , then Equations (3) do not define all dependent functions. Since a partially invariant solution satisfies the restrictions (2), this algorithm cuts out some particular solutions from the set of all solutions.

After constructing the representation of an invariant or partially invariant solution (3), it must be substituted into the original system of equations. The system of equations obtained for the functions  $W$  and superfluous functions  $u^k, (k=l+1, 2, \dots, m)$  is called the reduced system. This system is overdetermined and requires an analysis of compatibility.

Compatibility analysis for invariant solutions is easier than for partially invariant solutions. Another case of partially invariant solutions which is easier than the general case occurs when  $\bar{J}$  only depends on the independent variables  $J^{l+1} = J^{l+1}(x), J^{l+2} = J^{l+2}(x), \dots, J^{m+n-r_*} = J^{m+n-r_*}(x)$ .

In this case, a partially invariant solution is called regular, otherwise it is irregular. The number  $\sigma - \rho$  is called the measure of irregularity.

The process of studying compatibility consists of reducing the overdetermined system of partial differential equations to an involutive system. During this process different subclasses of  $H(\sigma, \delta)$  partially invariant solutions can be obtained. Some of these subclasses can be  $H_1(\sigma_1, \delta_1)$ -solutions with subalgebra  $H_1 \subset H$ . In this case  $\sigma_1 \geq \sigma, \delta_1 \leq \delta$ . The study of compatibility of the partially invariant solutions with the same rank  $\sigma_1 = \sigma$ , but with smaller defect  $\delta_1 < \delta$  is simpler than the study of compatibility for  $H(\sigma, \delta)$ -solutions. In many applications, there is a reduction of a  $H(\sigma, \delta)$ -solution to a  $H_1(\sigma, 0)$  solution. In this case the  $H(\sigma, \delta)$ -solution is called reducible to an invariant solution. The problem of reduction to an invariant solution is important since the invariant solutions are usually studied first.

### 3. THE UNSTEADY NAVIER-STOKES EQUATIONS

Unsteady motion of incompressible viscous fluid is governed by the

Navier-Stokes equations

$$u_t + u \cdot \nabla u = -\nabla p + \Delta u, \quad \nabla \cdot u = 0, \tag{4}$$

where  $u = (u_1, u_2, u_3) = (u, v, w)$  is the velocity field,  $p$  is the fluid pressure,  $\nabla$  is the gradient operator in the three-dimensional space  $x = (x_1, x_2, x_3) = (x, y, z)$  and  $\Delta$  is the Laplacian. A group classification of the Navier-Stokes equations in the three-dimensional case<sup>1</sup> was done in [2]. The Lie group admitted by the Navier-Stokes equations is infinite. Its Lie algebra can be presented in the form of the direct sum  $L^\infty \oplus L^5$ , where the infinite-dimensional ideal  $L^\infty$  is generated by the operators<sup>2</sup>

$$X_{\phi_i} = \phi_i(t)\partial_{x_i} + \phi_i'(t)\partial_{u_i} - \phi_i''(t)x_i\partial_p, \quad X_\psi = \psi(t)\partial_p$$

with arbitrary functions  $\phi_i(t), (i=1,2,3)$  and  $\psi(t)$ . The subalgebra  $L^5$  has the following basis:

$$Y = 2t\partial_t + x_i\partial_{x_i} - u_i\partial_{u_i} - 2p\partial_p, \quad Z_0 = \partial_t,$$

$$Z_{ik} = x_i\partial_{x_k} - x_k\partial_{x_i} + u_i\partial_{u_k} - u_k\partial_{u_i}, \quad (i < k \leq 3).$$

The Galilean algebra  $L^{10}$  is contained in  $L^\infty \oplus L^5$ .

Several articles [1,3,4,8,13,15,25,27] are devoted to invariant solutions of the Navier-Stokes equations<sup>3</sup>. While partially invariant solutions of the Navier-Stokes equations have been less studied<sup>4</sup>, there has been substantial progress in studying such classes of solutions of inviscid gas dynamics equations [5,9,17,18,19,21,23,24,28].

#### 4. PARTIALLY INVARIANT SOLUTIONS BY USING POLAR COORDINATE SYSTEM

In this section analysis of compatibility of regular partially invariant solutions with defect 1 and rank 1 of the subalgebra based on the 4-dimensional algebra is given. This subalgebra is taken from the optimal system constructed for the gas dynamics equations [22]. Partially invariant solutions of the gas dynamics equations for this subalgebra were considered in [24].

The Navier-Stokes equations are used in the component form:

$$u_t + uu_x + vu_y + wu_z = -p_x + u_{xx} + u_{yy} + u_{zz}, \tag{5}$$

$$v_t + uv_x + vv_y + wv_z = -p_y + v_{xx} + v_{yy} + v_{zz}, \tag{6}$$

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<sup>1</sup>A classification of the two-dimensional Navier-Stokes equations was studied in [26].  
<sup>2</sup>There is still no complete classification of the subalgebras of the Lie algebra. Classification of infinite-dimensional subalgebras of this algebra was studied in [14].  
<sup>3</sup>Short reviews devoted to invariant solutions of the Navier-Stokes equations can be found in [4,6,7,16,27].  
<sup>4</sup>Firstly the approach of partially invariant solutions to the Navier-Stokes equations was applied in [27].

$$w_t + uw_x + vw_y + ww_z = -p_z + w_{xx} + w_{yy} + w_{zz}, \tag{7}$$

$$u_x + v_y + w_z = 0. \tag{8}$$

The dependent variables  $u, v, w$  and  $p$  are functions of the space variables  $x, y, z$  and time  $t$ .

