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Cubature Formula for Double Integrals Based on Ostrowski Type Inequality

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Abstract

A general Ostrowski's type inequality in two dimensions for double integrals over a rectangular region is developed. We utilize a function whose second order partial derivatives are bounded. Applications to cubature rules are given.

Keywords: Ostrowski inequality, Functions of two independent variables, Cubature Formulae.

1. INTRODUCTION

In 1975, G. Milovanović [2] proposed the following generalization of Ostrowski's inequality [1] for a function f of several variables whose first order partial derivatives are bounded:

Theorem 1 *Let* $f\colon\ \mathbb{R}^m \to \mathbb{R}$ *be a differentiable function defined on* $\overline{D}$ *and let* $\left|\frac{\partial f}{\partial x_i}\right| \leq M_i$ $(M_i > 0;\ i = 1, \cdots, m)$ *in* D*. Then, for every* $X = (x_1, \cdots, x_m) \in \overline{D}$*,*

$$\left| f(x_1, \cdots, x_m) - \frac{1}{\prod\limits_{i=1}^{m} (b_i - a_i)} \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} f(y_1, \cdots, y_m)\, dy_1 \cdots dy_m \right|$$
$$\leq \sum_{i=1}^{m} \left[\frac{1}{4} + \frac{\left(x_i - \frac{a_i + b_i}{2}\right)^2}{(b_i - a_i)^2} \right] (b_i - a_i) M_i. \tag{1}$$

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In 1998, N. Barnett and S. Dragomir [3] established the next Ostrowski's type inequality for real functions of two variables with bounded second order partial derivatives.

Theorem 2 *Let $f\colon [a,b]\times[c,d]\to\mathbb{R}$ continuous on $[a,b]\times[c,d]$, $f''_{x,y}=\frac{\partial^2 f}{\partial x\partial y}$ exists on $(a,b)\times(c,d)$ and is bounded, i.e.,*

$$\left\|f''_{s,t}\right\|_{\infty}=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|<\infty.$$

Then we have the inequality

$$\begin{aligned}&\left|\int_a^b\int_c^d f(s,t)dsdt-\left[(b-a)\int_c^d f(x,t)dt+(d-c)\int_a^b f(s,y)ds-(d-c)(b-a)f(x,y)\right]\right|\\&\le\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]\left\|f''_{s,t}\right\|_{\infty},\end{aligned}\tag{2}$$

for all $(x,y)\in[a,b]\times[c,d]$.

In a recent paper, W. Alshanti [4] proposed the next Ostrowski's type inequality for double integrals utilizing a function whose partial derivative of order four exists and is bounded.

Theorem 3 *Let $f\colon [a,b]\times[c,d]\to\mathbb{R}$ such that $f\in C^4([a,b]\times[c,d])$ be an absolutely continuous function such that the partial derivative of order four exists and is bounded, i.e.,*

$$\left\|\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}\right\|_{\infty}=\sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}\right|<\infty,$$

for all $(t,s)\in[a,b]\times[c,d]$. Then for all $(x,y)\in[a,b]\times[c,d]$, we have

$$\begin{aligned}&\left|f(x,y)+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)f_{ts}(x,y)+\left(\frac{a+b}{2}-x\right)f_t(x,y)\right.\\&+\left(\frac{c+d}{2}-y\right)f_s(x,y)-\frac{1}{(b-a)}\left(\frac{c+d}{2}-y\right)\left[\int_a^b f_s(t,y)\,dt\right]\\&-\frac{1}{(d-c)}\left(\frac{a+b}{2}-x\right)\left[\int_c^d f_t(x,s)\,ds\right]\\&\left.-\left[\frac{1}{(b-a)}\int_a^b f(t,y)\,dt+\frac{1}{(d-c)}\int_c^d f(x,s)\,ds\right]+\int_a^b\int_c^d f(t,s)\,dsdt\right|\\&\le\frac{1}{4}\left(\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2\right)\left(\frac{(d-c)^2}{12}+\left(y-\frac{c+d}{2}\right)^2\right)\left\|\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}\right\|_{\infty}.\end{aligned}\tag{3}$$

For other related work, we refer the reader to [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. In the present paper, we point out a new inequality of Ostrowski's type for double integrals and apply it in numerical integration to obtain a cubature rule. We employ the kernel given in [8] to develop our Ostrowski's type inequality of two dimensional integrals.

