International Journal of Difference Equations (IJDE).
ISSN 0973-6069 Volume 17, Number 2 (2022), pp. 267-279

https://www.ripublication.com/ijde.htm

# Existence and Uniqueness Results of the Implicit Fractional Differential Equations by Topological Degree Methods

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**Abstract**

This paper attempts to study the existence and uniqueness of solving implicit fractional differential equations. Depends on the applications of some fixed point theorems with topological degree techniques for some conditions in Banach space. An example of this can be confirmed in the results.

**Keywords:** Fractional derivatives and integrals; Topological properties of mappings, Implicit fractional differential equations and Fixed point theorems.
***2010 MSC*:** 26A33; 34A08; 47H10; 58K15.

## 1 Introduction

Fractional calculus has proven to be valuable tools in modeling many phenomena in various fields of applied sciences and engineering such as astrophysics, acoustic control, chaotic dynamics, chemical engineering, electro chemistry, economics, optics, medicine, porous media and polymer physics. See for example [1, 2, 3, 4, 5, 6, 7, 8, 16, 18, 19, 20, 23, 25, 28, 29] and the references therein.

Topological degree methods have emerged as the most important tool in studying a large number of problems that occurs in nonlinear analysis. For example, Isia applied the aforesaid degree theorem to establish the necessary conditions for the existence of solutions of some nonlinear integral equations, see [17, 26, 27, 31]. The topological degree theory has named some authors the priori estimate method that Wang and his coauthors used to find out the conditions for a solution to a nonlinear differential equation of the fractional order, see [13]. Due to its importance in various fields, it has received increasing attention and occupied a central place in the interest of researchers and mathematicians.

Recently, great attention has been paid to the existence of solutions to the problem of boundary value problem and boundary conditions of implicit fractional differential equations and integral equations with a partial Caputo derivative. See [9, 10, 11, 12, 32]. In [14], Benchohra and Lazreg established the existence and uniqueness of the solution to a class of boundary value problems of implicit fractional differential equations with the Caputo fractional derivative. The arguments are based on the Banach contraction principle, Schauder's fixed point theorem and the nonlinear alternative of Leray-Schauder type.

In [15], Some sufficient conditions for the existence and uniqueness of the solution to the following boundary value problem of the implicit fractional differential equations are obtained as a result of applying the Arzela Ascoli's theorem and Banach contraction theorem with the topological degree theorem.

$$\begin{cases} {}^{c}\mathcal{D}^{\gamma}v(t) = \mathcal{F}(t, v(t), {}^{c}\mathcal{D}^{\gamma}v(t)), & t \in I = [0,1], \gamma \in (0,1], \\ v(0) = \lambda b(v) + \lambda v(\xi), \quad \xi, \lambda \in (0,1) \end{cases}$$

where $\mathcal{F} \in C([0,1] \times R \times R)$ and the nonlocal function $b : C(J, R) \to R$ is continuous. They have also developed results for Hyers-Ulam and generalized Hyers-Ulam type stabilization for the aforementioned to above BVP under some specific conditions.

In [14],using the Banach contraction principle, Schauder's fixed point theorem and a nonlinear alternative of type Leray Schauder , Benchohra and Lazreg investigated the existence and uniqueness of the results of nonlinear implicit fractional differential equations with boundary conditions of the type:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}y(t) = f(t, y(t), {}^{c}\mathcal{D}^{\alpha}x(t)), & t \in J = [0,T], T > 0, \ 1 < \alpha \le 2, \\ y(0) = y_0, \ y(T) = y_1, \end{cases}$$

where ${}^{c}\mathcal{D}^{\alpha}$ is the Caputo fractional derivative, $f : J \times R \times R \to R$ is a given function and $y_0, y_1 \in R$.

In [24], Tate and Dind investigated the existence and uniqueness of solutions to the boundary value problem of nonlinear implicit fractional differential equations involving the standard Riemann–Liouville fractional derivative. The results were based on Banach's contraction mapping principal and Krasnoselskii's fixed point theorem.

