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Forms of Solution and Qualitative Behavior of Twelfth-Order Rational Difference Equation

Turki D. Alharbi[1,2] and Elsayed M. Elsayed[1,3]

[1]Department of Mathematics, Faculty of Science,
King Abdulaziz University, Jeddah, Saudi Arabia
[2]Department of Mathematics, University College in Al-Leeth,
Umm Al-Qura University, Makkah, Saudi Arabia
[3]Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura, Egypt

Abstract

The qualitative studies of high-order nonlinear difference equations are investigated in this paper. We analysis the local asymptotic, global attractivity stability, and the periodicity of solutions. Additionally, the exact solutions of a special case equation are explored. Some numerical simulations are shown to confirm the theoretical analysis by using MATLAB programming.

Keywords: Stability, Periodicity of Solution, Difference Equation, Qualitative Behavior

1 Introduction

This paper is dealt with the following twelfth-order non linear difference equations

$$Z_{n+1} = aZ_{n-5} - \frac{bZ_{n-5}}{cZ_{n-5} - dZ_{n-11}}, \quad (1)$$

where the initial conditions Z_{-i} for $i = 0,1,...,11$ are arbitrary nonzero real numbers and a,b,c,and d are positive constants.

In the last decades, the use of mathematical models that consist of difference equations is increased to describe many problems in biology, economics, physics, ecology, engineering, economics, probability theory, etc(see, for example, [1], [4], [6], [13], [16], [21], [23]). Studying the qualitative and quantitative behaviors of the problems, could help the researchers to understand the long behavior of such models. This can be easily achieved by exploring the stability, periodicity, and boundedness of the solutions. Recently, the qualitative study of the high-order difference equations is a fertile research area and has increasingly attracted many mathematicians. It is very interesting to investigate the behavior of solutions of higherorder rational difference equations and to discuss the local asymptotic stability and the global behavior of its equilibrium points. For instance, Alayachi et al. [2] investigated analytically and numerically the global attractivity, local stability and explored the boundedness of the solution. As well as the exact solutions to three equations have been shown by using the Fibonacci sequence for the following sixth-order difference equations

$$y_{n+1} = Ay_{n-1} + \frac{By_{n-1}y_{n-3}}{Cy_{n-3} + Dy_{n-5}}.$$

In [3] some qualitative properties of the positive solutions were demonstrated for the nonlinear difference equation

$$x_{n+1} = \frac{ax_{n-m} + \delta x_n}{\beta + \gamma x_{n-k}x_{n-1}(x_{n-k} + x_{n-1})}.$$

Ma [18] analysed the higher-order nonlinear difference equation and assort all its positive solutions into three groups. Also, he studied the global attractivity character of the behavior of the following equation

$$z_{n+1} = \frac{(c+1)z_n z_{n-k} + C[f(z_n, z_{n-k}, w_3, ..., w_l) - z_n - z_{n-k}] + 2c^2}{z_n z_{n-k} + f(z_n, z_{n-k}, w_3, ..., w_l) + c}.$$

Elsayed et al. [7] examined the local asymptotic stability of the fixed point and presented the formula of solutions for the difference equations

$$x_{n+1} = \frac{x_{n-1}x_{n-6}}{x_{n-4}(\pm 1 \pm x_{n-1}x_{n-6})}.$$

Yang et al. [24] studied the global convergence and the local stability of the fixed points of the difference equation

$$x_{n+1} = \frac{ax_{n-1} + bx_{n-2}}{c + dx_{n-1} - dx_{n-2}}.$$

Garic-Demirovic et al. [9] demonstrated the dynamics behavior, periodicity of the solution, and the stability anylasis of the fractional difference equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2}.$$

There are several studies about the qualitative behavior of the high-order difference equations, for more information see [1]-[25]

2 Essential Theorems

Some fundamental theorems are stated in this section that we would use in the next section.