2. MAIN RESULTS

Theorem 4 *Let f: $[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order two exists and is bounded, i.e.,*

$$\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty := \sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right| < \infty,$$

for all $(t,s)\in[a,b]\times[c,d]$. Then for α_i, $\beta_i\in[0,\infty)$ such that $\alpha_i+\beta_i\neq 0$ $(i=1,2)$ and for all $(x,y)\in[a,b]\times[c,d]$, we have

$$\begin{aligned}
&\Bigg|(\alpha_1+\beta_1)(\alpha_2+\beta_2)f(x,y)\\
&-(\alpha_2+\beta_2)\left[\frac{\alpha_1}{(x-a)}\int_a^x f(t,y)dt+\frac{\beta_1}{(b-x)}\int_x^b f(t,y)dt\right]\\
&-(\alpha_1+\beta_1)\left[\frac{\alpha_2}{(y-c)}\int_c^y f(x,s)ds+\frac{\beta_2}{(d-y)}\int_y^d f(x,s)ds\right]\\
&+\Bigg[\frac{\alpha_1\alpha_2}{(x-a)(y-c)}\int_a^x\int_c^y f(t,s)dsdt+\frac{\alpha_1\beta_2}{(x-a)(d-y)}\int_a^x\int_y^d f(t,s)dsdt\\
&+\frac{\beta_1\alpha_2}{(b-x)(y-c)}\int_x^b\int_c^y f(t,s)dsdt+\frac{\beta_1\beta_2}{(b-x)(d-y)}\int_x^b\int_y^d f(t,s)dsdt\Bigg]\Bigg|\\
\leq\ &\frac{1}{4}\left[\alpha_1(x-a)+\beta_1(b-x)\right]\left[\alpha_2(y-c)+\beta_2(d-y)\right]\|f_{st}\|_\infty. \qquad (4)
\end{aligned}$$

Proof: Define the mappings $K_1(t;x):[a,b]\times[a,b]\to\mathbb{R}$ and $K_2(s;y):[c,d]\times[c,d]\to\mathbb{R}$ as

$$K_1(t;x)=\begin{cases}\frac{\alpha_1}{\alpha_1+\beta_1}\left(\frac{t-a}{x-a}\right), & t\in[a,x],\\ \frac{-\beta_1}{\alpha_1+\beta_1}\left(\frac{b-t}{b-x}\right), & t\in(x,b],\end{cases} \qquad (5)$$

and

$$K_2(s;y)=\begin{cases}\frac{\alpha_2}{\alpha_2+\beta_2}\left(\frac{s-c}{y-c}\right), & s\in[c,y],\\ \frac{-\beta_2}{\alpha_2+\beta_2}\left(\frac{d-s}{d-y}\right), & s\in(y,d].\end{cases} \qquad (6)$$

where α_i, $\beta_i\in[0,\infty)$ such that $\alpha_i+\beta_i\neq 0$ and $i=1,2$. Then, by definitions of both

$K_1(t;x)$ and $K_2(s;y)$, we have

$$\begin{aligned}
&\int_a^b\int_c^d K_1(t;x)K_2(s;y)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
=\ & \frac{\alpha_1\alpha_2}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\int_a^x\int_c^y\left(\frac{t-a}{x-a}\right)\left(\frac{s-c}{y-c}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
&+\frac{-\alpha_1\beta_2}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\int_a^x\int_y^d\left(\frac{t-a}{x-a}\right)\left(\frac{d-s}{d-y}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
&+\frac{-\beta_1\alpha_2}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\int_x^b\int_c^y\left(\frac{b-t}{b-x}\right)\left(\frac{s-c}{y-c}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
&+\frac{\beta_1\beta_2}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\int_x^b\int_y^d\left(\frac{b-t}{b-x}\right)\left(\frac{d-s}{d-y}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt. \qquad (7)
\end{aligned}$$

Therefore, we deduce:

$$\begin{aligned}
&\int_a^b\int_c^d K_1(t;x)K_2(s;y)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
=\ & f(x,y)-\frac{1}{(\alpha_1+\beta_1)}\left[\frac{\alpha_1}{x-a}\int_a^x f(t,y)dt+\frac{\beta_1}{b-x}\int_x^b f(t,y)dt\right] \\
&-\frac{1}{(\alpha_2+\beta_2)}\left[\frac{\alpha_2}{y-c}\int_c^y f(x,s)ds+\frac{\beta_2}{d-y}\int_y^d f(x,s)ds\right]+\frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\times \\
&\left[\frac{\alpha_1\alpha_2}{(x-a)(y-c)}\int_a^x\int_c^y f(t,s)dsdt+\frac{\alpha_1\beta_2}{(x-a)(d-y)}\int_a^x\int_y^d f(t,s)dsdt\right. \\
&\left.+\frac{\beta_1\alpha_2}{(b-x)(y-c)}\int_x^b\int_c^y f(t,s)dsdt+\frac{\beta_1\beta_2}{(b-x)(d-y)}\int_x^b\int_y^d f(t,s)dsdt\right]. \qquad (8)
\end{aligned}$$

