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The Global Behavior of a Cenrtain Difference Polynormial Equation

Prof. Dr. Vahidin Hadžiabdić*[1], Prof. Dr. Jasmin Bektešević[1], Prof. Dr. Midhat Mehuljić[1], Prof. Dr. Sadjit Metović[1], and Prof. Dr. Nedim Pervan[1]

[1]*University of Sarajevo, Faculty of Mechanical Engineering, Sarajevo, Bosnia and Herzegovina.*

Abstract

In this paper we will present a local dynamics and investigate the global behavior of a polynomial second order difference equation of type

$$x_{n+1} = x_{n-1}P\left(x_n\right)P\left(x_{n-1}\right)$$

where $P\left(0\right) > 0$ and $P\left(x\right)$ is polynomial with nonnegative coeffcients and initial conditions x_{-1} and x_0 arbitrary nonnegative numbers. This difference equation represents an example of difference equation for which the boundary of the point at infinity can be found explicitly and represent a planar curve.

Keywords: Basin of Attraction, Period-two solutions, Julia set, Difference equation.

1. INTRODUCTION

In this paper we studied the local and global stability character, the periodic nature and the boundedness of solutions of polynomial second order difference equation of type

$$x_{n+1} = x_{n-1}P\left(x_n\right)P\left(x_{n-1}\right) \tag{1}$$

where $P\left(0\right) > 0$ and $P\left(x\right)$ is polynomial with nonnegative coeffcients in the first quadrant of initial conditions, which will make our results more special but also more precise and applicable. Some of our results are based on number of theorems which hold for monotone difference equations. To investigate global behavior of Eq.(1) we are forced to avoid well known theory of monotone maps, and in particular competitive

*Corresponding author, e-mail: hadziabdic@mef.unsa.ba

and cooperative maps applied to polynomial maps, which guarantee the existence and uniqueness of the stable and unstable manifolds for the fixed points and periodic points (see [1]). It is very important to mention that Eq.(1) has infinitely many period-two solutions and we have exposed the explicit form of curve that separates first quadrant into two basins of attraction of a locally stable equilibrium point and the point at infinity. The polynomial difference equations and polynomial maps in the plane have been studied in both the real and complex domains (see [2, 3]). First results on quadratic polynomial difference equation have been obtained in [4, 5] but these results gave us only a part of the basins of attraction of equilibrium points and period-two solutions. In [6], the general second order difference equation is completely investigated and described the regions of initial conditions in the first quadrant for which all solutions tend to equilibrium points, period-two solutions, or the point at infinity, except for the case of infinitely many period-two solutions. In [7], case of infinitely many period-two solutions is completely investigated and corresponding difference equation is special case of equation $x_{n+1} = ax_n^m x_{n-1} + ax_{n-1}^{m+1} + bx_{n-1}$ for $m = 1$. In [8] we have extended our research to equation of type $x_{n+1} = ax_n^m x_{n-1} + ax_{n-1}^{m+1} + bx_{n-1}$ for $m > 1$. Since equation of type $x_{n+1} = x_{n-1}^{m+1} x_n^m$ can be solved explicitly (see [9]), we decided to consider the case difference equation of type (1). In the most of cases of polynomial difference equation the principal tool is the theory of monotone maps, and in particular cooperative maps, applied to the system

$$\begin{aligned} u_{n+1} &= v_n, \\ v_{n+1} &= f(v_n, u_n), \end{aligned}$$

where f is a continuous and increasing function in both variables, which guarantee the existence and uniqueness of the stable and unstable manifolds for the fixed points and periodic points (see [10]). If we set $u_n = x_{n-1}$ and $v_n = x_n$ for $n = 0, 1, 2, \ldots$, we obtain the results that are based on the theorems which hold for monotone difference equations. Hence, the method we discussed in this paper is applicable to some special types of difference systems. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{2}$$

where f is a continuous and increasing function in both variables. The following result has been obtained in [4]:

Theorem 1 *Let $I \subseteq R$ and let $f \in C[I \times I, I]$ be a function which increases in both variables. Then for every solution of Eq.(2) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following:*

(i) *Eventually they are both monotonically increasing.*

(ii) *Eventually they are both monotonically decreasing.*

(iii) *One of them is monotonically increasing and the other is monotonically decreasing.*

The following result can be proved by using the techniques of proof of Theorem 11 in [10].

