

OSCILLATION BEHAVIOR OF FIRST ORDER NONLINEAR FUNCTIONAL NEUTRAL DELAY DIFFERENCE EQUATIONS

A. Murugesan* and R. Suganthi

¹ *Department of Mathematics, Government Arts College (Autonomous),
Salem – 636 007, Tamil Nadu, India.
Email: amurugesan3@gmail.com*

² *Department of Mathematics, Vysya College,
Salem-636 103, Tamil Nadu, India.
Email: suganthivarshaa@gmail.com*

ABSTRACT

We study the oscillatory properties of first order functional neutral delay difference equation of the form

$$\Delta[r(n)(a(n)x(n) - p(n)x(n - \tau))] + q(n)f(x(n - \sigma)) = 0, \quad n \geq n_0; \quad (*)$$

Where $\{r(n)\}$, $\{a(n)\}$, $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive real numbers, and τ, σ are positive integers. We establish some new sufficient conditions which ensure that all solutions of equation (*) are oscillatory. These results improve some known results in the literature. Examples are provided to illustrate the main results.

AMS Subject Classification 2010: 39A10, 39A12

Keywords and Phrases: Neutral, delay difference equation, nonlinear, oscillation, nonoscillation.

1. INTRODUCTION

We consider the following first order nonlinear functional neutral delay difference equation with variable coefficients of the form

$$\Delta[r(n)(a(n)x(n) - p(n)x(n - \tau))] + q(n)f(x(n - \sigma)) = 0, \quad n \geq n_0; \quad (1)$$

where Δ is the forward difference operator defined by $\Delta x(n) = x(n + 1) - x(n)$. Throughout the paper the following conditions are assumed to hold:

(C₁) $\{a(n)\}$ is a sequence of positive real numbers defined on $N(n_0) = \{n_0, n_0 + 1, \dots\}$;

(C₂) $\{r(n)\}$ is a nondecreasing sequence of positive real numbers defined on $N(n_0)$;

(C₃) $\{p(n)\}$ and $\{q(n)\}$ are sequence of nonnegative real numbers defined on $N(n_0)$, $\{q(n)\}$

is not identically zero for large values of n ;

(C₄) there exists a constants a_0 and λ such that $a(n) \leq a_0 < \infty$ and $\frac{p(n)}{a(n)} \leq \lambda < 1$;

(C₅) τ and σ are positive integers.

(C₆) $f: R \rightarrow R$ is a continuous function satisfies $uf(u) > 0$ for $u \neq 0$ and there exist a positive

$$k_0 > 0 \text{ such that } \frac{f(u)}{u} > k_0 > 0.$$

Let $n^* = \max\{\tau, \sigma\}$. By a solution of (1) on $N(n_0) = \{n_0, n_0 + 1, \dots\}$, as a real sequence $\{x(n)\}$ which is defined on $n \geq n_0 - n^*$ and which satisfies (1) for $n \in N(n_0)$. A solution $\{x(n)\}$ of (1) on $N(n_0)$ is said to be oscillatory if for every positive integers $N_0 > n_0$ there exists $n \geq N_0$ such that $x(n)x(n + 1) \leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory.

The oscillation theory of difference equation and their applications have been receiving intensive attention in the last few decades, see for example [1-4] and the reference cited therein. Especially the study of oscillatory behavior of first order neutral delay difference equations of various types occupied a great deal of interest. These type of equations have wide applications in the areas such as economics, mathematical biology, and many other areas of mathematics. The oscillatory behavior of difference equations has been intensively studied in recent years.

Most of the literature has been concerned with equation of type (1) with $r(n) = 1$ and $a(n) = 1$ (see [5-14] and references cited therein). But very little is known regarding the oscillation of first order neutral delay difference equations

similar to (1). In [4], we established sufficient conditions for oscillation of all solutions of equations of the type (1) with positive variable coefficient in the neutral term. The purpose of this paper is to study the oscillatory properties of (1).

In the sequel, unless otherwise specified, when we write a functional inequality we shall assume that it holds for all sufficiently large values of n .

2. AUXILIARY LEMMAS

In this section we give some useful Lemmas which will play an important role in the study of the oscillation of (1).

Lemma 2.1: [5] Assume that $\{q(n)\}$ is a sequence of positive real numbers and σ is a positive integer. Then the difference inequality

$$\Delta x(n) + q(n)x(n - \sigma) = 0, \quad n \geq n_0 \quad (2)$$

has an eventually positive solution if and only if the equation

$$\Delta x(n) + q(n)x(n - \sigma) = 0, \quad n \geq n_0. \quad (3)$$

Lemma 2.2. [11] Assume that

$$\lim_{n \rightarrow \infty} \sup \sum_{s=n}^{n+\sigma} q(s) > 0. \quad (4)$$

If $\{x(n)\}$ is an eventually positive solution of the delay difference equation

$$\Delta x(n) + q(n)x(n - \sigma) = 0, \quad n \geq n_0, \quad (5)$$

then

$$\lim_{n \rightarrow \infty} \inf \frac{x(n - \sigma)}{x(n)} < \infty. \quad (6)$$

Lemma 2.3. Let $\{x(n)\}$ be an eventually positive solution of (1) and $\{z(n)\}$ be its associated sequence defined by

$$z(n) = a(n)x(n) - p(n)x(n - r). \quad (7)$$

Then $\{z(n)\}$ is eventually nonincreasing and $z(n) > 0$, eventually.