In [22], Nanware et al. studied the existence and uniqueness of the following nonlinear initial value problem for a system of implicit fractional differential equations

$$\begin{cases} \mathcal{D}_1^{\alpha}x(t) = f(t, x(t), \mathcal{D}_1^{\alpha}x(t)), & t \in J = [1,T], T > 1, \\ x^{(k)}(1) = x_k, \quad x_k \in R^n, \ k = 0, 1, 2, ..., m-1, \end{cases}$$

for some $\alpha \in (m-1, m], m \in N$, where $f : J \times R^n \times R^n \to R^n$ be a nonlinear continuous function, $x : J \to R^n$ and $\mathcal{D}_1^\alpha$ denotes the Caputo-Hadamard derivative of order $\alpha$.

In this paper, we demonstrate the existence and uniqueness of the results of the following implicit fractional differential equation in Banach space $\mathcal{X}$:

$$\begin{cases} {}^c\mathcal{D}^q x(t) = \xi(t, x(t), {}^c\mathcal{D}^q x(t)), \quad t \in \mathcal{J} = [0, T], T > 0, \ 1 < q \leq 2, \\ x(0) = \alpha, \ x(T) = \beta, \end{cases} \tag{1}$$

where ${}^c\mathcal{D}^q$ is the fractional Caputo derivative, $\xi : \mathcal{J} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a given function. $\mathcal{C}(\mathcal{J}, \mathcal{X})$ will be a Banach space of all continuous functions from $\mathcal{J}$ into $\mathcal{X}$ with the norm $\|x\| := sup\{\|x(t)\| : x \in \mathcal{C}(\mathcal{J}, \mathcal{X})\}$ for $t \in \mathcal{J}$, and $\alpha, \beta \in \mathcal{X}$.

## 2 Preliminaries

In this section, we recall some of the basic definitions, propositions, lemma and basic theorems that will be used in this paper.

**Definition 2.1.** *[21] The $q^{th}$ fractional order integral of a continuous function $\xi$ in the closed interval $[a, b]$, is defined by*

$$\mathcal{I}_a^q \xi(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \xi(s) ds, \tag{2}$$

*Where $\Gamma$ is the gamma function.*

**Definition 2.2.** *[21] The $q^{th}$ Riemann- Liouville fractional-order derivative of a continuous function $\xi$ in the closed interval $[a, b]$, is defined by*

$$(\mathcal{D}_{a+}^q \xi)(t) = \frac{1}{\Gamma(n-q)} (\frac{d}{dt})^n \int_a^t (t-s)^{n-q-1} \xi(s) ds. \tag{3}$$

*Where $n = [q] + 1$ and $[q]$ is the integer part of $q$.*

**Definition 2.3.** *[21] For a given continuous function $\xi$ in the closed interval $[a, b]$, the Caputo fractional order derivative of $\xi$, is defined by*

$$({}^c\mathcal{D}_{a+}^q \xi)(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} \xi^{(n)}(s) ds, \tag{4}$$

*where $n = [q] + 1$ .*

**Lemma 2.1.** *[30] Let $n-1 < q \leq n$ , then*

$$\mathcal{I}^q({}^c\mathcal{D}^q \xi)(t) = \xi(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

*, for some $c_i \in \mathcal{X}, i = 0, 1, 2, ..., n-1, n = [q] + 1$ .*

**Definition 2.4.** *([25], [33]) Let $\Omega \subset \mathcal{X}$ and $\mathcal{F} : \Omega \to \mathcal{X}$ be a continuous bounded map. One can say that $\mathcal{F}$ is $\alpha$-Lipschitz if there exists $k \geq 0$ such that*

$$\alpha(\mathcal{F}(B)) \leq k\alpha(B) \quad (\forall)\ B \subset \Omega \text{ bounded.}$$

*In case, $k < 1$, then we call $\mathcal{F}$ is a strict $\alpha$-contraction. One can say that $\mathcal{F}$ is $\alpha$-condensing if*

$$\alpha(\mathcal{F}(B)) < \alpha(B) \quad (\forall)\ B \subset \Omega \text{ bounded with } \alpha(B) > 0.$$