Theorem 2 *Consider Eq.*(2) *where* f *is increasing function in its arguments and assume that there is no minimal period-two solution. Assume that* $E_1(x_1, y_1)$ *and* $E_2(x_2, y_2)$ *are two consecutive equilibrium points in North-East ordering that satisfy*

$$(x_1, y_1) \preceq_{ne} (x_2, y_2)$$

and that E_1 *is a local attractor and* E_2 *is a saddle point or a non-hyperbolic point with second characteristic root in interval* $(-1, 1)$*, with the neighborhoods where* f *is strictly increasing. Then the basin of attraction* $\mathcal{B}(E_1)$ *of* E_1 *is the region below the global stable manifold* $\mathcal{W}^s(E_2)$*. More precisely*

$$\mathcal{B}(E_1) = \{(x, y) : \exists y_u : \; y < y_u, \quad (x, y_u) \in \; \mathcal{W}^s(E_2)\}.$$

The basin of attraction $\mathcal{B}(E_2) = \mathcal{W}^s(E_2)$ *is exactly the global stable manifold of* E_2*. The global stable manifold extend to the boundary of the domain of Eq.*(2)*. If there exists a period-two solution, then the end points of the global stable manifold are exactly the period two solution.*

Now, the theorems that are applied in [10] provided the two continuous curves $\mathcal{W}^s\left(E_2\right)$ (stable manifold) i $\mathcal{W}^u\left(E_2\right)$ (unstable manifold), both passing through the point $E_2(x_2, y_2)$ from Theorem 2, such that $\mathcal{W}^s\left(E_2\right)$ is a graph of decreasing function and $\mathcal{W}^u\left(E_2\right)$ is a graph of an increasing function. The curve $\mathcal{W}^s\left(E_2\right)$ splits the first quadrant of initial conditions into two disjoint regions, but we do not know the explicit form of the curve $\mathcal{W}^u\left(E_2\right)$. The basins of attraction may have very complicated structures even for very simple looking maps. As a rule in the case of chaotic maps the basins of attraction are as complicated as Cantor sets (see [11]). In complex domain, if $f\left(z\right) = \frac{P(z)}{Q(z)}$, where $z \in \mathbb{C} \cup \{\infty\}$ and P and Q are polynomials without common divisors, then Julia set J_f is the set of points z which do not approach infinity after $f\left(z\right)$ is repeatedly applied (corresponding to a attractor). At the same way, in real domain, corresponding Julia set J_f is connected and it is boundary of set of initial conditions for which the orbit of $f(n)$ does not tend to infinity. One of the major problems in the dynamics of polynomial maps in real domain is determining the basin of attractions of the point at infinity and in particular the boundary of the that basin known as the Julia set. We precisely determined the Julia set of Eq.(1) (boundary of set of initial conditions in the first quadrant for which the solutions of Eq.(1) does not tend to infinity) and we obtained the global dynamics in the interior of the Julia set, which includes all the points for which solutions are not asymptotic to the point at infinity. It turned out that the Julia set for Eq.(1) is the union of the stable manifolds of some saddle equilibrium points, nonhyperbolic equilibrium points or period-two points. In general, there is no explicit form of stable and unstable manifolds for the fixed points and periodic points of any difference equation (or system of difference equations), so the disadvantage of all results is that these manifolds are continuous decreasing (increasing) functions of which the parametrization is uncomfortable and we can only obtain their asymptotic formulas by using the method of undetermined coefficients. So the advantage of our results is that we obtain the exact formula of our Julia set of Eq.(1). We first list some results needed for the proofs of our theorems. The main result for studying local stability of equilibria is linearized stability theorem (see Theorem 1.1 in [12]).