The basis of generator that admit the Navier-Stokes equations is

$$\begin{aligned} X_1 &= \partial_x, & X_2 &= \partial_y, & X_3 &= \partial_z, \\ X_4 &= t\partial_x + \partial_u, & X_5 &= t\partial_y + \partial_v, & X_6 &= t\partial_z + \partial_w, \\ X_7 &= y\partial_z - z\partial_y + v\partial_w - w\partial_v, & X_8 &= z\partial_x - x\partial_z + w\partial_u - u\partial_w, \\ X_9 &= x\partial_y - y\partial_x + u\partial_v - v\partial_u, & X_{10} &= t\partial_t. \end{aligned} \tag{9}$$

The construction of a partially invariant solution consists of several steps. First, choose a subgroup from the equation (9). Then find a representation of a partially invariant solution. After that substitute the representation of the solution into the Navier-Stokes equations. Finally, one needs to study the compatibility of the obtained (reduced) system of equations. As a result, one obtains an exact solution of the Navier-Stokes equations.

For the sake of convenience, some problems require a special coordinate system. A coordinate transformation is a conversion from one system of the independent variables to another. In this paper, the polar coordinate system is used.

The simplest one is the Cartesian coordinate system. In this system  $\mathbf{x} = (x, y, z)$  is the vector of the independent variables,  $\mathbf{u} = (u, v, w)$  is the vector of the dependent variables. Relations of polar coordinate systems with the Cartesian coordinate system are given as follows.

The transformation from the Cartesian coordinate system to the polar coordinate system is presented by

$$\mathbf{x} = (x, y, z), \quad \mathbf{u} = (u, q \cos \varphi, q \sin \varphi)$$

or

$$u = u, \quad q = \sqrt{v^2 + w^2}, \quad \varphi = \arctan(w/v).$$

This coordinate system is used for convenience of writing a representation of partially invariant solutions. Analysis of partially invariant solutions is presented in detail for four examples.

**Subalgebra generated by:**  $\{X_2, X_3, X_4, X_7\}$

The basis of this subalgebra is  $\partial_y, \partial_z, t\partial_x + \partial_u, y\partial_z - z\partial_y + v\partial_w - w\partial_v$ .

In order to find an invariant solution, one needs to find a universal invariant of this subalgebra. Let a function  $J = J(t, x, y, z, u, v, w, p)$  be an invariant of the generator  $X_2$  and  $X_3$ . This means that  $(\partial_y)J = 0$  and  $(\partial_z)J = 0$ . So, the function  $J$  does not depend on  $y$  and  $z$ , it

means that  $J = J(t, x, u, v, w, p)$ . After substituting it into the equation  $(X_4)J = (t\partial_x + \partial_u)J = 0$ , one obtains the equation  $tJ_x + J_u = 0$ . The characteristics system is

$$\frac{dt}{0} = \frac{dx}{t} = \frac{du}{1} = \frac{dv}{0} = \frac{dw}{0} = \frac{dp}{0}.$$

Then  $J = J(t, u - x/t, v, w, p)$ . After substituting it into the equation

$$(X_7)J = (y\partial_z - z\partial_y + v\partial_w - w\partial_v)J = 0,$$

one obtains the equation  $vJ_w - wJ_v = 0$ . The characteristics system is

$$\frac{dt}{0} = \frac{d(u - x/t)}{1} = \frac{dv}{-w} = \frac{dw}{v} = \frac{dp}{0}.$$

Then  $J = J(t, u - x/t, q, p)$ , where

$$q = \sqrt{v^2 + w^2}, \quad v = q \cos \varphi, \quad w = q \sin \varphi, \quad \varphi = \arctan(w/v).$$

Thus, invariants of the Lie group corresponding to this subalgebra in the polar coordinate system are

$$t, u - x/t, q, p.$$

Hence, a representation of the partially invariant solution is

$$u = U(t) + \frac{x}{t}, \quad p = p(t), \quad q = q(t),$$

and the function  $\varphi(t, x, y, z)$  depends on all independent variables.

After substituting this representation into the Navier-Stokes equations (5)-(8), it can be written in the polar coordinate system as

$$tU' + U = 0, \tag{10}$$

$$\begin{aligned} & [(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - q\varphi_z \sin \varphi)t - (x + tU)\varphi_x]q \sin \varphi \\ & + t[q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + q' - q^2\varphi_y \sin \varphi] \cos \varphi = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} & [(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - q\varphi_z \sin \varphi)t - (x + tU)\varphi_x]q \cos \varphi \\ & - t[q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + q'] \sin \varphi + q^2\varphi_y \cos^2 \varphi = 0, \end{aligned} \tag{12}$$

$$qt(\varphi_y \sin \varphi - \varphi_z \cos \varphi) - 1 = 0. \tag{13}$$

Solving the equation (10), we have  $U = C_1/t$  where  $C_1$  is constant. Taking the combination of equations (11) and (12) by excluding the derivative  $\varphi_t$ , one has

$$q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + q' = 0. \tag{14}$$

For the compatibility analysis of systems (11)-(14), it is convenient to use an implicit representation of the function  $\varphi(t, x, y, z)$ . Assume that there is a function  $F(\varphi, t, s, y, z)$  such

that  $F_\varphi \neq 0$  and

$$F(\varphi(t, s, y, z), t, s, y, z) = 0. \quad (15)$$

Taking the total derivatives of equation (15) with respect to  $t, x, y, z$ , one has

$$\begin{aligned} D_t F &= F_\varphi \varphi_t + F_t = 0, \quad D_x F = F_\varphi \varphi_x + F_x = 0, \\ D_y F &= F_\varphi \varphi_y + F_y = 0, \quad D_z F = F_\varphi \varphi_z + F_z = 0, \\ D_x^2 F &= F_\varphi \varphi_{xx} + \varphi_x^2 F_{\varphi\varphi} + 2F_{x\varphi} \varphi_x + F_{xx} = 0, \\ D_y^2 F &= F_\varphi \varphi_{yy} + \varphi_y^2 F_{\varphi\varphi} + 2F_{y\varphi} \varphi_y + F_{yy} = 0, \\ D_z^2 F &= F_\varphi \varphi_{zz} + \varphi_z^2 F_{\varphi\varphi} + 2F_{z\varphi} \varphi_z + F_{zz} = 0. \end{aligned}$$

