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# Certain Definite Integral involving Modified Struve Function, Log Function and Hypergeometric Function

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## Abstract

Special functions are very useful in the field of Mathematical Analysis, Functional Analysis, Geometry, physics, and Satellite Communication. They are also used in the field of transmission of wave propagation. So they have a unique identity. The Struve function is one of the important special functions. It is used in electrodynamics, potential theory, and optics. In this paper, we have developed certain definite integral involving Modified Struve Function, Log function in the form of Hypergeometric function.

**Key Words :** Bessel Function, Hypergeometric Function, Struve function, Modified Struve function.

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## 1. Introduction

Struve functions are solutions of the non-homogeneous Bessel's differential equation:

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \alpha^2)y = \frac{4(\frac{z}{2})^{\alpha+1}}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \text{...........(1.1)}$$

and are defined as:

$$H_\alpha(z) = \frac{2(\frac{z}{2})^{\alpha}}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\frac{\pi}{2}} sin(z\cos\theta) sin^{2\alpha}(\theta)\, d\theta \text{..........(1.2)}$$

Modified Struve function is:

$$L_\alpha(z) = I_{-\alpha}(z) - \frac{2(\frac{z}{2})^{\alpha}}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\infty} sin(zu)(1+u^2)^{\alpha-\frac{1}{2}}\, du \text{............(1.3)}$$

Bessel functions of the first kind, denoted as $J_\alpha(z)$, are solutions of Bessel's differential equation that are finite at the origin ($z=0$) for integer or positive $\alpha$, and diverge as $z$ approaches zero for negative non-integer $\alpha$ ( See[15]). It is possible to define the function by its Taylor series expansion around $z=0$.

$$J_\alpha(z)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\left(\frac{z}{2}\right)^{2m+\alpha}\text{.......................}(1.4)$$

where $\Gamma(\eta)$ is the gamma function. The Bessel function of the first kind is an entire function if $\alpha$ is an integer.

The Bessel functions are valid even for complex arguments $z$, and an important special case is that of a purely imaginary argument( See[15]). In this case, the solutions to the Bessel equation are called the modified Bessel of the first and second kind. The first kind of modified Bessel function is defined as

$$I_\alpha(z)=\iota^{-\alpha}J_\alpha(\iota z)=\sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma(m+\alpha+1)}\left(\frac{z}{2}\right)^{2m+\alpha}\text{...........}(1.5)$$

A generalized hypergeometric function ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k}=\frac{P(k)}{Q(k)}=\frac{(k+a_1)(k+a_2)\ldots(k+a_p)}{(k+b_1)(k+b_2)\ldots(k+b_q)(k+1)}z\text{.........}(1.6)$$

Where $k+1$ in the denominator is present for historical reasons of notation[ Koepf p.12(2.9)], and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{matrix}a_1,a_2,\cdots,a_p & ; & \\ & & z\\ b_1,b_2,\cdots,b_q & ; & \end{matrix}\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k z^k}{(b_1)_k(b_2)_k\cdots(b_q)_k k!}\text{.........}(1.7)$$

where the parameters $b_1,b_2,\cdots,b_q$ are positive integers.

The ${}_pF_q$ series converges for all finite z if $p\le q$, converges for $|z|<1$ if $p=q+1$, diverges for all z , $z\neq 0$ if $p>q+1$[Luke p.156(3)].

Pochhammer symbol is defined as [ Steffensen p.8]

$$(\zeta)_n=\zeta(\zeta-1)(\zeta-2)\text{.......}(\zeta-n+1)=\prod_{k=1}^{n}(\zeta-k+1)=\prod_{k=0}^{n-1}(\zeta-k)\text{.......}(1.8)$$

## 2. Main Formulae of the Integration

$$\int_0^1 log\,x\,L_0(ax)dx=-\frac{1}{2\pi}[a\;{}_3F_4(1,1,1;\frac{3}{2},\frac{3}{2},2,2;\frac{a^2}{4})]\text{................}(2.1)$$

$$\int_0^1 x\, log\, x\, L_0(ax)dx = \frac{1}{\pi a^2}[2a - \pi\, L_0(a)]........................(2.2)$$

$$\int_0^1 x^2\, log\, x\, L_0(ax)dx = -\frac{1}{2\pi a}[{}_3F_4(1,1,1;\frac{1}{2},\frac{1}{2},2,2;\frac{a^2}{4})-1]..............(2.3)$$

