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Certain Applications of Fixed Points to Game Theory

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Abstract

Browuer Fixed Point Theorem and Nash Equilibrium play a vital role in game theory and while making economic models in particular. A lot of work have been done with regard to the application of fixed point results in Game Theory. In our present work, we examine the applicability of certain fixed point results to non-cooperative games.

Key words: Fixed Points, Games, Nash Equilibrium,
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1. INTRODUCTION

Fixed point theory forms one of the foundation of mathematical analysis. Fixed point theorems are those concerning the existence and properties of fixed points. Fixed point theory aims to find out self-correspondences in which atleast one of the elements does not vary. A fixed point is that point in the domain of a function getting mapped to itself. If u is called a fixed point of T, then $Tu = u$.
The concept of Fixed Points has its roots to way back in 1886 with the study of Poincare followed by Brouwer [1], S Banach [2], Kakutani [4] to name a few. Brouwer [1] came up with the fixed point result to solve Tx = x. Kakutani extended the results of Brouwer. In 1922, the foundation of present day's fixed point results was laid by S Banach [2] when he proved the fixed point results of contractive mapping in a complete metric space. Subsequently Von Nemann [5] presented his minimax theorem for two person zero sum games and Nash Equilibrium [6] for three or more persons games. Kakutani [4] discovered a fixed point theorem in 1941 to deduce Von Neumann's minimax theorem and intersection lemma. John Nash [8]introduced his well-received equilibrium theorem in 1951.

Having vast applications in various fields not just limited to Mathematics and Engineering but also to Game Theory, Artificial Intelligence, Image processing etc., to name a few, fixed point results are useful in finding coincidence points of the functions representing the supply and demand in economics, nash-equilibrium in game theory with finite strategic games. Inspired by the multifarious applications of fixed point theorems, in our present work, we attempt to find certain applications to real world problems of Game Theory.

2. PRELIMINARIES

Let us review some vital basic concepts, which are derived from certain monographs [10] - [13] which are required in the sequel.

Definition 2.1. *Let $X \neq \phi$. $T : X \to X$ is a self-map on X and a point $u \in X$ is a fixed pint of T, if $Tu = u$.*

Remark 2.1. *Fixed point exists for every function $T : [0, 1] \to [0, 1]$ which is continuous, .*

Definition 2.2. *A topological space (X, T) is Hausdorff, if for $p, q \in X$, $p \neq q$, $\exists$ open nbhds U and V such that $p \in U$ and $q \in V$ and $U \bigcap V$.*

Definition 2.3. *A subset E of a Topological Space (X, T) is compact if it has finite open cover property. $(X.T)$ is compact if X is compact by itself.*

Definition 2.4. *The "$Graph$" of a map $\rho : X \to Y$ is the collection $\{(x, y) : \rho(x) = y\}$*

Definition 2.5. *Let $V \subseteq R^m$ is Convex, if for p, $q \in V$, there exists $s \in [0, 1]$, such that $sp + (1 - s)q \in V$*

Remark 2.2. *If V is convex, then then the vectors with tips on the line segment joining two vectors of V will also be in V.*

Definition 2.6. *Let U and V be two topological spaces. Consider the correspondence $F : U \to 2^V$.*
(i) F is 'upper semi-continuous' if for each $u \in U$ and each open set $H \subset V$, with $F(U) \subset H$,then, $\exists$ an open nbhd G of u in U, such that $F(v) \in H$ for every $y \in G$

(ii) F is 'lower semi-continuous' if for each $u \in U$ and open set $H \subset V$ such that $F(x) \cap H \neq \phi$, $\exists$ an open neighborhood G of $u \in U$ such that, for all $v \in G$, $F(v) \cap H \neq \phi$.

(iii) F is said to have 'open lower sections' if $F^{-1}(v) = \{u \in U : v \in F(u)\}$ is open in U for every v in V.

In 1912, Brouwer [1] gave the following theorem:

Theorem 2.3. *[1] Let $X \subseteq R^n$ be non-empty, compact and convex. A self-map $T : X \to X$ is continuous, then there exists $x \in X$ such that $T(x) = x$.*

The proof is a consequence as in [11].