All derivatives of the function  $\varphi(t, x, y, z)$  can be found through the derivatives of the function  $F(\varphi, t, x, y, z)$  from these equations:

$$\begin{aligned} \varphi_t &= -F_t / F_\varphi, \quad \varphi_x = -F_x / F_\varphi, \quad \varphi_y = -F_y / F_\varphi, \quad \varphi_z = -F_z / F_\varphi, \\ \varphi_{xx} &= -(F_{\varphi\varphi} \varphi_x^2 + 2F_{\varphi x} \varphi_x + F_{xx}) / F_\varphi, \\ \varphi_{yy} &= -(F_{\varphi\varphi} \varphi_y^2 + 2F_{\varphi y} \varphi_y + F_{yy}) / F_\varphi, \\ \varphi_{zz} &= -(F_{\varphi\varphi} \varphi_z^2 + 2F_{\varphi z} \varphi_z + F_{zz}) / F_\varphi. \end{aligned}$$

Substituting these derivatives into equation (13), one obtains

$$F_y \sin \varphi - F_z \cos \varphi + a F_\varphi = 0, \quad (16)$$

where the function  $a = (qt)^{-1}$  only depends on  $t$ .

The characteristics system of the last equation is

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{\sin \varphi} = \frac{dz}{-\cos \varphi} = \frac{d\varphi}{a}.$$

The universal invariant is  $t, x, j_1, j_2$  where  $j_1 = \cos \varphi + ay$  and  $j_2 = \sin \varphi + az$ . Thus, the general solution of equation (16) is  $F = \phi(t, x, j_1, j_2)$ . Changing the function  $\varphi(t, x, y, z)$  in the equation (14) through the function  $F = \phi(t, x, j_1, j_2)$ , it becomes in this form

$$a_1 \cos^2 \varphi + a_2 \cos \varphi \sin \varphi + a_3 = 0 \quad (17)$$

where

$$\begin{aligned} a_1 &= (ta' + a) \left( \frac{\partial \phi}{\partial j_1} + \frac{\partial \phi}{\partial j_2} \right) \left( \frac{\partial \phi}{\partial j_1} - \frac{\partial \phi}{\partial j_2} \right), \\ a_2 &= (ta' + a) \frac{\partial \phi}{\partial j_1} \frac{\partial \phi}{\partial j_2}, \\ a_3 &= t \left( a \left( \frac{\partial \phi}{\partial x} \right)^2 - a' \left( \frac{\partial \phi}{\partial j_1} \right)^2 + a^3 \left( \frac{\partial \phi}{\partial j_2} \right)^2 \right) + a(ta^2 - 1) \left( \frac{\partial \phi}{\partial j_1} \right)^2. \end{aligned}$$

After splitting equation (17) with respect to  $\cos^2 \varphi$  and  $\cos \varphi \sin \varphi$  we get  $a_i = 0$ , ( $i=1,2,3$ ).

Consider case  $q \neq 0$ .

If  $ta' + a = 0$ , then  $a = C_2 / t$  where  $C_2$  is constant. Then the equation  $a_3 = 0$  becomes  $t^2 (\frac{\partial \phi}{\partial x})^2 + C_2^2 ((\frac{\partial \phi}{\partial j_1})^2 + (\frac{\partial \phi}{\partial j_2})^2) = 0$ . We get that  $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial j_1} = \frac{\partial \phi}{\partial j_2} = 0$ , which is contradiction.

If  $(\frac{\partial \phi}{\partial j_1} + \frac{\partial \phi}{\partial j_2})(\frac{\partial \phi}{\partial j_1} - \frac{\partial \phi}{\partial j_2}) = 0$ , then  $\frac{\partial \phi}{\partial j_1} = \pm \frac{\partial \phi}{\partial j_2}$ . Thus the equation  $a_2 = 0$  and  $a_3 = 0$  become  $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial j_1} = \frac{\partial \phi}{\partial j_2} = 0$ , which is contradiction.

So, there is no partially invariant solution in this case.

Consider case  $q = 0$ .

A representation of the partially invariant solution is

$$u = U(t) + \frac{x}{t}, \quad p = p(t), \quad q = 0,$$

and the function  $\varphi(t, x, y, z)$  depends on all independent variables. After substituting this representation into the Navier-Stokes equations (5)-(8), one obtains  $tU' + U = 0$  and  $-1 = 0$ , which is also contradiction. Thus, there is no partially invariant solution of the Navier-Stokes equations in this subgroup.

**Subalgebra generated by:**  $\{X_1, X_2, X_3, \beta X_4 + X_7, \beta \neq 0\}$

The basis of subalgebra is  $\partial_x, \partial_y, \partial_z, \beta(t\partial_x + \partial_u) + y\partial_z - z\partial_y + v\partial_w - w\partial_v$ .