Theorem A [15]

Assume that $Q_i \in R$, where $i = 1,2,3,...$ and $s \in \{0,1,2,3,..\}$. Then

$$\sum_{i=1}^{s} |Q_i| < 1.$$

is a sufficient condition for the asymptotic stability of the following the difference equation

$$Zn+1 + Q1Zn+s +... + QsZn = 0.$$

Theorem B [8]

Assume that h is a continuous function such that $h : [\alpha,\beta]^{s+1} \to [\alpha,\beta]$, where s is a positive integer and $[\alpha,\beta]$ is a real numbers interval. And consider the difference equation

$$Z_{n+1} = h(Z_n, Z_{n-1}, ..., Z_{n-s}),\ n = 0,1,2,... \qquad (2)$$

Now, let h satisfies the following

1. For all $1 \le i \le s + 1$ where i is an integer, the function $h(z_1,z_2,...,z_{s+1})$ is weakly monotonic in z_i for each $z_1,z_2,...,z_{s+1}$.
2. Assume (m,M) is a solution of the the system

$$m = h(m_1,m_2,...,m_{s+1}),$$

$$M = h(M_1,M_2,...,M_{s+1}).$$

Then, foe each $(i = 1,2,...,s + 1)$

$$m = M.$$

So, there exists a unique fixed point z^* of the equation (2) and any solution of (2) converges to z^*

3 Mathematical Analysis

3.1 Calculation The Equilibrium Point

In this section, the equilibrium point is found of the following difference equation

$$Z_{n+1} = aZ_{n-5} - \frac{bZ_{n-5}}{cZ_{n-5} - dZ_{n-11}}, \tag{3}$$

now, we can write (3) as

$$Z^* = aZ^* - \frac{bZ^*}{cZ^* - dZ^*},$$

and so,

$$Z^* = \frac{aZ^{*2} - abZ^{*2} - bZ^*}{cZ^* - dZ^*},$$

$$Z^*[c - d - ac - ad] + b = 0,$$

which implies

$$Z^* = \frac{-b}{(1-a)(c-d)}.$$

Therefore, the difference equation (3) has a unique equilibrium point Z^*

3.2 The Local Stability

The local stability of the unique fixed point of the equation (3) is demonstrated in this section, we state the condition of Z^* to be locally asymptotically stable.

Suppose that a function $g(0,\infty)^2 \to (0,\infty)$ is defined as follows

$$g(f,h) = af - \frac{bf}{cf - dh}, \tag{4}$$

after differentiating $g(f,h)$ with respect to f and h. We get

$$\frac{\partial g(f,h)}{\partial f} = a + \frac{bdh}{(cf-dh)^2}, \tag{5}$$

and

$$\frac{\partial g(f,h)}{\partial h} = \frac{-bdf}{(cf-dh)^2}, \tag{6}$$

now, substituting Z^* into (5) and (6). We get

$$\frac{\partial g(Z^*,Z^*)}{\partial f} = a + \frac{d(1-a)}{(c-d)} = -Q_1,$$

$$\frac{\partial g(Z^*,Z^*)}{\partial h} = \frac{-d(1-a)}{(c-d)} = -Q_2 \,.$$

Thus, the linearized equation of (3) about the equilibrium point Z^* is

$$Z_{n+1} + Q_1 Z_{n-5} + Q_2 Z_{n-11} = 0. \tag{7}$$

Theorem 3.2.1. The equilibrium point Z^* of the difference equation (3) is locally asymptotically stable if

$$|ac + d| + d|1 - a| < |c - d|.$$

Proof. By using **Theorem A** that the fixed point of the difference equation (3) is locally asymptotically stable if

$$|Q_1| + |Q_2| < 1.$$

After substituting the values of Q_1 and Q_2,

$$|a + \frac{d(1-a)}{c-d}| + |\frac{-d(1-a)}{c-d}| < 1,$$

$$|a(c - d) + d(1 - a)| + |- d(1 - a)| < |c - d|,$$

thus,

$$|ac + d| + d|1 - a| < |c - d|. \tag{8}$$

Therefore, the fixed point of (3) is locally stable if the inequality (8) is satisfied. The proof is completed.

3.3 Global Stability Character

Now, we devote this part to presenting the condition under which the equilibrium point Z^* of (3) is globally asymptotically stable.

Theorem 3.3.1. The unique fixed point Z^* of the twelfth-order difference equation(3) is globally asymptotically stable if

$$c > ac.$$

Proof. Let r and s be real numbers and suppose that a function $g(r,s)^2 \to (r,s)$ is defined by (4). Now, it is clear from (5) and (6) that g is increasing in f and it is decreasing in h. Then, assume that (σ,μ) is a solution of the following system