The paper is organised as follows. In Section-3 we present an analogy for Kakutani's result. In section-4 we analyse Nash Equilibrium and present some applications of Nash Equilibrium

3. MAIN RESULT

In 1941, S.Kakutani [4] gave the following:

Theorem 3.1. *Let X be a non-empty, compact and convex subset of R^n. Let $T : X \to 2^X$ be a map such that:*
(a) $T(x)$ is non-empty and convex valued for $x \in X$
(b) T has a closed graph.
Then T has a fixed point.

Now, we present an analogous proof of Kakutani's Fixed point theorem.

Lemma 3.2. *If P and Q are compact Hausdorff topological space and $F : P \times Q \to R$ is continuous, then the functions $\phi(P) = minF(p, Q)$, where $p \in P$ and $\psi(q) = maxF(P, q)$, where $q \in Q$ are continuous as well.*

In 2006, [18] gave an analogy of Kakutani's Fixed Point Theorem, which is as under:

Theorem 3.3. *[18]: An element $u \in U$is called a fixed point of a set-valued mapping $\Psi : U \rightrightarrows V$ if $u \in U$. If Ψ is single valued then the usual notion of fixed point*

Proposition 3.4. *Let U and V be two topological spaces. Consider the set valued mapping $f : U \to V$.*
(i) f has a Closed Graph if V is Regular, f is upper semi-continuous $(u.s.c)$ and for every $u \in U$, the set $f(u) \neq \phi$ and also closed.
(ii) Conversely if V is a Compact Hausdorff space with the map f having a closed graph, then f is $u.s.c.$.

Proposition 3.5. *If $U_u = \{v \in Q : F(u,v) = f(u)\}$ and $V_v = \{v \in P : F(u,v) = g(v)\}$ and let $X = P \times Q$ for all $u \in P$ and $v \in Q$. Clearly, U_u and V_v are closed and non-void for all $(u,v) \in P \times Q$.*

Now we give the analogy of Kakutani's fixed point theorem.

Theorem 3.6. *Let P and Q are compact Hausdorff topological spaces with $X = P \times Q$. Let $\psi : X \to X$ such that $\psi(u,v) = V_v \times U_u$ for all $(u,v) \in X$ and $\psi(x,y)$ is a non-void convex subset of X, $\{< (u,v), \psi(u,v) >\}$ is closed, then there exists $u^* = (u_0, v_0)$ such that $u^* \in \psi(u^*)$.*

Proof. Since U_u and V_v are closed and non-void for all $(u,v) \in P \times Q$, from Proposition 1, we have $\psi(u,v)$ is non-void and closed subset of X.
Let $G - \psi = \{< (u,v), \psi(u,v) > \ inX \times X\}$ be a graph.
Since $\psi(u,v)$ is closed, we have $\psi(u,v)$ is also closed. Hence ψ is u.s.c. and has a fixed point $u^* \in \psi(u^*)$, i.e.,
$u^* = (u_0, v_0) \in \psi(u_0, v_0) \Rightarrow u^* \in \psi(u^*)$.

□

4. NASH EQUILIBRIUM AND ITS APPLICATIONS

A game is a set of actions done by the participants defined by a set of rules. The Participants are called players who make decisions, or choices, that determine the outcome of the game. The final outcome of the game is determined by the combination of choices made by all players. An action of a player is called move. Though there are various types of games, in the present work, we consider non-cooperative game-a game where each player acts independently for maximizing gains or minimising losses.

Equilibrium in general is a state of rest, where all counteracting forces balance where as in game theory, it is a state where all players play their strategy without changing, when the strategies of the opponent is given. The primary question in game theory is to ascertain if equilibrium can be obtained or not. If so, to decide the strategy one must play to obtain it.

John Nash used the generalised [4] and proved the existence of equilibrium in the case of n-player non-cooperative game in 1950 [7]. Later, in 1951 [8], he gave a considerably improvised version of his earlier result by using Brouwer fixed point theorem.