Do the same as the previous example, so invariants of the Lie group corresponding to this subalgebra in the polar coordinate system are

$$t, u - \beta\varphi, q, p.$$

Hence, a representation of the partially invariant solution is

$$U = U(t) + \beta\varphi, \quad p = P(t), \quad q = q(t),$$

where  $q = \sqrt{v^2 + w^2}$ ,  $v = q \cos \varphi$ ,  $w = q \sin \varphi$  and  $\varphi = \arctan(w/v)$ .

After substituting this representation into the Navier-Stokes equations (5)-(8), one obtains the system of ordinary differential equations

$$U' + \beta(q(\varphi_z \sin \varphi + \varphi_y \cos \varphi) - \varphi_{xx} - \varphi_{yy} - \varphi_{zz} + \varphi_t + (\beta\varphi + U)\varphi_x) = 0, \quad (18)$$

$$\begin{aligned}
 & (\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - q\varphi_z \sin \varphi - (\beta\varphi + U)\varphi_x)q \sin \varphi \\
 & + (q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + q' - q^2\varphi_y \sin \varphi) \cos \varphi = 0,
 \end{aligned} \tag{19}$$

$$\begin{aligned}
 & [(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - q\varphi_z \sin \varphi - (\beta\varphi + U)\varphi_x]q \cos \varphi \\
 & - [(q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + q') \sin \varphi + q^2\varphi_y \cos^2 \varphi] = 0,
 \end{aligned} \tag{20}$$

$$\beta\varphi_x - q(\varphi_y \sin \varphi - \varphi_z \cos \varphi) = 0. \tag{21}$$

Taking the combination of equations (19) and (20) by excluding the derivative  $\varphi_t$ , one has

$$q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + q' = 0. \tag{22}$$

Changing of the function  $\varphi(t, x, y, z)$  through the function  $F(\varphi, t, s, y, z)$ , the equation (21) becomes,

$$F_y \sin \varphi - F_z \cos \varphi - aF_x = 0, \tag{23}$$

where the function  $a = \beta q^{-1}$  only depends on  $t$ .

The characteristics system of the last equation is

$$\frac{dt}{0} = \frac{dx}{a} = \frac{dy}{\sin \varphi} = \frac{dz}{-\cos \varphi} = \frac{d\varphi}{0}.$$

The universal invariants are  $t, j_1, j_2, \varphi$  where  $j_1 = x \sin \varphi + ay$  and  $j_2 = x \cos \varphi + az$ .

Changing the function  $\varphi(t, x, y, z)$  in the equation (21) through the function  $F = \varphi(t, j_1, j_2, \varphi)$ , it becomes in this form

$$a_1 \cos^2 \varphi + a_2 \cos \varphi \sin \varphi + a_3 = 0 \tag{24}$$

where

$$\begin{aligned}
 a_1 &= (x^2 a' + a) \left( \frac{\partial \varphi}{\partial j_1} + \frac{\partial \varphi}{\partial j_2} \right) \left( \frac{\partial \varphi}{\partial j_1} - \frac{\partial \varphi}{\partial j_2} \right), \\
 a_2 &= (x^2 a' + a) \frac{\partial \varphi}{\partial j_1} \frac{\partial \varphi}{\partial j_2}, \\
 a_3 &= \beta \left( (x^2 a' + a^3) \left( \frac{\partial \varphi}{\partial j_2} \right)^2 - a(a^2 + 1) \left( \frac{\partial \varphi}{\partial j_1} \right)^2 \right).
 \end{aligned}$$

After splitting the equation (24) with respect to  $\cos^2 \varphi$  and  $\cos \varphi \sin \varphi$  we get  $a_i = 0$ , ( $i=1, 2, 3$ ).

Consider case  $q \neq 0$ .

If  $x^2 a' + a = 0$ , then  $a = 0$ . This is contradiction.

If  $(\frac{\partial\phi}{\partial j_1} + \frac{\partial\phi}{\partial j_2})(\frac{\partial\phi}{\partial j_1} - \frac{\partial\phi}{\partial j_2}) = 0$ , then  $\frac{\partial\phi}{\partial j_1} = \pm \frac{\partial\phi}{\partial j_2}$ . From the equation  $a_2 = 0$ , we get  $\frac{\partial\phi}{\partial j_1} = \frac{\partial\phi}{\partial j_2} = 0$ , which is contradiction.

Consider case  $q = 0$ .

A representation of the partially invariant solution is

$$u = U(t) + \beta\varphi, \quad p = p(t), \quad q = 0,$$

and the function  $\varphi(t, x, y, z)$  depends on all independent variables.

After substituting this representation into the Navier-Stokes equations (18)-(21), one obtains

$$U' - \beta(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - (\beta\varphi + U)\varphi_x) = 0 \quad (25)$$

$$\beta\varphi_x = 0. \quad (26)$$

So  $\varphi$  does not depend on  $x$ , that is  $\varphi = \psi(t, y, z)$ . Thus the equation (25) becomes

$$U' - \beta(\psi_{yy} + \psi_{zz} - \psi_t) = 0. \quad (27)$$

In particular case, if  $U' = 0$  then we get  $U = C_1$  where  $C_1$  is constant, thus the last equation is

$$\psi_{yy} + \psi_{zz} - \psi_t = 0. \quad (28)$$

Solving the equation (28), one obtains

$$\psi = e^{C_2 t} (C_4 e^{\sqrt{C_3} y} + C_5 e^{-\sqrt{C_3} y}) (C_6 \sin \sqrt{C_3 - C_2} z + C_7 \cos \sqrt{C_3 - C_2} z)$$

where  $C_i, i = 2, \dots, 7$  are constants.