*We recall that $\mathcal{F} : \Omega \to \mathcal{X}$ is Lipschitz if there exists $k > 0$ such that*

$$\|\mathcal{F}_x - \mathcal{F}_y\| \leq k\|x - y\| \quad (\forall)\ x, y \subset \Omega,$$

*and if $k < 1$ then $\mathcal{F}$ is a strict contraction.*

**Theorem 2.1.** *(Banach contraction mapping principle)[33]*
*Let $\mathcal{X}$ be a complete metric space, and $\psi : \mathcal{X} \to \mathcal{X}$ is a contraction mapping with a contraction constant $\mathcal{K}$, then $\psi$ has a unique fixed point.*

**Theorem 2.2.** *(Schaefer's fixed point theorem)[33]*
*Let $\mathcal{K}$ be a nonempty convex, closed and bounded subset of a Banach space $\mathcal{X}$. If $\psi : \mathcal{K} \to \mathcal{K}$ is a complete continuous operator such that $\psi(\mathcal{K}) \subset \mathcal{X}$, then $\psi$ has at least one fixed point in $\mathcal{K}$.*

**Proposition 2.1.** *([25], [33]) If $\mathcal{F}, \mathcal{G} : \Omega \to \mathcal{X}$ are $\alpha$-Lipschitz maps with the constants $k$, $k^{'}$ respectively, then $\mathcal{F} + \mathcal{G} : \Omega \to \mathcal{X}$ is $\alpha$-Lipschitz with constant $k + k^{'}$.*

**Proposition 2.2.** *([25], [33]) If $\mathcal{F} : \Omega \to \mathcal{X}$ is compact , then $\mathcal{F}$ is $\alpha$-Lipschitz with zero constant.*

**Proposition 2.3.** *([25], [33])) If $\mathcal{F} : \Omega \to \mathcal{X}$ is Lipschitz with a constant $k$, then $\mathcal{F}$ is $\alpha$-Lipschitz with the same constant $k$.*

# 3 Main Results

**Definition 3.1.** *The function $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ is said to be the solution to the problem (1) if it satisfies the equation ${}^c\mathcal{D}^q x(t) = \xi(t, x(t), {}^c\mathcal{D}^q x(t))$ on $\mathcal{J}$, and conditions $x(0) = \alpha,\ x(T) = \beta$.*

In order to discuss the existence and uniqueness of the IFDE (1) solution, we need the following assumptions:

**H1** The function $\xi : \mathcal{J} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is continuous.

**H2** There exist constants $K > 0$ and $L > 0$ such that

$$\|\xi(t,x,y) - \xi(t,\bar{x},\bar{y})\| \leq K\|x-\bar{x}\| + L\|y-\bar{y}\|$$

for any $x, y, \bar{x}, \bar{y} \in \mathcal{X}$ and $t \in \mathcal{J}$.

**H3** There exist positive values $p, r, s \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ with $s^* = sup_{t\in\mathcal{J}} s(t) < 1$ such that for $t \in \mathcal{J}$ and $x, y \in \mathcal{X}$

$$\|\xi(t,x,y)\| \leq p(t) + r(t)\|x\| + s(t)\|y\|,$$

where $p^* = sup_{t\in\mathcal{J}} p(t)$ and $r^* = sup_{t\in\mathcal{J}} r(t)$.