Theorem 3 *(linearized stability): Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}) \tag{3}$$

and let $\bar{x}$ be an equilibrium point of difference equation (3). Let $p = \frac{\partial f(\bar{x},\bar{x})}{\partial u}$ and $q = \frac{\partial f(\bar{x},\bar{x})}{\partial v}$ denote the partial derivatives of $f(u,v)$ evaluated at the equilibrium $\bar{x}$. Let λ_1 and λ_2 roots of the quadratic equation $\lambda^2 - p\lambda - q = 0$.

a) If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then the equilibrium $\bar{x}$ is locally asymptotically stable (sink).

b) If $|\lambda_1| > 1$ or $|\lambda_2| > 1$, then the equilibrium $\bar{x}$ is unstable.

c) $|\lambda_1| < 1$ and $|\lambda_2| < 1 \Leftrightarrow |p| < 1 - q < 2$. Equilibrium $\bar{x}$ is a sink.

d) $|\lambda_1| > 1$ and $|\lambda_2| > 1 \Leftrightarrow |q| > 1$ and $|p| < |1 - q|$. Equilibrium $\bar{x}$ is a repeller.

e) $|\lambda_1| > 1$ and $|\lambda_2| < 1 \Leftrightarrow |p| > |1 - q|$. Equilibrium $\bar{x}$ is a saddle point.

f) $|\lambda_1| = 1$ or $|\lambda_2| = 1 \Leftrightarrow |p| = |1 - q|$ or $q = -1$ and $|p| \leq 2$. Equilibrium $\bar{x}$ is called a non-hyperbolic point.

The next theorem (Theorem 1.4.1. in [13]) is a very useful tool in establishing bounds for the solutions of nonlinear equations in terms of the solutions of equations with known behaviour.

Theorem 4 *Let I be an interval of real numbers, let k be a positive integer, and let $F : I^{k+1} \to I$ be a function which is increasing in all its arguments. Assume that $\{x_n\}_{n=-k}^{\infty}$, $\{y_n\}_{n=-k}^{\infty}$ and $\{z_n\}_{n=-k}^{\infty}$ are sequences of real numbers such that*

$$x_{n+1} \leq F(x_n, \ldots, x_{n-k}), \quad n = 0, 1, \ldots$$

$$y_{n+1} = F(y_n, \ldots, y_{n-k}), \quad n = 0, 1, \ldots$$

$$z_{n+1} \geq F(z_n, \ldots, z_{n-k}), \quad n = 0, 1, \ldots$$

and

$$x_n \leq y_n \leq z_n, \quad for \quad all \quad -k \leq n \leq 0.$$

Then

$$x_n \leq y_n \leq z_n, \quad for \quad all \quad n > 0.$$

The following well known theorem are very useful in determining the number of positive zero of polynomial.

Theorem 5 *Let $P(x) = a_0 x^{b_0} + a_1 x^{b_1} + \ldots + a_n x^{b_n}$ where a_i, $i = \overline{0, n}$ are nonzero real numbers and $0 \leq b_0 < b_1 < \ldots < b_n$ are integers.The number of positive zeros of $P(x) = 0$, counting multiplicities, is either equal to $v(P)$ or less than that by an even number, where $v(P)$ denotes the number of sign changes in the sequence $a_0, a_1, \ldots, a_n$.*