So, there is no partially invariant solution of the Navier-Stokes equations in case  $q \neq 0$ . For  $q = 0$ , the partially invariant solution of this subgroup is

$$u = C_1 + \beta e^{C_2 t} (C_4 e^{\sqrt{C_3} y} + C_5 e^{-\sqrt{C_3} y}) (C_6 \sin \sqrt{C_3 - C_2} z + C_7 \cos \sqrt{C_3 - C_2} z), \\ v = 0, \quad w = 0, \quad p = p(t),$$

where  $C_i, i = 1, \dots, 7$  are constants.

**Subalgebra generated by:**  $\{X_2, X_3, X_4, X_1 + X_7\}$

The basis of this subalgebra is  $\partial_y, \partial_z, t\partial_x + \partial_u, \partial_x + y\partial_z - z\partial_y + v\partial_w - w\partial_v$ .

Invariants of the Lie group corresponding to this subalgebra in the polar coordinate system are

$$t, u + (\varphi - x)/t, q, p.$$

Hence, a representation of the partially invariant solution is

$$U = U(t) - (\varphi - x)/t, \quad p = P(t), \quad q = q(t),$$

where  $q = \sqrt{v^2 + w^2}$ ,  $v = q \cos \varphi$ ,  $w = q \sin \varphi$  and  $\varphi = \arctan(w/v)$ .

After substituting this representation into the Navier-Stokes equations (5)-(8), it can be written in the polar coordinate system as

$$\begin{aligned} t(tU' + U + \varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - q\varphi_y \cos \varphi - q\varphi_z \sin \varphi) \\ - (x + tU - \varphi)\varphi_x = 0, \end{aligned} \quad (29)$$

$$\begin{aligned} [(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - q\varphi_z \sin \varphi)t - (x + tU - \varphi)\varphi_x]q \sin \varphi \\ + t[q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + q' - q^2 \varphi_y \sin \varphi] \cos \varphi = 0, \end{aligned} \quad (30)$$

$$\begin{aligned} [(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - q\varphi_z \sin \varphi)t - (x + tU - \varphi)\varphi_x]q \cos \varphi \\ - t[[q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + q'] \sin \varphi + q^2 \varphi_y \cos^2 \varphi] = 0, \end{aligned} \quad (31)$$

$$qt(\varphi_y \sin \varphi - \varphi_z \cos \varphi) + \varphi_x - 1 = 0. \quad (32)$$

Taking the combination of equations (30) and (31) by excluding the derivative  $\varphi_t$ , one has

$$q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + q' = 0. \quad (33)$$

Changing of the function  $\varphi(t, x, y, z)$  through the function  $F(\varphi, t, x, y, z)$ , the equation (32) becomes,

$$F_y \sin \varphi - F_z \cos \varphi + aF_x + aF_\varphi = 0, \quad (34)$$

where the function  $a = (qt)^{-1}$  only depends on  $t$ .

The characteristics system of the last equation is

$$\frac{dt}{0} = \frac{dx}{a} = \frac{dy}{\sin \varphi} = \frac{dz}{-\cos \varphi} = \frac{d\varphi}{a}.$$

The universal invariants are

$$t, j_1, j_2, j_3$$

where  $j_1 = \cos \varphi + ay$ ,  $j_2 = \sin \varphi + az$  and  $j_3 = \varphi - x$ .

Changing the function  $\varphi(t, x, y, z)$  in the equation (33) through the function  $F = \phi(t, j_1, j_2, j_3)$ , it becomes in this form

$$a_1 \cos^2 \varphi + a_2 \cos \varphi \sin \varphi + a_3 \cos \varphi + a_4 \sin \varphi + a_5 = 0 \quad (35)$$

where

$$\begin{aligned}
 a_1 &= (ta' + a)\left(\frac{\partial\phi}{\partial j_1} + \frac{\partial\phi}{\partial j_2}\right)\left(\frac{\partial\phi}{\partial j_1} - \frac{\partial\phi}{\partial j_2}\right), \\
 a_2 &= (ta' + a)\frac{\partial\phi}{\partial j_1}\frac{\partial\phi}{\partial j_2}, \\
 a_3 &= (ta' + a)\frac{\partial\phi}{\partial j_2}\frac{\partial\phi}{\partial j_3}, \quad a_4 = (ta' + a)\frac{\partial\phi}{\partial j_1}\frac{\partial\phi}{\partial j_3}, \\
 a_5 &= a((ta^2 - 1)\left(\frac{\partial\phi}{\partial j_1}\right)^2 + ta^2\left(\frac{\partial\phi}{\partial j_2}\right)^2 + (t-1)\left(\frac{\partial\phi}{\partial j_3}\right)^2) - ta'\left(\left(\frac{\partial\phi}{\partial j_1}\right)^2 + \left(\frac{\partial\phi}{\partial j_3}\right)^2\right).
 \end{aligned}$$

After splitting the equation (35) with respect to  $\cos^2\varphi$ ,  $\cos\varphi\sin\varphi$ ,  $\cos\varphi$  and  $\sin\varphi$ , we get  $a_i = 0$ , ( $i=1,2,3,4,5$ ).

Consider case  $q \neq 0$ .

If  $ta' + a = 0$ , then  $a = C_1/t$  where  $C_1$  is a constant. Then equation  $a_5 = 0$  becomes

$$C_1^2\left(\left(\frac{\partial\phi}{\partial j_1}\right)^2 + \left(\frac{\partial\phi}{\partial j_2}\right)^2\right) + t^2\left(\frac{\partial\phi}{\partial j_3}\right)^2 = 0.$$

It implies that  $\frac{\partial\phi}{\partial j_1} = \frac{\partial\phi}{\partial j_2} = \frac{\partial\phi}{\partial j_3} = 0$ , which is contradiction.