**Lemma 3.1.** *Let $1 < q \leq 2$ and $\psi(t) =^c \mathcal{D}^q x(t)$ be continuous. The function $x$ is the solution to the IFDE (1) if and only if, $x$ is the solution to the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\psi(s)ds - \frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\psi(s)ds + (1-\frac{t}{T})\alpha + \frac{t}{T}\beta. \quad (5)$$

*Proof.* First, suppose that $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ satisfies the IFDE(1), then we need to show that $x$ is also a solution of FIE(5). By Lemma (2.1), we can reduce problem (1) to equation

$$x(t) = \mathcal{I}^q\psi(t) + c_0 + c_1 t = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\psi(s)ds + c_0 + c_1 t \quad (6)$$

For the constants $\alpha, \beta \in \mathcal{X}$, the conditions $x(0) = \alpha$ and $x(T) = \beta$ give $x(0) = \alpha = c_0$ and

$$x(T) = \beta = \frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\psi(s)ds + \alpha + c_1 T$$
$$\Rightarrow c_1 = \frac{1}{T}\beta - \frac{1}{T}\alpha - \frac{1}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\psi(s)ds$$

Then the solution to problem (1) is given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\psi(s)ds + \alpha + t[\frac{1}{T}\beta - \frac{1}{T}\alpha - \frac{1}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\psi(s)ds] \\ &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\psi(s)ds - \frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\psi(s)ds + \alpha + \frac{t}{T}\beta - \frac{t}{T}\alpha \\ &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\psi(s)ds - \frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\psi(s)ds + (1-\frac{t}{T})\alpha + \frac{t}{T}\beta \end{aligned}$$

Conversely, suppose $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ satisfies FIE(5). If $t = 0$ then $x(0) = \alpha$ and $x(T) = \beta$. For $t < T \in \mathcal{J}$ using the facts that the Caputo fractional derivative ${}^c\mathcal{D}_t^q$ is the left inverse of the fractional integral $\mathcal{I}_t^q$ and the Caputo derivative of the constant equal to zero. Hence we get ${}^c\mathcal{D}^q x(t) = \xi(t, x(t), {}^c\mathcal{D}^q x(t))$ which completes the proof.

□

From expression (5), we can turn the problem (1) into a fixed point problem. Define the operator $\mathcal{H} : \mathcal{C}(\mathcal{J}, \mathcal{X}) \to \mathcal{C}(\mathcal{J}, \mathcal{X})$ by;

$$\mathcal{H}(x)(t) = \iota(t) + \int_0^T \varpi(t,s)\psi(s)ds, \tag{7}$$

where $\psi(t) \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ which satisfies the implicit functional equation $\psi(t) = \xi(t, x(t), \psi(t))$,

$$\iota(t) = \alpha + \frac{(\beta-\alpha)t}{T}, \tag{8}$$

and

$$\varpi(t,s) = \frac{1}{\Gamma(q)} \left\{ \begin{array}{ccc} (t-s)^{q-1} & if & 0 \le s < t \\ \frac{-t}{T}(T-s)^{q-1} & if & t \le s < T \end{array} \right\} \tag{9}$$

**Lemma 3.2.** *The operator $\iota(t) \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ is $\alpha-$ Lipschitz with zero constant. Moreover, $\iota$ is completely continuous.*

*Proof.* Using the definition (2.4), we have $\|\iota_x(t) - \iota_y(t)\| = 0$ for each $x, y \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ which means $\iota$ is a Lipschitz with a constant zero. By Proposition (2.3), $\iota$ is $\alpha-$ Lipschitz with zero constant. It is clear that for every sequence $\{x_n\}$ of a bounded set $\mathbb{B}_k(k > 0) \subseteq \mathcal{C}(\mathcal{J}, \mathcal{X})$, we get $\iota$ is continuous and bounded for $0 \le t \le T$.
For $0 \le t_1 \le t_2 \le T$, we have

$$\|\iota(t_2) - \iota(t_1)\| = \frac{|\beta-\alpha|}{|T|}\|t_2 - t_1\|$$

As $t_2 \to t_1$, the right hand side of the inequality tends to zero. Hence, $\iota$ is equicontinuous. As a result of the Arzela Ascoli theorem, we conclude that $\iota$ is completely continuous. □

**Lemma 3.3.** *The operator $\zeta : \mathcal{J} \times \mathcal{X} \to \mathcal{X}$ defined by;*

$$\zeta(x)(t) = \int_0^T \varpi(t,s)\psi(s)ds,$$

*is continuous and compact. Consequently $\zeta$ is $\alpha$-Lipschitz with the zero constant.*