2. MAIN RESULTS

As a consequence of Theorem 1 the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution of Eq.(1) are both monotonic. Thus, every bounded solution of Eq.(1) approaches either an equilibrium solution or period-two solution and every unbounded solution is asymptotic to the point at infinity in a monotonic way. Thus the major problem in dynamics of Eq.(1) is the problem of determinig the basins of attraction of three different types of attractors: the equilibrium solutions, period-two solution(s) and the point(s) at infinity. Since $P(0) > 0$ and $P(x)$ is polynomial with nonnegative coeffcients and initial conditions x_{-1} and x_0 arbitrary nonnegative numbers, in a view of this restriction, we deduce that the equilibrium points of Eq.(1) are the nonnegative solutions of equation

$$\overline{x} = \overline{x}\left(P\left(\overline{x}\right)\right)^2$$

or equivalently

$$\overline{x}\left(P\left(\overline{x}\right)-1\right)\left(P\left(\overline{x}\right)+1\right)=0.$$

One can see that Eq.(1) has always zero equilibrium point. It is clear that $P\left(\overline{x}\right)+1>0$. In addition to the zero equilibrium, Eq.(1) has a positive equilibrium if and only if $P\left(\overline{x}\right)=1$. Since all coeffcients of the polynomial $P\left(x\right)$ are nonnegative, by applying Theorem 5, we deduce that the polynomial $P\left(x\right)-1$ has at most one positive zero. Here we investigate the stability of the zero equilibrium. Set $f\left(x,y\right)=yP\left(x\right)P\left(y\right)$ and observe that $\frac{\partial f(x,y)}{\partial x}=yP'\left(x\right)P\left(y\right)$ and $\frac{\partial f(x,y)}{\partial y}=P\left(x\right)\left(P\left(y\right)+yP'\left(y\right)\right)$. If $\overline{x}$ denotes an equilibrium point of Eq..(1), then the linearized equation associated with Eq.(1) about the equilibrium point $\bar{x}$ is

$$z_{n+1}=pz_n+qz_{n-1},$$

where $p=\frac{\partial f(\bar{x},\bar{x})}{\partial x}$ and $q=\frac{\partial f(\bar{x},\bar{x})}{\partial y}$ denote the partial derivatives of $f(x,y)$ evaluated at the equilibrium $\bar{x}$. The local stability character of $\overline{x}$ is now described by linearized stability Theorem 3. For the zero eqiulibrium of Eq.(1) we have $p=0$ and $q=\left(P\left(0\right)\right)^2$

Since $P\left(0\right)>0$, by using the Theorem 3, we obtained the following result on local stability of the zero equilibrium of Eq.(1):

Proposition 1 *The zero equilibrium of Eq.(1) is one of the following:*

a) locally asymptotically stable if $P\left(0\right)<1$,

b) non-hyperbolic and locally stable if $P\left(0\right)=1$,

c) unstable if $P\left(0\right)>1$.

If the positive equilibrium exists, then $P\left(\overline{x}\right)=1$ and the linearized equation at the positive equilibrium $\bar{x}$ is

$$\begin{aligned} z_{n+1} &= pz_n+qz_{n-1}, \\ p &= \bar{x}P'\left(\bar{x}\right)P\left(\bar{x}\right)=\bar{x}P'\left(\bar{x}\right)\geq 0, \\ q &= P\left(\bar{x}\right)\left(P\left(\bar{x}\right)+\bar{x}P'\left(\bar{x}\right)\right)=1+\bar{x}P'\left(\bar{x}\right)\geq 1, \\ p+q &= 1+2\bar{x}P'\left(\bar{x}\right)\geq 1, \\ q-p &= 1. \end{aligned}$$

Now, in view of Theorem 3 we obtain the following results on local stability of the positive equilibrium of Eq.(1):

Proposition 2 *The positive equilibrium of Eq.(1) is unstable and non-hyperbolic.*

Theorem 6 *If $P(0) \geq 1$ then every solution $\{x_n\}$ of Eq.(1) satisfies $\lim_{n\to\infty} x_n = \infty$.*