$$\sigma = g(\sigma,\mu),\ \mu = g(\mu,\sigma).$$

And so,

$$\sigma = a\sigma - \frac{b\sigma}{c\sigma - d\mu},$$

$$\mu = a\mu - \frac{b\mu}{c\mu - d\sigma},$$

this gives,

$$c\sigma^2 - d\sigma\mu = ac\sigma^2 - ad\sigma\mu - b\sigma, \qquad (9)$$

$$c\mu^2 - d\sigma\mu = ac\mu^2 - ad\sigma\mu - b\mu, \qquad (10)$$

after subtracting (10) from (9). We get

$$c(\sigma^2 - \mu^2) = ac(\sigma^2 - \mu^2) - b(\sigma - \mu),$$

thus,

$$(\sigma - \mu)[(c - ac)(\sigma - \mu) + b] = 0.$$

When

$$c > ac,$$

$$\sigma = \mu.$$

Therefore, by using **Theorem B** the fixed point Z^* is globally asymptotically stable. The proof is completed.

3.4 Periodicity of Behavior

In this section, we prove that the difference equation (3) has no solution of period two.

Theorem.3.4.1 The difference equation (3) has no period two solution

Proof. Assume that the nonlinear difference equation (3) has a behavior of period two

$$\ldots,\kappa,\beta,\kappa,\beta,\ldots$$

with $\kappa \neq \beta$

$$\kappa = a\kappa - \frac{b\kappa}{c\kappa - d\kappa},$$

$$\beta = a\beta - \frac{b\beta}{c\beta - d\beta}.$$

So,

$$\kappa(1 - a) = \frac{b}{c - d},$$

$$\beta(1 - a) = \frac{b}{c - d}.$$

Consequently,

$$\kappa = \frac{b}{(1-a)(c-d)} ,$$
$$\beta = \frac{b}{(1-a)(c-d)} .$$

Therefore, our assumption is contradicted by the fact that $\kappa = \beta$. The proof is completed

3.5 Special Case Equation

We solve a special case equation from the difference equation (3) and explore the forms of solutions for the following equation

$$Z_{n+1} = Z_{n-5} - \frac{Z_{n-5}}{Z_{n-5} - Z_{n-11}} , n = 0, 1,.... \tag{11}$$

Theorem 3.5.1 let$\{Z_n\}_{n=-11}^{\infty}$ is a solution of (11) and suppose that $Z_0 = e$, $Z_{-1} = u$, $Z_{-2} = z$, $Z_{-3} = r$, $Z_{-4} = p$, $Z_{-5} = k$, $Z_{-6} = h$, $Z_{-7} = a$, $Z_{-8} = b$, $Z_{-9} = d$, $Z_{-10} = c$ and $Z_{-11} = s$.

Then, for $n \geqslant 0$ the solutions of equation (11) can be formed as follows

$$Z_{12n-11}=\frac{[(n-1)s-nk][s-k+n]}{k-s}, \quad Z_{12n-10}=\frac{[(n-1)c-np][c-p+n]}{p-c},$$
$$Z_{12n-9}=\frac{[(n-1)d-nr][d-r+n]}{r-d}, \quad Z_{12n-8}=\frac{[(n-1)b-nz][b-z+n]}{z-b},$$
$$Z_{12n-7}=\frac{[(n-1)a-nu][a-u+n]}{u-a}, \quad x_{12n-6}=\frac{[(n-1)h-ne][h-e+n]}{e-h},$$
$$Z_{12n-5}=\frac{[ns-(n+1)k][s-k+n]}{k-s}, \quad Z_{12n-4}=\frac{[nc-(n+1)p][c-p+n]}{p-c},$$
$$Z_{12n-3}=\frac{[nd-(n+1)r][d-r+n]}{r-d}, \quad Z_{12n-2}=\frac{[nb-(n+1)z][b-z+n]}{z-b},$$
$$Z_{12n-1}=\frac{[na-(n+1)u][a-u+n]}{u-a}, \quad Z_{8n}=\frac{[nh-(n+1)e][h-e+n]}{e-h}.$$

Proof. It is clear that at $n = 0$ the results are true. Now for $n > 0$, assume the results are true at $n - 1$ and they are given as follows