*Proof.* Clearly, $\varpi$ is continuous on $[0,T] \times [0,T]$. Denote by

$$\varpi^* = sup\{\|\varpi(t,s)\|, \quad (t,s) \in \mathcal{J} \times \mathcal{X}\}.$$

In order to proof the compactness of $\zeta$ , we consider several steps, as follows:
**Step 1:** $\zeta$ is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $\mathcal{C}(\mathcal{J}, \mathcal{X})$. Then for each $t \in \mathcal{J}$

$$\|(\zeta x_n)(t) - (\zeta x)(t)\| = \int_0^T \|\varpi(t,s)\|\|\psi_n(s) - \psi(s)\|ds \tag{10}$$

where $\psi_n, \psi \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ such that $\psi_n(t) = \xi(t, x_n(t), \psi(t))$ and $\psi(t) = \xi(t, x(t), \psi(t))$. By (H2), we get

$$\begin{aligned}\|\psi_n(t) - \psi(t)\| &= \|\xi(t, x_n(t), \psi(t)) - \xi(t, x(t), \psi(t))\| \\ &\leq K\|x_n(t) - x(t)\| + L\|\psi_n(t) - \psi(t)\|\end{aligned}$$

Then

$$\|\psi_n(t) - \psi(t)\| \leq \frac{K}{1-L}\|x_n(t) - x(t)\|.$$

Since $x_n \to x$ then for each $t \in \mathcal{J}$, we get $\psi_n(t) \to \psi(t)$ as $n \to \infty$. Let $\mu > 0$ be integrable on $\mathcal{J}$ such that, for each $t \in \mathcal{J}$, we have $\|\psi_n(t)\| \leq \mu$ and $\|\psi(t)\| \leq \mu$. Then we have

$$\|\varpi(t,s)\|\|\psi_n(s) - \psi(s)\| \leq \|\varpi(t,s)\|[\|\psi_n(s)\| + \|\psi(s)\|] \leq 2\mu\|\varpi(t,s)\|$$

Thus for each $t \in \mathcal{J}$, the function $s \to 2\mu\|\varpi(t,s)\|$ is integrable on $\mathcal{J}$. The sense of Bochner integral and (10) indicate that $\|(\zeta x_n)(t) - (\zeta x)(t)\| \to 0$ as $n \to \infty$. Hence $\zeta$ is continuous.

**Step 2:** $\zeta$ maps bounded sets into bounded sets in $\mathcal{C}(\mathcal{J}, \mathcal{X})$.
It suffices to show that for any $\delta > 0$, there is a positive $\varepsilon$ such that for $x \in \mathbb{B}_\delta = \{x \in \mathcal{C}(\mathcal{J}, \mathcal{X}) : \|x\| \leq \delta\}$, we have $\|\zeta(x)\| \leq \varepsilon$. For $x \in \mathbb{B}_\delta$, we have, for each $t \in \mathcal{J}$.

$$\|(\zeta x)(t)\| \leq \int_0^T \|\varpi(t,s)\|\|\psi(s)\|ds$$

Then

$$\|(\zeta x)(t)\| \leq \varpi^* \int_0^T \|\psi(s)\|ds \tag{11}$$

By (H3), for each $t \in \mathcal{J}$, we have

$$\begin{aligned}\|\psi(t)\| &= \|\xi(t, x(t), \psi(t))\| \\ &\leq p(t) + r(t)\|x(t)\| + s(t)\|\psi(t)\| \\ &\leq p(t) + r(t)\delta + s(t)\|\psi(t)\| \\ &\leq p^* + r^*\delta + s^*\|\psi(t)\|\end{aligned}$$

Then

$$\|\psi(s)\| \leq \frac{p^* + r^*\delta}{1 - s^*} := M^*$$

Thus (11) implies that

$$\|(\zeta x)(t)\| \leq \varpi^* M^* T := \varepsilon.$$

Hence $\zeta$ maps bounded sets into bounded sets in $\mathcal{C}(\mathcal{J}, \mathcal{X})$.