Proof. *Since $P(x)$ is polynomial with nonnegative coeffcients, we deduce that $P(x)$ is non-decreasing function and $P(x) \geq P(0)$ for all $x > 0$. If $\{x_n\}$ is a solution of Eq.(1), then $\{x_n\}$ satisfies the inequality*

$$x_{n+1} = P(x_n)\,P(x_{n-1})\,x_{n-1} \geq (P(0))^2\,x_{n-1}, \quad n = 0, 1, \ldots$$

which in view of the result on difference inequalities, see Theorem 4, implies that $x_n \geq y_n$, $n \geq 1$ where $\{y_n\}$ is a solution of the initial value problem

$$y_{n+1} = (P(0))^2\,y_{n-1}, \quad y_{-1} = x_{-1} \text{ and } y_0 = x_0 \quad n = 0, 1, \ldots$$

Consequently, $x_0, x_{-1} > 0$ then $y_0, y_{-1} > 0$, $y_n \geq 0$ for all n, and

$$\begin{aligned} x_n &\geq y_n = \lambda_1 \sqrt{(P(0))^2}^{\,n} + \lambda_2 \left(-\sqrt{(P(0))^2}\right)^n, \\ x_n &\geq y_n = (P(0))^n\,(\lambda_1 + \lambda_2 (-1)^n), \quad n = 1, 2, \ldots \end{aligned}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $y_n \geq 0$ for all n, which implies $\lim_{n\to\infty} x_n = \infty$. ■

Theorem 7 *Consider the difference equation (1) in the first quadrant of initial conditions, where $P(0) \in (0, 1)$ and $P(x)$ is polynomial with nonnegative coeffcients. Then Eq.(1) has a zero equilibrium and a unique positive equilibrium $\bar{x}_+$. The curve $P(x)\,P(y) = 1$ is the Julia set and separates the first quadrant into two regions: the region below the given curve is the basin of attraction of point $E_0(0, 0)$, the region above the curve is the basin of attraction of the point at infinity and every point on the curve except $E_+(\bar{x}_+, \bar{x}_+)$ is a period-two solution of Eq.(1)*

Proof. The equilibrium points of Eq.(1) are the solutions of equation $x\,(P(x))^2 = x$ or equivalently

$$x\,(P(x) - 1)\,(P(x) + 1) = 0. \tag{4}$$

Set $h(x) = P(x) - 1$, then $h(0) = P(0) - 1 < 0$ and $h(\infty) = \infty$ which yields Eq.(1) always has positive solution $\bar{x}_+$. By applying Theorem 5, we obtain that the polynomial $h(x)$ has at most one positive zero. All this implies that Eq.(4) has two equilibria: zero equilibrium and unique positive equilibrium $\bar{x}_+$. Since $P(0) \in (0,1)$, then by applying Proposition (1) the zero equilibrium is locally asymptotically stable. In a view of Proposition (2), the positive equilibrium is an unstable non-hyperbolic point. Period-two solution u, v satisfies the system

$$\begin{aligned} u &= P(u)\,P(v)\,u, \\ v &= P(v)\,P(u)\,v. \end{aligned}$$

Obviously, the point $(0,0)$ is solution of the system above, but it is not period-two solution. Hence, it has to be $v > 0$ which implies $P(v)\,P(u) = 1$. Therefore every point of the set $\{(x,y) : P(x)\,P(y) = 1\}$ is a period-two solution of Eq.(1), except point E_+. Now, we have to show that the curve $P(x)\,P(y) = 1$ is a graph of the decreasing function in the first quadrant. Let for some $x > 0$ there are y_1 and y_2 $(0 < y_1 < y_2)$ such that $P(x)\,P(y_1) = P(x)\,P(y_2) = 1$. As $P(x)$ is increasing function, then

$$1 = P(x)\,P(y_1) < P(x)\,P(y_2) = 1,$$

which is impossible. Thus the curve $P(x)\,P(y) = 1$ is the graph of function in the first quadrant. Furtherover