$$Z_{12n-23}=\frac{[(n-2)s-(n-1)k][s-k+n-1]}{k-s}, \quad Z_{12n-22}=\frac{[(n-2)c-(n-1)p][c-p+n-1]}{p-c},$$
$$Z_{12n-21}=\frac{[(n-2)d-(n-1)r][d-r+n-1]}{r-d}, \quad Z_{12n-20}=\frac{[(n-2)b-(n-1)z][b-z+n-1]}{z-b},$$
$$Z_{12n-19}=\frac{[(n-2)a-(n-1)u][a-u+n-1]}{u-a}, \quad Z_{12n-18}=\frac{[(n-2)h-(n-1)e][h-e+n-1]}{e-h},$$
$$Z_{12n-17}=\frac{[(n-1)s-nk][s-k+n-1]}{k-s}, \quad Z_{12n-16}=\frac{[(n-1)c-np][c-p+n-1]}{p-c},$$
$$Z_{12n-15}=\frac{[(n-1)d-nr][d-r+n-1]}{r-d}, \quad Z_{12n-14}=\frac{[(n-1)b-nz][b-z+n-1]}{z-b},$$
$$Z_{12n-13}=\frac{[(n-1)a-nu][a-u+n-1]}{u-a}, \quad Z_{12n-12}=\frac{[(n-1)h-ne][h-e+n-1]}{e-h}.$$

Now, the first form will be proven.

After substituting $12n-11$ into equation (11). We get

$$Z_{12n-11} = aZ_{12n-17} - \frac{bZ_{12n-17}}{cZ_{12n-17} - dZ_{12n-23}}$$

So,

$$Z_{12n-11} = \frac{[(n-1)s-nk][s-k+n-1]}{k-s} - \frac{\frac{[(n-1)s-nk][s-k+n-1]}{k-s}}{\frac{[(n-1)s-nk][s-k+n-1]}{k-s} - \frac{[(n-2)s-(n-1)k][s-k+n-1]}{k-s}}$$

$$Z_{12n-11} = \frac{[(n-1)s-nk][s-k+n-1]}{k-s} - \frac{(n-1)s-nk}{s-k}$$

$$Z_{12n-11} = \frac{[(n-1)s-nk][s-k+n-1]}{k-s} + \frac{(n-1)s-nk}{k-s}$$

and so,

$$Z_{12n-11} = \frac{[(n-1)s-nk][s-k+n]}{k-s}.$$

In a similar way, we can prove the second form.

Substituting $12n-10$ into equation (11). We get

$$Z_{12n-10} = aZ_{12n-16} - \frac{bZ_{12n-16}}{cZ_{12n-16} - dZ_{12n-22}},$$

So,

$$Z_{12n-10} = \frac{[(n-1)c-np][c-p+n-1]}{p-c} - \frac{\frac{[(n-1)c-np][c-p+n-1]}{p-c}}{\frac{[(n-1)c-np][c-p+n-1]}{p-c} - \frac{[(n-2)c-(n-1)p][c-p+n-1]}{p-c}}$$

$$Z_{12n-10} = \frac{[(n-1)c-np][c-p+n-1]}{p-c} - \frac{(n-1)c-np}{c-p}$$

$$Z_{12n-10} = \frac{[(n-1)c-np][c-p+n-1]}{p-c} + \frac{(n-1)c-np}{p-c}$$

and so,

$$Z_{12n-10} = \frac{[(n-1)c-np][c-p+n]}{p-c}.$$

Similarly, the remaining forms can be proven.

4 Numerical Simulation

In this section, MATLAB programming is used to simulate numerically the behavior of the high-order difference equation (3). The initial values and parameters are assigned for all figures.

In figure (1), the local asymptotic stability of the fixed point is confirmed under the random values "$Z_0 = 0.04$, $Z_{-1} = 0.09$, $Z_{-2} = 0.03$, $Z_{-3} = -0.05$, $Z_{-4} = 0.13$, $Z_{-5} = -0.01$, $Z_{-6} = 0.02$, $Z_{-7} = 0.01$, $Z_{-8} = -0.02$, $Z_{-9} = 0.05$, $Z_{-1}0 = 0.1$ and $Z_{-1}1 = -0.06$".

Figure (2) demonstrates the global stability more succinctly, the behavior of equation (3) approaches the fixed point z^* as n goes to ∞.

Figure (3) displays the stability when the parameters are assigned to be a = 0.5, b = 10, c = 0.7 and d = 1. The fixed point Z^* is no longer stable since it does not meet the local stability condition. Finally, figure (4) illustrates the behavior of the special case equation.