**Step 3:** $\zeta$ maps bounded sets into equicontinuous sets of $\mathcal{C}(\mathcal{J}, \mathcal{X})$.
Let $t_1, t_2 \in \mathcal{J}, t_1 < t_2$, and let $x \in \mathbb{B}_\delta$.

Then

$$\begin{aligned}\|(\zeta x)(t_2)-(\zeta x)(t_1)\| &\leq \int_0^T \|\varpi(t_2,s)-\varpi(t_1,s)\|\|\psi(s)\|ds\\ &\leq \frac{p^*+r^*\delta}{1-s^*}\int_0^T \|\varpi(t_2,s)-\varpi(t_1,s)\|ds\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the inequality tends to zero. Hence $\zeta(\mathbb{B}_\delta)$ is relatively compact.

As a result of step 1 to 3 together with the Arzela Ascoli theorem, we conclude that $\zeta$ is continuous and compact. As consequence of proposition (2.2), we conclude that $\zeta$ is $\alpha-$ Lipschitz with a constant of zero. □

**Theorem 3.1.** *Assume (H1)-(H3) hold. If*

$$\frac{r^*T\varpi^*}{1-s^*}<1, \tag{12}$$

*then the IFDE (1) has at least one solution and the set of solutions is bounded.*

*Proof.* Obviously, that having a fixed point that satisfies the operator $\mathcal{H}$ is also equivalent to having a solution for IFDE (1). As we explained in Lemma (3.2), $\iota$ is $\alpha-$ Lipschitz with a constant zero and by Lemma (3.3), we get $\zeta$ is also $\alpha-$ Lipschitz with a constant zero therefore, one can get that $\mathcal{H}$ is $\alpha-$ condensing with a constant zero. As a result of Krasnoselskii's fixed point theorem, we conclude that $\mathcal{H}$ has at least one fixed point that corresponds to the IFDE (1) solution.

Now, consider

$$\mathcal{S}=\{x\in\mathcal{C}(\mathcal{J},\mathcal{X}) : \exists\lambda\in[0,1]\ \textit{such that}\ x=\lambda\mathcal{H}x\}$$

For $x\in\mathcal{S}$, we have

$$\begin{aligned}\|x(t)\| &= \lambda\|\iota(t)+\int_0^T \varpi(t,s)\psi(s)ds\|\\ &\leq \lambda[2|\alpha|+|\beta|+\varpi^*M^*T] =: K^*.\end{aligned}$$

The above inequality shows that $\mathcal{S}$ is bounded which means that the set of IFDE (1) solutions is bounded.

□

**Theorem 3.2.** *Assume (H1) and (H2) hold. If*

$$\frac{KT\varpi^*}{1-L}<1, \tag{13}$$

*then there exists a unique solution for IFDE (1).*

*Proof.* Obviously, the fixed points of the operator $\mathcal{H}$ are solutions to problem (1). Let $u, v \in \mathcal{C}(\mathcal{J}, \mathcal{X})$. Then for $t \in \mathcal{J}$, we have

$$\|(\mathcal{H}u)(t) - (\mathcal{H}v)(t)\| = \int_0^T \|\varpi(t,s)\| \|\psi(s) - \varphi(s)\| ds, \tag{14}$$

where $\psi, \varphi \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ such that $\psi(t) = \xi(t, u(t), \psi(t))$, and $\varphi(t) = \xi(t, v(t), \varphi(t))$. By (H2) we have

$$\begin{aligned}\|\psi(t) - \varphi(t)\| &= \|\xi(t, u(t), \psi(t)) - \xi(t, v(t), \varphi(t))\| \\ &\leq K\|u(t) - v(t)\| + L\|\psi(t) - \varphi(t)\|.\end{aligned}$$

Thus $\|\psi(t) - \varphi(t)\| \leq \frac{K}{1-L}\|u(t) - v(t)\|$.
By (14), we have

$$\begin{aligned}\|(\mathcal{H}u)(t) - (\mathcal{H}v)(t)\| &\leq \frac{K}{1-L} \int_0^T \|\varpi(t,s)\| \|u(t) - v(t)\| ds \\ &\leq \frac{KT\varpi^*}{1-L}\|u(t) - v(t)\|.\end{aligned}$$