But,

$$\begin{aligned}
\left|\int_a^b\int_c^d K_1(t;x)K_2(s;y)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right| &\leq \int_a^b\int_c^d |K_1(t;x)|\,|K_2(s;y)|\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|dsdt \\
&\leq \left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\int_a^b |K_1(t;x)|\,dt\cdot\int_c^d |K_2(s;y)|\,ds, \qquad (9)
\end{aligned}$$

such that

$$\int_a^b |K_1(t;x)|\,dt = \frac{\alpha_1}{\alpha_1+\beta_1}\int_a^x \left|\frac{t-a}{x-a}\right| dt + \frac{\beta_1}{\alpha_1+\beta_1}\int_x^b \left|\frac{b-t}{b-x}\right| dt$$
$$= \frac{1}{2(\alpha_1+\beta_1)}\left[\alpha_1(x-a)+\beta_1(b-x)\right], \tag{10}$$

and

$$\int_c^d |K_2(s;y)|\,ds = \frac{\alpha_2}{\alpha_2+\beta_2}\int_c^y \left|\frac{s-c}{y-c}\right| ds + \frac{\beta_2}{\alpha_2+\beta_2}\int_y^d \left|\frac{d-s}{d-y}\right| ds$$
$$= \frac{1}{2(\alpha_2+\beta_2)}\left[\alpha_2(y-c)+\beta_2(d-y)\right]. \tag{11}$$

Now, by considering (8)-(11), inequality (4) can be obtained and hence, the proof is completed.

Theorem 5 *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order two exists with for all $(t,s)\in[a,b]\times[c,d]$. Then for α_i, $\beta_i\in[0,\infty)$ such that $\alpha_i=\beta_i\neq 0$ for $i=1,2$ and for all $(x,y)\in[a,b]\times[c,d]$ we have*

$$\Bigg| f(x,y) - \frac{1}{2}\left[\frac{1}{(x-a)}\int_a^x f(t,y)dt + \frac{1}{(b-x)}\int_x^b f(t,y)dt\right]$$
$$-\frac{1}{2}\left[\frac{1}{(y-c)}\int_c^y f(x,s)ds + \frac{1}{(d-y)}\int_y^d f(x,s)ds\right]$$
$$+\frac{1}{4}\left[\frac{1}{(x-a)(y-c)}\int_a^x\int_c^y f(t,s)dsdt + \frac{1}{(x-a)(d-y)}\int_a^x\int_y^d f(t,s)dsdt\right.$$
$$\left.+\frac{1}{(b-x)(y-c)}\int_x^b\int_c^y f(t,s)dsdt + \frac{1}{(b-x)(d-y)}\int_x^b\int_y^d f(t,s)dsdt\right]\Bigg|$$
$$\leq \begin{cases} \frac{(b-a)(d-c)}{16}\|f_{st}\|_\infty, & \frac{\partial^2 f(t,s)}{\partial t\partial s}\in L_\infty([c,d]\times[c,d]); \\ \left[\frac{(b-a)(d-c)}{(q+1)^2}\right]^{\frac{1}{q}}\frac{\|f_{st}\|_p}{4}, & \frac{\partial^2 f(t,s)}{\partial t\partial s}\in L_p([c,d]\times[c,d]), \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \|f_{st}\|_1, & \frac{\partial^2 f(t,s)}{\partial t\partial s}\in L_1([c,d]\times[c,d]). \end{cases} \tag{12}$$

where $\|\omega\|_.$ are the classical Lebesgue norms for $\omega\in L_.([c,d]\times[c,d])$ with

$$\|\omega\|_\infty := ess \sup_{(m,n)\in(a,b)\times(c,d)} |\omega(m,n)| < \infty,$$

and

$$\|\omega\|_p := \left(\int_a^b \int_c^d |\omega(m,n)|^p \, dndm\right)^{\frac{1}{p}}, 1 \leq p \leq \infty.$$

Proof: By considering both *Theorem* 4 and the result obtained by [8] in *Theorem* 2, it is easy to show that (12) holds.