Proof. From (1) and (7), we have

$$\Delta(r(n)z(n)) \leq 0,$$

Which implies in view of the hypothesis $q(n) \neq 0$, that $\{r(n)z(n)\}$ is nonincreasing and does not equal to a constant eventually. Then it follows that $\{r(n)z(n)\}$ is either eventually positive or eventually negative. Now assume that $r(n)z(n) < 0$, eventually and so $z(n) < 0$, eventually. Then

$$x(n) < \frac{p(n)}{a(n)}x(n - \tau) \leq \lambda x(n - \tau). \quad (8)$$

Then by induction, we obtain

$$x(n + j\tau) < \lambda^j x(n) \quad (9)$$

for all positive integers j . Hence $x(n) \rightarrow 0$ as $n \rightarrow \infty$. But this together with the fact that $\{p(n)\}$ and $\{a(n)\}$ are bounded implies that $z(n)$ decreases to 0 as $n \rightarrow \infty$. This is a contradiction to the assumption $z(n) < 0$ and completes the proof of the Lemma.

Lemma 2.4. Assume that $r(n) \equiv 1$ and

$$\limsup_{n \rightarrow \infty} \sum_{s=n}^{n+\sigma} q(s) > 0. \quad (10)$$

Let $\{x(n)\}$ be an eventually positive solutions of (1) and $\{z(n)\}$ be its associated sequence defined by (7). Then

$$\liminf_{n \rightarrow \infty} \frac{z(n - \sigma)}{z(n)} < \infty. \quad (11)$$

Proof. By Lemma 2.3, $\{z(n)\}$ is eventually positive and nonincreasing. From (7), we have,

$$z(n) \geq a_0 x(n)$$

or

$$x(n) \geq \frac{z(n)}{a_0}$$

or

$$x(n - \sigma) \geq \frac{z(n - \sigma)}{a_0}. \quad (12)$$

Using this and (7) in (1), we have

$$\Delta z(n) + \frac{k_0}{a_0} q(n) z(n - \sigma) \leq 0. \quad (13)$$

By Lemma 2.1 we find the equation

$$\Delta z(n) + \frac{k_0}{a_0} q(n) z(n - \sigma) = 0. \tag{14}$$

has an eventually positive solution as well. As a result, by Lemma 2.2 and (11), we get

$$\liminf_{n \rightarrow \infty} \frac{z(n - \sigma)}{z(n)} < \infty,$$

which is the desired result and hence the proof is completed.

Lemma 2.5. Assume that $r(n) \equiv 1$. If (1) has an eventually positive solution, then

$$\sum_{s=n}^{n+\sigma} q(s) \leq \frac{a_0}{k_0} \tag{15}$$

for all sufficiently large n .

Proof. Proceeding as in the proof of Lemma 2.3, we again obtain the inequality (13).

Summing the inequality from n to $n + \sigma$, we get

$$z(n + \sigma + 1) - z(n) + \frac{k_0}{a_0} \sum_{s=n}^{n+\sigma} q(s) z(s - \sigma) \leq 0.$$

By applying the decreasing nature of $\{z(n)\}$, we have

$$z(n + \sigma - 1) - z(n) + \frac{k_0}{a_0} z(n) \sum_{s=n}^{n+\sigma} q(s) \leq 0.$$

Then

$$z(n + \sigma + 1) + \left(\frac{k_0}{a_0} z(n) \sum_{s=n}^{n+\sigma} q(s) - 1 \right) z(n) \leq 0 \tag{16}$$

Since $z(n) > 0$, (16) implies

$$\frac{k_0}{a_0} \sum_{s=n}^{n+\sigma} q(s) - 1 \leq 0$$

Hence for all sufficiently large n , we have

$$\sum_{s=n}^{n+\sigma} q(s) \leq \frac{a_0}{k_0},$$

Which is the desired result. The proof is completed.

3. OSCILLATION OF SOLUTION

Theorem 3.1. Assume that $r(n) \equiv 1$ and (11) hold. If

$$\sum_{s=n}^{\infty} q(n) \ln \left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} q(s) \right) = \infty. \quad (17)$$

Then every solution of (1) oscillates.