If  $\left(\frac{\partial\phi}{\partial j_1} + \frac{\partial\phi}{\partial j_2}\right)\left(\frac{\partial\phi}{\partial j_1} - \frac{\partial\phi}{\partial j_2}\right) = 0$ , then  $\frac{\partial\phi}{\partial j_1} = \pm\frac{\partial\phi}{\partial j_2}$ . Thus the equation  $a_2 = 0$  becomes

$\frac{\partial\phi}{\partial j_1} = \frac{\partial\phi}{\partial j_2} = 0$ . Since  $a_3 = 0$ ,  $a_4 = 0$  and  $a_5 = 0$ , we get  $\frac{\partial\phi}{\partial j_3} = 0$  which is contradiction.

Consider case  $q = 0$ .

Hence, a representation of the partially invariant solution is

$$u = U(t) - (\varphi - x)/t, \quad p = p(t), \quad q = 0,$$

and the function  $\varphi(t, x, y, z)$  depends on all independent variables.

After substituting this representation into the Navier-Stokes equations (18)-(21), we obtain

$$t(tU' + U + \varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t) - (x + tU - \varphi)\varphi_x = 0, \quad (36)$$

$$\varphi_x - 1 = 0. \quad (37)$$

Solving the last equation, one obtains  $\varphi = x + \psi(t, y, z)$ . Substituting  $\varphi$  in the equation (36), we get

$$t^2U' + t(\psi_{yy} + \psi_{zz} - \psi_t) + \psi = 0. \quad (38)$$

In particular case, if  $U' = 0$  then we find  $U = C_1$  where  $C_1$  is a constant, thus the last equation becomes

$$t(\psi_{yy} + \psi_{zz} - \psi_t) + \psi = 0. \quad (39)$$

Solving the equation (39), one obtains

$$\psi = te^{C_2 t} (C_4 e^{\sqrt{C_3} y} + C_5 e^{-\sqrt{C_3} y}) (C_6 \sin \sqrt{C_3 - C_2} z + C_7 \cos \sqrt{C_3 - C_2} z)$$

where  $C_i, i = 2, \dots, 7$  are constants.

So, there is no partially invariant solution of the Navier-Stokes equations in case  $q \neq 0$ . For  $q = 0$ , the partially invariant solution of this subgroup is

$$u = C_1 - e^{C_2 t} (C_4 e^{\sqrt{C_3} y} + C_5 e^{-\sqrt{C_3} y}) (C_6 \sin \sqrt{C_3 - C_2} z + C_7 \cos \sqrt{C_3 - C_2} z), \\ v = 0, w = 0, p = p(t),$$

where  $C_i, i = 1, \dots, 7$  are constants.

### Subalgebra generated by: $\{X_2, X_3, X_7, X_{10}\}$

The basis of this subalgebra is  $\partial_y, \partial_z, \partial_t, y\partial_z - z\partial_y + v\partial_w - w\partial_v$ .

Invariants of the Lie group corresponding to this subalgebra in the polar coordinate system are

$$x, u, q, p.$$

Hence, a representation of the partially invariant solution is

$$u = U(x), p = P(x), q = q(x),$$

where  $q = \sqrt{v^2 + w^2}$ ,  $v = q \cos \varphi$ ,  $w = q \sin \varphi$ ,  $\varphi = \arctan (w/v)$ .

After substituting this representation into the Navier-Stokes equations (5)-(8), one obtains the system of ordinary differential equations

$$P' - U'' + UU' = 0, \quad (40)$$

$$[(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - \varphi_z q \sin \varphi)q + (2q' - Uq)\varphi_x] \sin \varphi \\ - [q'' - Uq' - q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + q^2 \varphi_y \sin \varphi] \cos \varphi = 0, \quad (41)$$

$$[(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - \varphi_z q \sin \varphi)q + (2q' - Uq)\varphi_x] \cos \varphi \\ + [q'' - Uq' - q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2)] \sin \varphi - q^2 \varphi_y \cos^2 \varphi = 0, \quad (42)$$

$$q(\varphi_y \sin \varphi - \varphi_z \cos \varphi) - U' = 0. \quad (43)$$

Notice that for  $q = 0$  the general solution of equations (40)-(43) is

$$U = C_1, P = C_2, v = 0, w = 0,$$

where  $C_1$  and  $C_2$  are constants.

Therefore, the partially invariant solution of this case is

$$u = C_1, v = 0, w = 0, p = C_2,$$

where  $C_1$  and  $C_2$  are constants.

Further consideration is given for  $q \neq 0$ .

Integrating equation (40) with respect to  $x$ , one finds  $P = U' - 2^{-1}U^2 + C_1$ .

Taking the combination of equations (41) and (42) by excluding the derivative  $\varphi_t$ , one has

$$q'' - Uq' - q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) = 0. \quad (44)$$

Changing of the function  $\varphi(t, x, y, z)$  through the function  $F(\varphi, t, x, y, z)$ , equation (43) becomes,

$$F_y \sin \varphi - F_z \cos \varphi + aF_\varphi = 0, \quad (45)$$

where the function  $a = q^{-1}U'$  only depends on  $x$ .

**Case 1.** Let  $a = 0$ , or  $U = C_1$ . Notice that if  $\varphi$  is a constant, then the general solution of equations (40)-(43) is  $P = C_2$ ,  $q = C_3 + C_4 e^{C_1 x}$ , where  $C_1, C_2, C_3$  and  $C_4$  are constants.