Corollary 1 *Under the assumption of Theorem 5 with* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$*, we have*

$$\Bigg| f(\frac{a+b}{2}, \frac{c+d}{2}) - \frac{1}{(b-a)} \int_a^b f(t, \frac{c+d}{2}) dt - \frac{1}{(d-c)} \int_c^d f(\frac{a+b}{2}, s) ds$$

$$+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s) dsdt \Bigg|$$

$$\leq \begin{cases} \frac{(b-a)(d-c)}{16} \|f_{st}\|_\infty, & \frac{\partial^2 f(t,s)}{\partial t \partial s} \in L_\infty([c,d] \times [c,d]); \\ \left[\frac{(b-a)(d-c)}{(q+1)^2}\right]^{\frac{1}{q}} \frac{\|f_{st}\|_p}{4}, & \frac{\partial^2 f(t,s)}{\partial t \partial s} \in L_p([c,d] \times [c,d]), \\ & \frac{1}{p} + \frac{1}{q} = 1, \; p > 1; \\ \|f_{st}\|_1, & \frac{\partial^2 f(t,s)}{\partial t \partial s} \in L_1([c,d] \times [c,d]). \end{cases} \tag{13}$$

This result (13) gives an Ostrowski's type inequality in which double and single integrals are involved together with a function evaluation at the middle point. We should point out that a similar inequality to the first inequality of (13) was obtained earlier by [3].

Remark 1 *In Theorem 4, if we choose* $f(t,s) = h(t)h(s)$ *such that* $h : [a,b] \to \mathbb{R}$ *is continuous and* $\|h'\|_\infty < \infty$*, then for* $x = y$ *we have*

$$\left[h(x) - \frac{1}{2}\left(\frac{1}{x-a}\int_a^x h(t)\,dt + \frac{1}{b-x}\int_x^b h(t)\,dt\right)\right]^2 \leq \frac{(b-a)^2}{16} \|h'\|_\infty^2. \tag{14}$$

3. APPLICATION FOR CUBATURE FORMULAE

Let $I_n : a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < y_2 < ... < y_{m-1} < y_m = d$ be two arbitrary partitions such that $\zeta_i \in [x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ and $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, 1, ..., m-1)$ be intermediate points. Also, assume that $\delta_i = x_{i+1} - x_i$ $(i = 0, 1, ..., n-1)$ and $\Delta_j = y_{j+1} - y_j$ $(j = 0, 1, ..., m-1)$. Consider the following sum:

$$
\begin{aligned}
S\left(f, I_n, J_m, \zeta, \eta\right) &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\delta_i \int_{y_j}^{y_{j+1}} f\left(\zeta_i, s\right) ds + \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Delta_j \int_{x_i}^{x_{i+1}} f\left(t, \eta_j\right) dt \\
&\quad -\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\delta_i\Delta_j f\left(\zeta_i, \eta_j\right), \qquad (15)
\end{aligned}
$$

for which we assume that the involved one-dimensional integrals can more easily be computed than the original double integral that is

$$
D := \int_a^b \int_c^d f(t,s) ds dt.
$$

Remark 2 *If the one-dimensional integrals, in (15), are not known or difficult to be evaluated, then appropriate rules can be employed to approximate them.*

Now, using the above assumption, we can state the next cubature rule.

Theorem 6 *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be as in theorem (4) and I_n, J_m, ζ and η be as above. Then for $\alpha_i = \beta_i$ $(i = 1, 2)$ with $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, we have the cubature formula*

$$
\int_a^b \int_c^d f(t,s) ds dt = S\left(f, I_n, J_m, \zeta, \eta\right) + R\left(f, I_n, J_m, \zeta, \eta\right), \qquad (16)
$$

where the remainder term $R\left(f, I_n, J_m, \zeta, \eta\right)$ satisfies the estimation

$$
\left|R\left(f, I_n, J_m, \zeta, \eta\right)\right| \leq \frac{\|f_{st}\|_\infty}{16}\sum_{i=0}^{n-1}\delta_i^2 \sum_{j=0}^{m-1}\Delta_j^2. \qquad (17)
$$

Proof: Applying Theorem 4 on the interval $[x_i, x_{i+1}]\times[y_j, y_{j+1}]$ with $\alpha_i = \beta_i$ $(i = 1, 2)$, $x = \frac{a+b}{2}$, and $y = \frac{c+d}{2}$ yields

$$
\begin{aligned}
&\left|\delta_i\Delta_j f(\zeta_i, \eta_j) - \Delta_j \int_{x_i}^{x_{i+1}} f(t, \eta_j) dt - \delta_i \int_{y_j}^{y_{j+1}} f(\zeta_i, s) ds + \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(t,s) ds dt\right| \\
\leq\ & \frac{\|f_{st}\|_\infty}{16}\delta_i^2\Delta_j^2, \qquad (18)
\end{aligned}
$$

for all $(i = 0, 1, ..., n-1)$ and $(j = 0, 1, ..., m-1)$. Summing over i from 0 to $n-1$ and over j from 0 to $m-1$ and using the generalized triangle inequality deduces (17).

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