Proof. Assume the contrary. Without loss of generality we may assume that $\{x(n)\}$ is an eventually positive solution of (1). Set $z(n)$ as in (7). Then $\{z(n)\}$ is eventually positive and decreasing. Also $\{z(n)\}$ satisfies the inequality (13). That is

$$\Delta z(n) + \frac{k_0}{a_0} q(n) z(n - \sigma) \leq 0.$$

Define the sequence $\{u(n)\}$ as

$$u(n) = -\frac{\Delta z(n)}{z(n)}.$$

Then $\{u(n)\}$ is eventually nonnegative. So, there exist $n_1 \geq n_0$ with $z(n_1) > 0$. We can easily show that

$$\Delta z(n) \leq z(n_1) \exp \left(-\sum_{s=n_1}^{n-1} q(s) \right).$$

Moreover, $\{u(n)\}$ satisfies

$$u(n) \geq \frac{k_0}{a_0} q(n) \exp \left(\sum_{s=n-\sigma}^{n-1} u(s) \right). \quad (18)$$

By using the inequality

$$e^{rx} \geq x + \frac{\ln(er)}{r}, \quad x, r > 0,$$

we have from (18)

$$\begin{aligned} u(n) &\geq \frac{k_0}{a_0} q(n) \exp\left(\frac{\alpha(n)}{\alpha(n)} \sum_{s=n-\sigma}^{n-1} u(s)\right) \\ &\geq \frac{k_0}{a_0} q(n) \left(\frac{1}{\alpha(n)} \sum_{s=n-\sigma}^{n-1} u(s) + \frac{\ln(e\alpha(n))}{\alpha(n)}\right), \end{aligned}$$

Where

$$\alpha(n) = \frac{k_0}{a_0} \sum_{s=n+1}^{n+\sigma} q(s)$$

Therefore

$$u(n) \sum_{s=n+1}^{n+\sigma} q(s) - q(n) \sum_{s=n-\sigma}^{n-1} u(s) \geq q(n) \ln\left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} q(s)\right).$$

Hence for $\xi > N + \sigma$

$$\begin{aligned} &\sum_{n=N}^{\xi-1} u(n) \left(\sum_{s=n+1}^{n+\sigma} q(s)\right) - \sum_{n=N}^{\xi-1} q(n) \left(\sum_{s=n+\tau-\sigma}^{n-1} u(s)\right) \\ &\geq \sum_{n=N}^{\xi-1} q(n) \ln\left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} q(s)\right) \end{aligned} \tag{19}$$

By interchanging the order of summation, we have

$$\sum_{n=N}^{\xi-1} q(n) \sum_{s=n-\sigma}^{n-1} u(s) \geq \sum_{n=N}^{\xi-\sigma-1} u(n) \sum_{s=n+1}^{n+\sigma} q(s). \tag{20}$$

Combing (19) and (20), leads to

$$\sum_{n=\xi-\sigma}^{\xi-1} u(n) \sum_{s=n+1}^{n+\sigma} q(s) \geq \sum_{n=N}^{\xi-1} q(n) \ln\left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} q(s)\right). \tag{21}$$

Using (16) of Lemma 2.4 in (21), we obtain

$$\sum_{n=\xi-\sigma}^{\xi-1} u(n) \geq \frac{k_0}{a_0} \sum_{n=N}^{\xi-1} q(n) \ln\left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} q(s)\right),$$

(ie)

$$\ln \frac{z(\xi - \sigma)}{z(\xi)} \geq \frac{k_0}{a_0} \sum_{n=N}^{\xi-1} q(n) \ln \left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} q(s) \right).$$

This result along with condition (17) leads to

$$\lim_{n \rightarrow \infty} \frac{z(n - \sigma)}{z(n)} = \infty,$$

which contradicts (11) and completes the proof.

Theorem 3.2. *Assume that*

$$0 < c \leq \liminf_{n \rightarrow \infty} \sum_{s=n}^{n+\sigma} \frac{q(s)}{r(s - \sigma)} \quad (22)$$

and

$$\sum_{n=n_0}^{\infty} \frac{q(n)}{r(n - \sigma)} \ln \left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s - \sigma)} \right) = \infty. \quad (23)$$

Then every solution of (1) oscillates.