Definition 4.1. *In an n-person game, when the strategy of other players being fixed, if the strategy s_i of the collection s of strategies, maximising the payoff i^{th} player, then* *the game is said to be in Nash Equilibrium. So, the collection $\{s_i|i \in N\}$ is a Nash equilibrium if for all i,*

$$\phi(s_1, ...s_{i-1}, s_i, s_{i+1}...., s_n) = max\phi\{(s_1, ...s_{i-1}, s_i, s_{i+1}...., s_n)\} \tag{1}$$

The above equilibrium is known as response equilibrium since ”no player can improve his expectation by changing his choice, the others being held fixed” [9]

We now present Nash Equilibrium of a non cooperative game which is a consequence of Kakutani fixed point theorem.

Example: 4.1. *Let N be finite player game. A normal form game is an ordered triple in such a way that, ν_i :$\prod S_i \to R$, where $\S_i$ is not empty for all i. Clearly S_i denotes the set of strategies and $i \in N$ is the gain. If $S_N = \prod S_i$, then $s \in S_N$ is the outcome of the game.*

Let the strategy of ith player be $s_i \in S_i$. The outcome s and the gain $i - \nu_i(s)$ is considered only when strategies are chosen by all players.
Let $s = \{s_1, s_1, s_2, s_3.....s_n\}$ be set of strategies
(ii)$s_{-i} = s_1, s_1, s_2, ...s_{i-1}, s_{i+1}.....s_n$
(iii) $s_{-i}, s^* = s_1, s_1, s_2, ...s_{i-1}, s^*, s_{i+1}.....s_n$ Clearly Nash Equilibrium is $s^* \in S$ in such a way that for $i \in N$, $\nu_i(s_i^*, s^*) \geq (s_{-i}^*, s_i^*)$, for $s_i \in S_i$

Theorem 4.2. *: [7, 8] If strategic sets of each player are such that:*
(i) they are Non-empty, convex and compact.
(ii) and their utility functions are continuous and quasi-concave.
Then, for the case of a normal form game in the type s_{-j}, equilibrium exists.

Proof. Define $\rho_i(s_{-i}) = argmax\{\nu_i(s_i, s_{-i}) : s_i \in S_i\}$ and $\rho(s) = \prod_{i=1}^{n} \rho_i(s_{-i})$
Clearly, ρ is well defined by Weierstrass theorem.
Since the given set is compact, it attains maxium and minimum values by virtue of Weierstrass theorem.
Thus, if $s^* \in \rho(s^*)$, then $s_i^* \in \rho(s_{-i})$ for each $i \in N$. Thus $s*$ is a Nash Equilibrium. □

We now present an example to check the existence and effect of the Nash equilibrium for a 2-person 2-choice game.

Example: 4.3. *Consider A and B to be two players with choices $\{R, S\}$ with their respective pure strategies being $\{\rho_{1,1}, \rho_{1,2}\}$ and $\{\rho_{2,1}, \rho_{2,2}\}$. Let us assume that A chooses R with a probability of p. Since the choices are limited to two for each player,* *the probability of choice Q chosen by A is $1-p$ Likewise B has the probability of choosing q and $1-q$ for each of the choices R and S respectively.*

Thus the $(p, 1-p)$ and $(q, 1-q)$ are the mixed strategies of A and B respectively. The Payoff-matrix is got by tabulating the associated payoff for each combination of strategies of A and B. It may be noted that (ϕ_1, ϕ_2) in the $\mathcal{P}\dashv\dagger\wr\{\{-\Updownarrow\dashv\sqcup\nabla\rangle\S$ is nothing but the payoff for each of A and B for a given choice.

Now we consider the following cases for the game:
Case(i) :

$$\begin{array}{c} \\ A \end{array}\begin{array}{c} B \\ \begin{pmatrix} Choice & R & S \\ R & (10,-10) & (-30,30) \\ S & (-30,30) & (20,-20) \end{pmatrix} \end{array}$$

It is to be noted that (10,-10) means that when the choice of both A and B are N, then A wins 10 points while B loses 10 points. For Nash Equilibrium, strategy of one player is Pure, while that of the other can be any mixed strategy. Clearly, the payoff of A for the pure strategy $\rho_{1,1} = (1,0)$ is dependent on the probability q or $(1,q)$ which B choose from either of R or S.
Thus, the payoff for A with regard to the pure strategy $\rho_{1,1}$ is,

$$\phi_1 = (10)q + (-30)(1-q) = 40q - 30 \tag{2}$$

while that of $\rho_{1,2}$ is,

$$\phi_1 = (-30)q + (20)(1-q) = -50q + 20 \tag{3}$$

Solving the simultaneous equations gives $q^* = \frac{5}{9}$.