Assume that  $\varphi$  is not a constant. The universal invariants are

$$t, x, \hat{y}, \varphi,$$

where  $\hat{y} = y \cos \varphi + z \sin \varphi$ .

Thus, the general solution of equation (45) is  $F = \phi(t, x, \hat{y}, \varphi)$ . Since  $F_\varphi = (-y \sin \varphi + z \cos \varphi)\phi_{\hat{y}} + \phi_\varphi \neq 0$ , this gives that  $\phi_{\hat{y}}^2 + \phi_\varphi^2 \neq 0$ . Substituting the function  $F = \phi(t, x, \hat{y}, \varphi)$  into equation (44), one obtains

$$a_1 z^2 + a_2 z + a_3 = 0 \quad (46)$$

where

$$\begin{aligned} a_1 &= -(q'' - C_1 q')\phi_{\hat{y}}^2, \\ a_2 &= -2(q'' - C_1 q')[\phi_\varphi \cos \varphi - \hat{y}\phi_{\hat{y}} \sin \varphi]\phi_{\hat{y}}, \\ a_3 &= -[(q'' - C_1 q')\phi_\varphi^2 - q\phi_x^2 - (\hat{y}^2(q'' - C_1 q') + q)\phi_{\hat{y}}^2]\cos^2 \varphi \\ &\quad + (q'' - C_1 q')(2\phi_\varphi \cos \varphi \sin \varphi - \hat{y}\phi_{\hat{y}})\hat{y}\phi_{\hat{y}}. \end{aligned}$$

The coefficients  $a_1$ ,  $a_2$  and  $a_3$  do not depend on  $z$ . Splitting (46) with respect to  $z$ , one finds  $a_i = 0$ , ( $i = 1, 2, 3$ ).

If  $\phi_{\hat{y}} \neq 0$ , then the equation  $a_1 = 0$  implies that  $q'' - C_1 q' = 0$ , and the equation  $a_3 = 0$

becomes  $q(\phi_y^2 + \phi_x^2) = 0$ . Because of  $\phi_y^2 + \phi_x^2 \neq 0$ , one has  $q = 0$  which contradicts the assumption  $q \neq 0$ .

If  $\phi_y = 0$ , the equation  $a_3 = 0$  implies that  $(q'' - C_1 q')\phi_\varphi^2 - q\phi_x^2 = 0$ .

The general solution of the last equation is  $\phi = \tilde{\phi}(t, \tilde{\varphi})$ , where  $\tilde{\varphi} = \varphi + H(x)$ ,

$$H(x) = \int h(x) dx, h = \pm((q'' - C_1 q')/q)^{1/2},$$

and  $\phi_\varphi = \tilde{\phi}_{\tilde{\varphi}} \neq 0$ . Substituting  $\phi = \tilde{\phi}(t, \tilde{\varphi})$  into equation (41), one obtains

$$b_1(x)\tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}} + b_2(x)\tilde{\phi}_{\tilde{\varphi}} + \tilde{\phi}_t = 0, \tag{47}$$

where  $b_1 = h^2$ ,  $b_2 = -q^{-1}(2q' - C_1 q)h - h'$ . Let us analyze this equation. Differentiating it with respect to  $x$ , one obtains  $b_1'\tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}} + b_2'\tilde{\phi}_{\tilde{\varphi}} = 0$ .

Assume first that  $b_1' \neq 0$ . Then the last equation can be rewritten as  $\tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}}/\tilde{\phi}_{\tilde{\varphi}} = -b_2'/b_1'$ . Differentiating it with respect to  $x$ , one obtains  $(-b_2'/b_1')' = 0$ . Thus  $b_2 = -C_2 b_1 + C_3$ . This relation, after substituting representations for the coefficients  $b_1$  and  $b_2$  becomes

$$\begin{aligned} & ((q'' - C_1 q')q)^{1/2}(C_2(q'' - C_1 q') + C_3 q) + (q'' - C_1 q')(2q' - C_1 q) \\ & - [(q''' - C_1 q'')q - (q'' - C_1 q')q']2^{-1} = 0. \end{aligned}$$

In this case the solution of equation (47) is  $\tilde{\phi} = C_4 e^{C_2(\tilde{\varphi} - C_3 t)} + C_5$ . Returning back to the solution of equations (41)-(43), one gets  $\varphi = C_3 t - H(x) + C$ , where  $C = C_5(C_2 C_4)^{-1}$ .

For the case  $b_1' = 0$ , then  $b_2' = 0$ , or  $b_1 = C_2 \geq 0$  and  $b_2 = C_3$ . This means that

$$(q'' - C_1 q')/q = C_2, ((2q' - C_1 q)C_2^{1/2})/q = \pm C_3.$$

Let  $C_2 = 0$ , then  $C_3 = 0$ , and hence  $q = C_4 + C_5 e^{C_1 x}$ . Equation (47) becomes  $\tilde{\phi}_t = 0$ . This means that the function  $\tilde{\phi}$  only depends on  $\tilde{\varphi}$ . Returning back to the solution of equations (41)-(43), one has  $\varphi = -H(x) + C_6$ .