Then $\|\mathcal{H}u - \mathcal{H}v\| \leq \frac{KT\varpi^*}{1-L}\|u - v\|$. The assumption (13) and the above inequality give that the operator $\mathcal{H}$ is a contraction map with the contraction constant $\frac{KT\varpi^*}{1-L}$. Thus, according to Banach's Contraction Principle, $\mathcal{H}$ has a unique fixed point that represents a unique solution to the problem (1). □

**Example 3.1.** *Consider the following fractional boundary value problem*

$$\begin{cases} {}^c\mathcal{D}^{\frac{3}{2}} x(t) = \frac{2 + |x(t)| + |{}^c\mathcal{D}^{\frac{3}{2}} x(t)|}{10e^t(1 + |x(t)| + |{}^c\mathcal{D}^{\frac{3}{2}} x(t)|)}, \quad t \in [0,1], \\ x(0) = 1, \ x(1) = 2 \end{cases} \tag{15}$$

*Set* $\xi(t,x,y) = \frac{2+|x|+|y|}{10e^t(1+|x|+|y|)}$, $t \in [0,1]$, $x, y \in R$. *It is clear that the function* $\xi$ *is jointly continuous. For each* $x, y, \bar{x}, \bar{y} \in R$ *and* $t \in [0,1]$, *we have*

$$\begin{aligned}|f(t,x,y) - f(t,\bar{x},\bar{y})| &\leq |\frac{2+|x|+|y|}{10e^t(1+|x|+|y|)} - \frac{2+|\bar{x}|+|\bar{y}|}{10e^t(1+|\bar{x}|+|\bar{y}|)}| \\ &= \frac{1}{10e^t}|\frac{|x|+|y|-|\bar{x}|-|\bar{y}|}{(1+|x|+|y|)(1+|\bar{x}|+|\bar{y}|)}|, \quad t \in [0,1] \\ &\leq \frac{1}{10}|x-\bar{x}| + \frac{1}{10}|y-\bar{y}| \Rightarrow K = L = \frac{1}{10}.\end{aligned}$$

*Also, we have, for each* $x, y \in R$ *and* $t \in [0,1]$

$$\begin{aligned}|f(t,x,y)| &= |\frac{2+|x|+|y|}{10e^t(1+|x|+|y|)}| \\ &\leq \frac{1}{10e^t}(2+|x|+|y|).\end{aligned}$$

*Thus the assumption (H3) is satisfied with* $p(t) = \dfrac{1}{5e^t}$, $r(t) = s(t) = \dfrac{1}{10e^t}$. *Then* $p^* = \frac{1}{5}$, $r^* = s^* = \frac{1}{10}$. *The function* $\varpi$ *is given by*

$$\varpi(t,s) = \frac{1}{\Gamma(\frac{3}{2})} \left\{ \begin{array}{ccc} (t-s)^{\frac{3}{2}-1} & if & 0 \leq s < t \\ -t(1-s)^{\frac{3}{2}-1} & if & t \leq s < 1 \end{array} \right\}$$

*It is clear that* $\varpi^* < \frac{2}{\sqrt{\pi}}$. *Thus condition*

$$\frac{r^* T \varpi^*}{1-s^*} = \frac{\frac{1}{10}(1)(\frac{2}{\sqrt{\pi}})}{1-\frac{1}{10}} < 1.$$

*is satisfied. It follows from theorem (3.1) that problem (15) has at least one solution in [0,1].*

# 4 Conclusion

This paper studies some of the conditions necessary for the existence and uniqueness of a solution to the implicit fractional differential equation problem. It deals with sense of Bochner integral, Arzela Ascoli's theorem, Banach contraction principle, Krasnoselskii's fixed point theorem as well as topological techniques for solutions, which are to get the result. Finally, an example is shown to understand our results.

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