Suppose, we fix $q^* < \frac{5}{9}$. The possible payoff of A lie between the payoff lines of two pure strategies of A. If B is playing mixed strategy say, $(q_1, 1-q_1)$, the best response of A is to play the pure strategy $\rho_{1,2}$ which is $(0,1)$. It must be noted that this is not Nash equilibrium strategy, because, if player A goes with $\rho_{1,2}$, it is obvious that B will go with $\rho_{2,1}$ and not $(q_1, 1-q_1)$.

Similarly we can show that with B playing $(q_1, 1-q_1)$ with $q^* > \frac{11}{18}$, also is not Nash equilibrium either. It has to be noted that when player B plays the mixed strategy

$(q_1, 1-q_1)$ with $q^* = \frac{5}{9}$, any mixed strategy of A gives the same payoff and the payoff is maximized.

Hence, ϕ_1 is maximized for the mixed strategy $(\frac{5}{9}, \frac{4}{9})$ for B if she chooses to play 5 out of 9 games. Similarly, it can be seen that when A plays $(\frac{5}{9}, \frac{4}{9})$, then also the payoff of B is maximized for all his mixed strategies.

Hence, the mixed strategy, $s = ((\frac{5}{9}, \frac{4}{9}), (\frac{5}{9}, \frac{4}{9}))$ is the Nash Equilibrium.

Case(ii): Now we consider the other case, where two players play a coin game such that if the coins match, each player gets double their coins. At the same time, if they do not match, the coins are exchanged. The payoff is given by:

$$\begin{array}{c} \qquad B \\ A\begin{pmatrix} Choice & R & S \\ R & (5,5) & (30,-30) \\ S & (-30,30) & (25,35) \end{pmatrix} \end{array}$$

While examining the best response of each of A and B, with regard to the strategies of the opponent,existence of two Nash equilibria can be found for this game.

This is due to the fact that if B plays R alone, then the payoff of A is maximum if he also choose to play only R and vice-versa. Hence the combination $((1,0),(1,0))$ of pur strategies forms Nash equilibrium. Similarly, there exists another Nash equilibrium of pure strategy at $((0,1),(0,1))$, for both A and B playing S always. Hence Nash equilibria of two pure strateigies exists at pure-strategy Nash equilibria for this game are at $(\rho_{1,1}, \rho_{2,1})$ and $(\rho_{1,2}, \rho_{2,2})$.
Thus, the payoff for A with regard to the pure strategy $\rho_{1,1}$ is,

$$\phi_1 = (5)q + (-30)(1-q) = 35q - 20 \tag{4}$$

while that of $\rho_{1,2}$ is,

$$\phi_1 = (-30)q + (25)(1-q) = -55q + 25 \tag{5}$$

In this case, B will opt to play R always because, by choosing S, he will lose 0 points. We know that in a 2-person game, the Nash Equilibrium is stable if the change of strategy of one player has no effect on the other player to change his strategy as well

and the former reverts to his original strategy. And on the other hand a nash equilibrium is said to be unstable if the change of strategy of one player forces the other to make changes as well.
While in $Case(i)$ no nash equilibrium exists, we find in $Case(ii)$ we find that the Nash Equilibrium exists for the pure strategies and is stable as well. For instance, while the players are playing the Nash Equilibrium combination of $(\rho_{1,1}, \rho_{2,1})$ and if B plays S, A will get 5 points while B loses 30 points, which will force B to switch back to R and vice-versa. It must also be noted mixed-strategy equilibrium is unstable, as, a small change in the strategy will force to make major changes and move away from the Nash Equilibrium strategy to one pure strategy states.