If  $C_2 > 0$ , then  $q = C_4 e^{\tilde{C}x/2}$ , where  $\tilde{C} = C_1 \pm C_3 C_2^{-1/2}$  and the constants  $C_1, C_2, C_3$  and  $C_4$  are related by the equation  $C_3^2 - C_2(C_1^2 + 4C_2) = 0$ . The solution of equation (47) is  $\phi = \tilde{\phi}(t, \tilde{\varphi})$  where the function  $\tilde{\phi}(t, \tilde{\varphi})$  is a solution of the linear parabolic equation with constant coefficients

$$C_2 \tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}} + C_3 \tilde{\phi}_{\tilde{\varphi}} + \tilde{\phi}_t = 0.$$

Solving the last equation, one obtains

$$\tilde{\phi} = e^{-c_5 t} (C_6 e^{-(c_3 - \sqrt{c_3^2 + 4c_2 c_5})\tilde{\varphi}/(2c_2)} + C_7 e^{-(c_3 + \sqrt{c_3^2 + 4c_2 c_5})\tilde{\varphi}/(2c_2)}),$$

where  $C_5, C_6$  and  $C_7$  are constants.

**Case 2.** Assume that  $a \neq 0$ .

The characteristic system of the equation (45) is

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{\sin \varphi} = \frac{dz}{-\cos \varphi} = \frac{d\varphi}{a}.$$

Hence, the universal invariants of this system are

$$t, x, y_1, y_2,$$

where  $y_1 = \cos \varphi + ay$  and  $y_2 = \sin \varphi + az$ .

Thus, the general solution of equation (43) is  $F = \phi(t, x, y_1, y_2)$ , where  $\phi_{y_1}^2 + \phi_{y_2}^2 \neq 0$ . Substituting the derivatives of the function  $\phi(t, x, y, z)$  through the derivatives of the function  $F(\varphi, t, x, y, z)$  into equation (42), one obtains

$$a_1 \cos^2 \varphi + a_2 \cos \varphi \sin \varphi + a_3 \cos \varphi + a_4 \sin \varphi + a_5 = 0 \quad (48)$$

where

$$\begin{aligned} a_1 &= [(q'' - Uq')a^2 + q(a')^2](\phi_{y_1} + \phi_{y_2})(\phi_{y_1} - \phi_{y_2}), \\ a_2 &= 2[(q'' - Uq')a^2 + q(a')^2]\phi_{y_1}\phi_{y_2}, \\ a_3 &= -2[(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_x]qa'\phi_{y_1}, \\ a_4 &= -2[(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_x]qa'\phi_{y_2}, \\ a_5 &= [[2(y_1\phi_{y_1} + y_2\phi_{y_2})a'\phi_x + (a^2\phi_{y_2}^2 + \phi_x^2)a]q - \\ &\quad (q'' - Uq' - qa^2)a\phi_{y_1}^2]a + [(y_1\phi_{y_1} + y_2\phi_{y_2})^2 + \phi_{y_2}^2]q(a')^2. \end{aligned}$$

Note that  $a_1, a_2, a_3, a_4$  and  $a_5$  does not depend on  $\varphi$ . Splitting (48) with respect to  $\cos^2 \varphi, \cos \varphi \sin \varphi, \cos \varphi, \sin \varphi$ , one obtains  $a_i = 0, (i=1, 2, \dots, 5)$ .

Noting that  $\phi_{y_1}^2 + \phi_{y_2}^2 \neq 0$ , the equations  $a_1 = 0, a_2 = 0$  imply that

$$(q'' - Uq')a^2 + q(a')^2 = 0,$$

and the equations  $a_3 = 0, a_4 = 0$  imply

$$[(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_x]a' = 0. \quad (49)$$

The last equation is split into two cases.

**Case 2.1.** Assume that  $a' \neq 0$ , then equation (49) gives

$$(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_x = 0.$$

The general solution of this equation is  $\phi = f(t, g_1, g_2)$ , where  $g_1 = y_2 / y_1, g_2 = a / y_1$  and  $f_{g_1}^2 + f_{g_2}^2 \neq 0$ . Substituting the function  $\phi$  into the equation  $a_5 = 0$ , one obtains  $(q'' - Uq' - qa^2)[(g_1f_{g_1} + g_2f_{g_2})^2 + f_{g_1}^2] = 0$ .

Since  $f_{g_1}^2 + f_{g_2}^2 \neq 0$ , then  $(g_1 f_{g_1} + g_2 f_{g_2})^2 + f_{g_1}^2 \neq 0$ . It means that  $q'' - Uq' - qa^2 = 0$ . Finding  $q''$  from this equation, and substituting it into the equation  $(q'' - Uq')a^2 + q(a')^2 = 0$ , one has  $(a')^2 + a^4 = 0$ . This gives  $a = 0$ , which contradicts the assumption  $a \neq 0$ .

Case 2.2. Assume that  $a' = 0$ . Substituting  $a' = 0$  into the equation  $a_5 = 0$ , one obtains  $(q'' - Uq' - qa^2)\phi_{y_1}^2 - q(a^2\phi_{y_2}^2 + \phi_x^2) = 0$ . Using here the equation  $(q'' - Uq')a^2 + q(a')^2 = 0$ , one finds  $a^2\phi_{y_1}^2 + a^2\phi_{y_2}^2 + \phi_x^2 = 0$ . Hence,  $\phi_{y_1} = \phi_{y_2} = \phi_x = 0$ , which is a contradiction to the condition  $\phi_{y_1}^2 + \phi_{y_2}^2 \neq 0$ . Therefore, there is no partially invariant solutions in case  $a \neq 0$ .

## 6. CONCLUSION

This manuscript devoted to the construction partially invariant solutions of the Navier-Stokes equations by using the polar coordinate system. This coordinate system is used for convenience of writing a representation of partially invariant solutions. Subgroups for studying are taken from the optimal system constructed for the gas dynamics equations [22].

The algorithm of obtaining partially invariant solutions were applied to the Navier-Stokes equations. As a result of this study some exact partially invariant solutions corresponding to some subalgebras are presented. These possibilities extend an area of using Group analysis for constructing new exact solutions.

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