$$\int_0^1 x^4\, log\, x\, L_0(ax)dx = \frac{1}{8\pi a^3}[-4\;{}_3F_4(1,1,1;-\frac{1}{2},-\frac{1}{2},2,2;\frac{a^2}{4})+a^2+4]........(2.4)$$

$$\int_0^1 x^6\, log\, x\, L_0(ax)dx = \frac{1}{72\pi a^5}[-324\;{}_3F_4(1,1,1;-\frac{3}{2},-\frac{3}{2},2,2;\frac{a^2}{4})+4a^4+9a^2+324]....(2.5)$$

$$\int_0^1 x^8 \log x\, L_0(ax)dx = \frac{1}{288\pi a^7}[-32400\;{}_3F_4(1,1,1;-\frac{5}{2},-\frac{5}{2},2,2;\frac{a^2}{4})+$$

$$+9a^6+16a^4+324a^2+32400]...........................(2.6)$$

$$\int_0^1 x^{10} \log x\, L_0(ax)dx = \frac{1}{800\pi a^9}[-4410000\;{}_3F_4(1,1,1;-\frac{7}{2},-\frac{7}{2},2,2;\frac{a^2}{4})+$$

$$+16a^8+25a^6+400a^4+22500a^2+4410000].............(2.7)$$

$$\int_0^1 x^{12} \log x\, L_0(ax)dx = \frac{1}{7200\pi a^{11}}[-3214890000\;{}_3F_4(1,1,1;-\frac{9}{2},-\frac{9}{2},2,2;\frac{a^2}{4})+$$

$$+100a^{10}+144a^8+2025a^6+90000a^4+9922500a^2+3214890000]............(2.8)$$

$$\int_0^1 x^{14} \log x\, L_0(ax)dx = \frac{1}{352800\pi a^{13}}[-19061082810000\;{}_3F_4(1,1,1;-\frac{11}{2},-\frac{11}{2},2,2;\frac{a^2}{4})+$$

$$+3600a^{12}+4900a^{10}+63504a^8+2480625a^6+$$

$$+216090000a^4+39382402500a^2+19061082810000]\ldots\ldots\ldots(2.9)$$

$$\int_0^1 x^{16}\log x\,L_0(ax)dx=$$

$$=\frac{1}{6272\pi a^{15}}[-57267964353600\;{}_3F_4(1,1,1;-\frac{13}{2},-\frac{13}{2},2,2;\frac{a^2}{4})+$$

$$+49a^{14}+64a^{12}+784a^{10}+28224a^8+2160900a^6+$$

$$+311169600a^4+84715923600a^2+57267964353600]\ldots\ldots\ldots\ldots(2.10)$$

$$\int_0^1 x^{18}\log x\,L_0(ax)dx+$$

$$=\frac{1}{508032\pi a^{17}}[-1043708650344360000\;{}_3F_4(1,1,1;-\frac{15}{2},-\frac{15}{2},2,2;\frac{a^2}{4})+$$

$$+3136a^{16}+3969a^{14}+46656a^{12}+1587600a^{10}+112021056a^8+14177664900a^6+$$

$$+3049773249600a^4+1159676278160400a^2+1043708650344360000]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.11)$$

$$\int_0^1 x^{20}\,log\,x\,L_0(ax)dx=-\frac{1}{242\pi}[a\;{}_3F_4(1,11,11;\frac{3}{2},\frac{3}{2},12,12;\frac{a^2}{4})]\ldots\ldots\ldots\ldots\ldots(2.12)$$

$$\int_0^1 x^{100}\,log\,x\,L_0(ax)dx=-\frac{1}{5202\pi}[a\;{}_3F_4(1,51,51;\frac{3}{2},\frac{3}{2},52,52;\frac{a^2}{4})]\ldots\ldots\ldots\ldots(2.13)$$

$$\int_0^1 x^{190}\,log\,x\,L_0(ax)dx=-\frac{1}{18432\pi}[a\;{}_3F_4(1,96,96;\frac{3}{2},\frac{3}{2},97,97;\frac{a^2}{4})]\ldots\ldots\ldots\ldots(2.14)$$

$$\int_0^1 x^{1190}\,log\,x\,L_0(ax)dx=-\frac{1}{710432\pi}[a\;{}_3F_4(1,596,596;\frac{3}{2},\frac{3}{2},597,597;\frac{a^2}{4})]\ldots\ldots\ldots(2.15)$$