$$P'(x)\,P(y) + P(x)\,P'(y)\,y' = 0$$

or

$$y' = -\frac{P'(x)\,P(y)}{P(x)\,P'(y)}.$$

By applying the fact that $P(x)$ is increasing function, we deduce that $y' < 0$ in the first quadrant. Hence, $P(x)\,P(y) = 1$ is the graph of the decreasing function in the first quadrant. Let $\{x_n\}$ be a solution of Eq.(1) for initial condition (x_0, x_{-1}) which lies below the curve $P(x)\,P(y) = 1$. Then $P(x_0)\,P(x_{-1}) < 1$ and

$$\begin{aligned} x_1 &= x_{-1}P(x_0)\,P(x_{-1}) < x_{-1}, \\ x_2 &= x_0P(x_1)\,P(x_0) < x_0P(x_{-1})\,P(x_0) < x_0. \end{aligned}$$

Thus (x_2, x_1) and (x_0, x_{-1}) are two points in North-East ordering $(x_2, x_1) \leq_{ne} (x_0, x_{-1})$ which means that the point (x_2, x_1) is also below the curve $P(x)\,P(y) = 1$ and also holds

$$P(x_2)\,P(x_1) < 1.$$

Similarly we find

$$\begin{aligned} x_3 &= P(x_2)\,P(x_1)\,x_1 < x_1, \\ x_4 &= P(x_3)\,P(x_2)\,x_2 < P(x_1)\,P(x_2)\,x_2 < x_2. \end{aligned}$$

Continuing on this way we get

$$(0,0) \leq_{ne} \ldots \leq_{ne} (x_4, x_3) \leq_{ne} (x_2, x_1) \leq_{ne} (x_0, x_{-1}),$$

which implies that both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically decreasing and bounded below by 0. Since below the curve $P(x)P(y) = 1$ there are no period-two solutions it must be $x_{2n} \to 0$ and $x_{2n+1} \to 0$. On the other hand, if we consider solution $\{x_n\}$ of Eq.(1) for initial condition (x_0, x_{-1}) which lies above the curve $P(x)P(y) = 1$, then $P(x_0)P(x_{-1}) > 1$ and by applying the method shown above we obtain the following condition:

$$(x_{-1}, x_0) \leq_{ne} (x_1, x_2) \leq_{ne} (x_3, x_4) \leq_{ne} \ldots$$

Therefore both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically increasing, hence $x_{2n} \to \infty$ and $x_{2n+1} \to \infty$ as $n \to \infty$. ■

The next Figure 1 is visual illustration of Theorem 7 obtained by using Mathematica 9.0, with the boundaries of the basins of attraction $\mathcal{B}(E_0)$, $\mathcal{B}(\infty)$ of point $E_0(0,0)$ and point at infinity, respectively, obtained by using the software package Dynamica [6] (see [11]).

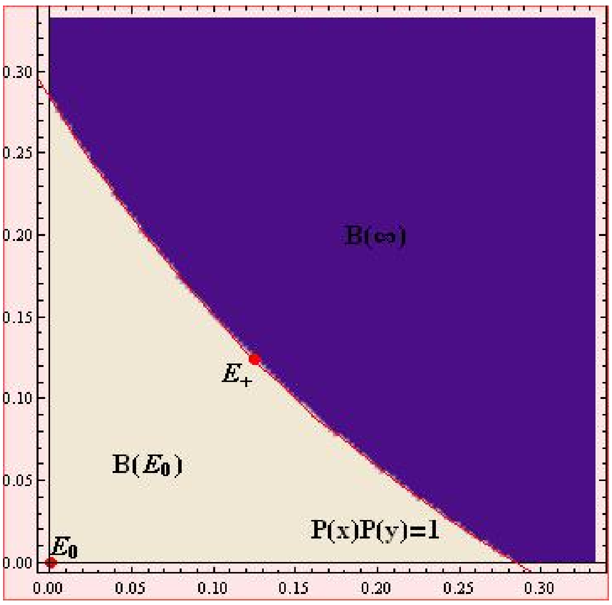

Figure 1: $P(x) = 8x^2 + 3x + \frac{1}{2}$

In view of Theorem 4 which implies results on difference inequalities we get the following:

Proposition 3 *Let* $P(x) = \sum_{i=0}^{s} u_i x^i$ *and* $Q(x) = \sum_{i=0}^{t} v_i x^i$ *be the polynomials with nonnegative coeffcients, where* $P(0)Q(0) < 1$. *Consider the difference equation of type*

$$x_{n+1} = x_{n-1} P(x_n) Q(x_{n-1}) \tag{5}$$

in the first quadrant of initial conditions. Let $r = \max\{s, t\}$ *and*

$$m(x) = \sum_{i=0}^{r} \min\{u_i, v_i\} x^i \text{ and } M(x) = \sum_{i=0}^{r} \max\{u_i, v_i\} x^i.$$

Then the global stable manifold of the positive equilibrium of Eq.(5) is between two curves

$$m(x) m(y) = 1 \text{ and } M(x) M(y) = 1.$$

Proof. The equilibrium points of Eq.(5) are the nonnegative solutions of the equations $\overline{x} = \overline{x} P(\overline{x}) Q(\overline{x})$ or equivalently

$$\overline{x}(P(\overline{x}) Q(\overline{x}) - 1) = 0.$$

Obviously, Eq.(5) has zero equilibrium. Set $g(x) = P(x)Q(x) - 1$, then $g(0) = P(0)Q(0) - 1 < 0$ and $g(\infty) = \infty$, which yields Eq.(5) has at least one positive solution $\bar{x}_+$. By applying Theorem 5, we obtain that the polynomial $g(x)$ has at most one positive zero. Hence, Eq.(5) has two equilibrium points: zero equilibrium and unique positive equilibrium $\bar{x}_+$. Next, we will investigate the local stability of the equilibrium points of Eq.(5). Set $f(x, y) = yP(x)Q(y)$ and observe that $\frac{\partial f(x,y)}{\partial x} = yP'(x)Q(y)$ and $\frac{\partial f(x,y)}{\partial y} = P(x)(Q(y) + yQ'(y))$. Let $p = \frac{\partial f(\bar{x},\bar{x})}{\partial x}$ and $q = \frac{\partial f(\bar{x},\bar{x})}{\partial y}$ denote the partial derivatives of $f(x, y)$ evaluated at the equilibrium $\bar{x}$. The local stability character of $\overline{x}$ is now described by linearized stability Theorem 3. For the zero eqiulibrium of Eq.(5) we have $p = 0$ and $q = P(0)Q(0)$. Since $P(0)Q(0) \in (0, 1)$, then by applying Proposition (1) the zero equilibrium is locally asymptotically stable. After straightforward calculation, it easy to verify that the following conditions hold:

$$\begin{aligned}
p &= \bar{x}_+ P'(\bar{x}_+) Q(\bar{x}_+) \geq 0, \\
q &= P(\bar{x}_+)(Q(\bar{x}_+) + \bar{x}_+ Q'(\bar{x}_+)) = P(\bar{x}_+) Q(\bar{x}_+) + \bar{x}_+ P(\bar{x}_+) Q'(\bar{x}_+) \\
&= 1 + \bar{x}_+ P(\bar{x}_+) Q'(\bar{x}_+) \geq 1, \\
p + q &= 1 + \bar{x}_+ (P'(\bar{x}_+) Q(\bar{x}_+) + P(\bar{x}_+) Q'(\bar{x}_+)) \geq 1 \\
q - p &= 1 + \bar{x}_+ (P(\bar{x}_+) Q'(\bar{x}_+) - P'(\bar{x}_+) Q(\bar{x}_+)).
\end{aligned}$$