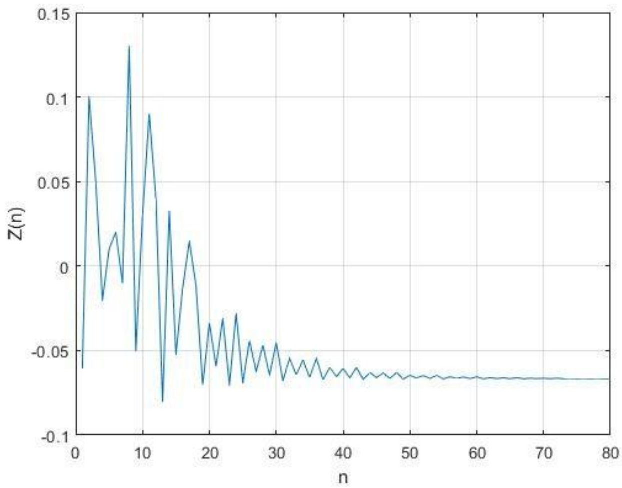

Figure 1: Local Stability of Equation (3)

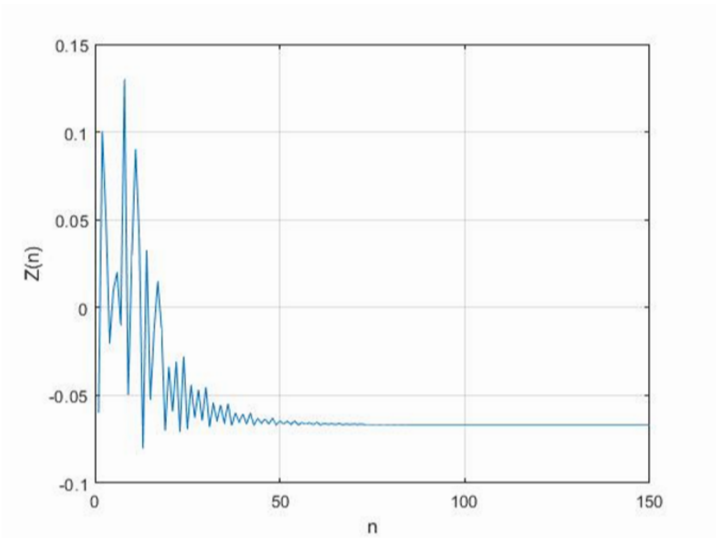

Figure 2: Global Stability of Equation (3)

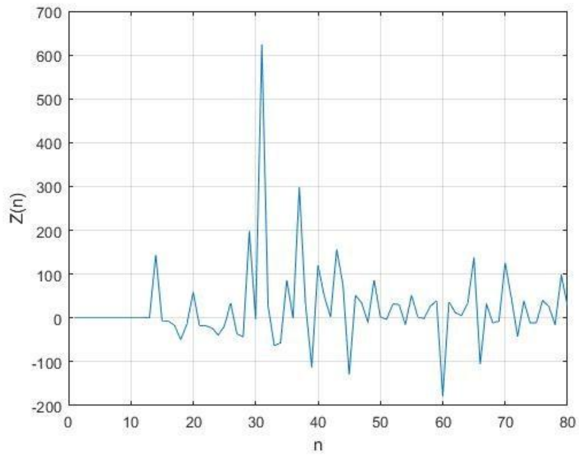

Figure 3: Instability Case of Equation (3)

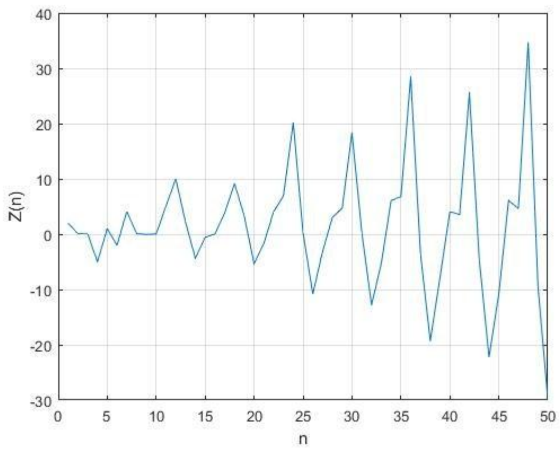

Figure 4: The Solution of The Special case Equation (11)

5 Conclusion

On the difference equation (3), the dynamics of its behavior were studied. Theorem 3.2.1 stated the condition of the fixed point to be locally asymptotic stable. While theorem 3.3.1 pointed out that the fixed point Z^* is globally asymptotically stable if $c > ac$. In section 3.4, we showed that equation (3) has no period tow solution. Furthermore, one special case of equation (3) was solved analytically and the forms of its solutions have been discovered. For verification, numerical simulation was used and figures 1,2,3,4 confirmed our results.

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