$$\int_0^1 x\, log\, x\, L_1(ax)dx = -\frac{1}{2\pi}[{}_3F_4(1,1,1;\frac{1}{2},\frac{3}{2},2,2;\frac{a^2}{4})-1]\text{.........}(2.16)$$

$$\int_0^1 x^3\, log\, x\, L_1(ax)dx = \frac{1}{8\pi a^2}[4\; {}_3F_4(1,1,1;-\frac{1}{2},\frac{1}{2},2,2;\frac{a^2}{4})+a^2-4]\text{...............}(2.17)$$

$$\int_0^1 x^9 \log x\, L_1(ax)dx = -\frac{1}{2400\pi a^8}[-1890000\; {}_3F_4(1,1,1;-\frac{7}{2},-\frac{5}{2},2,2;\frac{a^2}{4})-$$

$$-48a^8+75a^6+400a^4+13500a^2+1890000]\text{.............}(2.18)$$

$$\int_0^1 x^{14} \log x\, L_2(ax)dx =$$

$$=\frac{1}{94080\pi a^{13}}[-6007129128000\; {}_3F_4(1,1,1;-\frac{13}{2},-\frac{9}{2},2,2;\frac{a^2}{4})+$$

$$+245a^{14}-960a^{12}+3920a^{10}+28224a^8+926100a^6+$$

$$+74088000a^4+12835746000a^2+6007129128000]\text{................}(2.19)$$

$$\int_0^1 x^{17}\, log\, x\, L_7(ax)dx = -\frac{1}{685134450\pi}[a^8\; {}_3F_4(1,13,13;\frac{3}{2},\frac{17}{2},14,14;\frac{a^2}{4})]\text{.........}(2.20)$$

$$\int_0^1 x^{21}\, log\, x\, L_7(ax)dx = -\frac{1}{912161250\pi}[a^8\; {}_3F_4(1,15,15;\frac{3}{2},\frac{17}{2},16,16;\frac{a^2}{4})]\text{.........}(2.21)$$

$$\int_0^1 x^{11}\, log\, x\, L_9(ax)dx = -\frac{1}{158444436\pi}[a^{10}\; {}_3F_4(1,11,11;\frac{3}{2},\frac{21}{2},12,12;\frac{a^2}{4})]\text{........}(2.22)$$

$$\int_0^1 x^{13}\, log\, x\, L_{11}(ax)dx = -\frac{1}{106887140410050\pi}[a^{12}\; {}_3F_4(1,13,13;\frac{3}{2},\frac{25}{2},14,14;\frac{a^2}{4})]\text{.........}(2.23)$$

## 3. Derivation of the Integration

Derivation of (2.2)

$$\int_0^1 x\, log\, x\, L_0(ax)dx = \frac{1}{\pi\, a^2}[ax(\pi\, log\, x\, L_1(ax)+2)-\pi\, L_0(ax)]_0^1 = \frac{1}{\pi a^2}[2a-\pi\, L_0(a)]\text{.........}(3.1)$$

Derivation of (2.3)

$$\int_0^1 x^2 \log x\, L_0(ax)dx = \frac{-1}{2\pi\, a}[x^2({}_3F_4(1,1,1;\frac{1}{2},\frac{1}{2},2,2;\frac{a^2x^2}{4})-$$

$$-2\log x\ {}_2F_3(1,1;\frac{1}{2},\frac{1}{2},2;\frac{a^2x^2}{4})+2\log x-1]_0^1$$

$$=-\frac{1}{2\pi a}[{}_3F_4(1,1,1;\frac{1}{2},\frac{1}{2},2,2;\frac{a^2}{4})-1]\text{..............}(3.2)$$

Derivation of (2.4)

$$\int_0^1 x^4 \log x\, L_0(ax)dx = [\frac{1}{8\pi a^3}x^2(-4\ {}_3F_4(1,1,1;-\frac{1}{2},-\frac{1}{2},2,2;\frac{a^2x^2}{4})+$$

$$+8\log x\ {}_2F_3(1,1;-\frac{1}{2},-\frac{1}{2},2;\frac{a^2x^2}{4})+a^2x^2-4a^2x^2\log x-8\log x+4]_0^1$$

$$=\frac{1}{8\pi a^3}[-4\ {}_3F_4(1,1,1;-\frac{1}{2},-\frac{1}{2},2,2;\frac{a^2}{4})+a^2+4]\text{..........}(3.3)$$

On the same way other integration can be derived.

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