In the next section we present application of Kakutani's fixed point theorem to Game Theory:

5. APPLICATION OF KAKUTANI'S FIXED POINT THEOREM

Now we present an application of the above analogy of Kakutani's fixed point theorem to find the existence of Nash Equilibrium in respect of finite players. Let U_1, U_2, $U_3 \cdots U_n$ be n players with their respective probability of actions sets being u_1, u_2,$\cdots$ u_n and π_1, π_2, $\cdots$ π_n the corresponding payoff.

Proof. For i = 1 to n, let $P_i(u^1, u_2 \cdots u_{i-1}, u_{i+1} \cdots u_n)$ represents the set of best-play u_i. Clearly, $P_i(u_1, u_2 \cdots u_{i-1}, u_{i+1} \cdots u_n)$ is convex,closed and non-void, for all i.

Let us now define a function

$$\Psi : \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ u_{i+1} \\ \vdots \\ u_n \end{pmatrix} \rightarrow \begin{pmatrix} P_1(u_2, u_3, \cdots, u_n) \\ P_2(u_1, u_3, \cdots, u_n) \\ \vdots \\ P_i(u_1, u_2, \cdots, u_{i-1}, u_{i+1}, \cdots u_n) \\ P_{i+1}(u_1, u_2, \cdots, u_i, u_{i+2}, \cdots u_n) \\ P_n(u_1, u_2, \cdots, u_{n-1}) \end{pmatrix} \tag{6}$$

From Theorem(4.5) we have,

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ u_{i+1} \\ \vdots \\ u_n \end{pmatrix} \in \Psi \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ u_{i+1} \\ \vdots \\ u_n \end{pmatrix} \tag{7}$$

This establishes the existence of Nash Equilibrium.
Now, we have to check the criteria for existence of fixed point given out in Theorem(3.5). Since the first two criteria are trivial, we are left to show $\{((u_1, u_2, \cdots u_n), \Psi(u_1, u_2, \cdots u_n))\}$ is closed, i.e.

$$(u, \Psi(u)) \to (u^*, v^*) \tag{8}$$

where $u = ((u_1, u_2, \cdots u_n)$, u^* = $(u_0, u_0, \cdots u_0)$ n times, and v^* = $\Psi(u^*)$.
Let u = (u_1, u_2,$\cdots$ u_n) and v = (v_1, v_2,$\cdots$ v_n), such that $\sum u = \sum v = 1$
Then,
$(u^*, v^*) \in \Psi(u^*) \Leftrightarrow v^* \in \Psi(u^*)$
$\Leftrightarrow \pi_1(v_1^*, u_2^*, u_3^*, \cdots, u_n^*) \geq \pi_1(u_1, u_2^*, u_3^*, \cdots, u_n^*)$ for all u_1
$\Leftrightarrow \pi_2(u_1^*, v_2^*, u_3^*, \cdots, u_n^*) \geq \pi_2(u_1^*, u_2, u_3^*, \cdots, u_n^*)$ for all u_2
$\cdots\cdots\cdots$
$\cdots\cdots\cdots$
$\cdots\cdots\cdots$
$\Leftrightarrow \pi_n(u_1^*, u_2^*, u_3^*, \cdots, v_n^*) \geq \pi_n(u_1^*, u_2^*, u_3^*, \cdots, u_n)$ for all u_n
Thus it is evident that $(u, T(u))$ is closed, which completes the proof.

□

6. CONCLUSION

In this paper, we presented simple application to find the Nash equilibrium of a 2-person 2-choice game as a consequence of Brouwer Fixed Point theorem and an analogous proof of Kakutani Fixed point theorem. The established result was supported with suitable example and supported with suitable application to establish Nash Equilibrium in respect of finite players.

The Banach Theorem applies only to functions that are contractions. An elementary example is the function $f(u) = 1 - u$, which has an obvious fixed point at $u = 1/2$. But for every u and $u^{'}$ in $[0, 1]$, $d(f(u), f(u^{'}) = d(u, u^{'})$ and hence, f is not a contraction and so Banach Contraction Principle cannot be applied here. However, all conditions of Brouwer fixed point theorem are satisfied in this case. Brouwer Theorem needs f to be continuous, not necessarily a contraction and hence we can have many situations, where Brouwer conditions are satisfied while Banach contraction principle cannot be applied. It will be interesting to apply contractive conditions of various topological spaces to find applications to games and economic models.

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