In view of Theorem 3, the positive equilibrium must be unstable equilibrium point, so we have at most two scenarios: either the positive equilibrium is a saddle point (non-hyperbolic) or a repeller. The theorems applied in [10] provided the following global behavior. More precisely, if the positive equilibrium $\bar{x}_+$ is a saddle point or a non-hyperbolic point, then the global behavior follows from Theorem 2. In this case, there exists a global stable manifold $\mathcal{W}^s(E_+)$ which contains point $E_+(\bar{x}_+, \bar{x}_+)$, where $\bar{x}_+$ is the positive equilibrium (see Theorem 9 in [6] for visual illustration). If the positive equilibrium is a repeller, then there exists a period-two solution and we obtain that the period-two solution is a saddle point and there are two global stable manifolds $\mathcal{W}^s(P_1)$ and $\mathcal{W}^s(P_2)$ which contain points $P_1(u, v)$ and $P_2(v, u)$ whre (u, v) is unique period-two solution of Eq.(5). In this case the global behavior of Eq.(5) is described by Theorem 10 in [6]. Although the Theorems 9 and 10 in [6] have been applied on a polynomial second order difference equation they are sepcial cases of general Theorems in [10] applied on function f, where f is increasing function in its arguments. So, the global dynamics of Eq.(5) is exactly the same as the global dynamics of equations decribed by Theorems 9 and 10 in [6]. Furthermore

$$x_{n+1} = x_{n-1}P(x_n)Q(x_{n-1}) \geq x_{n-1}m(x_n)m(x_{n-1}),$$

and

$$x_{n+1} = x_{n-1}P(x_n)Q(x_{n-1}) \leq x_{n-1}M(x_n)M(x_{n-1})$$

for all n, by applying Theorem 4 for solution $\{x_n\}$ of Eq.(5) the following inequality holds

$$y_n \leq x_n \leq z_n,$$

for all n, where $\{y_n\}$ is a solution of the difference equation

$$y_{n+1} = x_{n-1}m(x_n)m(x_{n-1}) \tag{6}$$

and $\{z_n\}$ is a solution of the difference equation

$$z_{n+1} = x_{n-1}M(x_n)M(x_{n-1}). \tag{7}$$

Since Eq.(6) and Eq.(7) satisfy all conditions of Theorem 7 this implies that the statement of Proposition 3 holds. Figure 2 is visual illustration of Proposition 3 obtained by using Mathematica 9.0, with the boundaries of the basins of attraction, obtained by using the software package Dynamica [6] where $P(x) = 3x^2 + \frac{1}{2}x + \frac{3}{4}$, $Q(x) = \frac{3}{8}x^2 + x + \frac{1}{2}$, $m(x) = \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{2}$, $M(x) = 3x^2 + x + \frac{3}{4}$, $E_+(\bar{x}_+, \bar{x}_+)$, $\bar{x}_+ \approx 0.307225$,

■

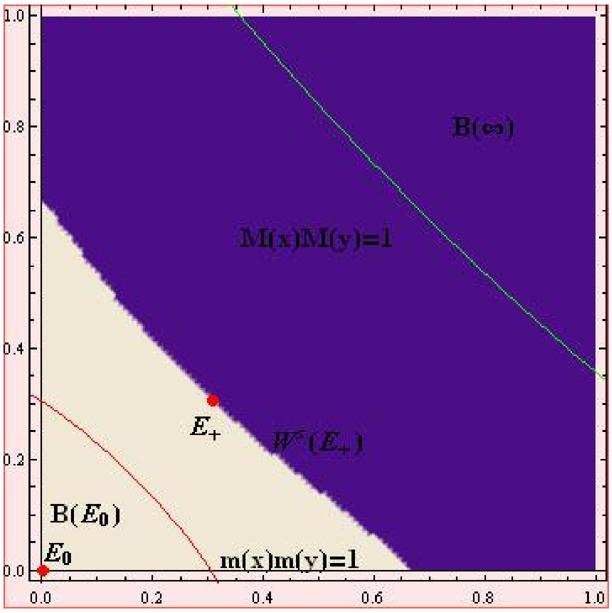

Figure 2: Visual illustration of Proposition 3.

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