Proof. For the benefit of acquiring a contradiction, without deficiency of generality, we may suppose that $\{x(n)\}$ is an eventually positive solution of (1). Set $z(n)$ as in (7). Then $\{z(n)\}$ is positive and decreasing. Proceeding as in the proof of Lemma 2.3, we get

$$\Delta(r(n)z(n)) + \frac{k_0}{a_0} q(n)z(n - \sigma) \leq 0. \quad (24)$$

Set

$$y(n) = r(n)z(n). \quad (25)$$

Using this in (24), we get

$$\Delta y(n) + \frac{k_0}{a_0} \frac{q(n)}{r(n - \sigma)} y(n - \sigma) \leq 0. \quad (26)$$

Set

$$\lambda(n) = -\frac{\Delta y(n)}{y(n)}. \quad (27)$$

Then $\lambda(n) > 0$ eventually and $\{\lambda(n)\}$ satisfies the inequality

$$\lambda(n) \geq \frac{k_0}{a_0} \frac{q(n)}{r(n-\sigma)} \exp\left(\sum_{s=n-\sigma}^{n-1} \lambda(s)\right). \tag{28}$$

Applying the inequality

$$e^{rx} \geq x + \frac{\ln(er)}{r}, \quad x, r > 0,$$

to (28) yields that

$$\begin{aligned} \lambda(n) &\geq \left(\frac{k_0}{a_0}\right) \frac{q(n)}{r(n-\sigma)} \exp\left(\frac{\beta(n)}{\beta(n)} \sum_{s=n-\sigma}^{n-1} \lambda(s)\right) \\ &\geq \left(\frac{k_0}{a_0}\right) \frac{q(n)}{r(n-\sigma)} \left(\frac{1}{\beta(n)} \sum_{s=n-\sigma}^{n-1} \lambda(s) + \frac{\ln(e\beta(n))}{\beta(n)}\right), \end{aligned}$$

where

$$\beta(n) = \frac{k_0}{a_0} \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} \tag{29}$$

Therefore,

$$\lambda(n) \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} - \frac{q(n)}{r(n-\sigma)} \sum_{s=n-\sigma}^{n-1} \lambda(s) \geq \frac{q(n)}{r(n-\sigma)} \ln\left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)}\right).$$

Hence, for $\eta > N_1 + \sigma$

$$\begin{aligned} &\sum_{n=N_1}^{\eta-1} \lambda(n) \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} - \sum_{n=N_1}^{\eta-1} \frac{q(n)}{r(n-\sigma)} \sum_{s=n-\sigma}^{n-1} \lambda(s) \\ &\geq \sum_{n=N_1}^{\eta-1} \frac{q(n)}{r(n-\sigma)} \ln\left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)}\right). \end{aligned} \tag{30}$$

By interchanging the order of summation, we have

$$\sum_{n=N_1}^{\eta-1} \left(\frac{q(n)}{r(n-\sigma)}\right) \left(\sum_{s=n-\sigma}^{n-1} \lambda(s)\right) \geq \sum_{n=N_1}^{\eta-1} \lambda(n) \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} \tag{31}$$

From (30) and (31), we have

$$\sum_{\eta=n-\sigma}^{\eta-1} \lambda(n) \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} \geq \sum_{s=N_1}^{\eta-1} \frac{q(n)}{r(n-\sigma)} \ln \left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} \right). \quad (32)$$

By employing (16) in (32), it follows that

$$\sum_{\eta=n-\sigma}^{\eta-1} \lambda(n) \geq \frac{k_0}{a_0} \sum_{n=N_1}^{\eta-1} \frac{q(n)}{r(n-\sigma)} \ln \left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} \right). \quad (33)$$

or

$$\ln \frac{y(\eta-\sigma)}{y(\eta)} \geq \frac{k_0}{a_0} \sum_{n=N_1}^{\eta-1} \frac{q(n)}{r(n-\sigma)} \ln \left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} \right).$$

From (33) and (23), we have

$$\lim \frac{y(n-\sigma)}{y(n)} = \infty. \quad (34)$$

On the other hand from condition (22), there exists a sequence $\{n_k\}$ of integers, $n_k \rightarrow \infty$ as $k \rightarrow \infty$, and there exists $n_k^* \in \{n_k, n_k + 1, \dots, n_k + \sigma - \tau\}$ for every k such that

$$\sum_{s=n_k}^{n_k^*} \frac{q(s)}{r(s-\sigma)} \geq \frac{c}{2} \quad \text{and} \quad \sum_{s=n_k^*}^{n_k+\sigma} \frac{q(s)}{r(s-\sigma)} \geq \frac{c}{2}. \quad (35)$$

Summing both sides of (25) from n_k to n_k^* and n_k^* to $n_k + \sigma$, we have

$$y(n_k^* + 1) - y(n_k) + \frac{k_0}{a_0} \sum_{s=n_k}^{n_k^*} q(s) \frac{y(s-\sigma)}{r(s-\sigma)} \quad (36)$$

and

$$y(n_k + \sigma + 1) - y(n_k^*) + \frac{k_0}{a_0} \sum_{s=n_k^*}^{n_k+\sigma} \frac{q(s)}{r(s-\sigma)} y(s-\sigma) \leq 0. \quad (37)$$

Using the decreasing nature of $\{y(n)\}$ and from (35), (36) and (37), we get

$$-y(n_k) + \frac{k_0 c}{2a_0} y(n_k^* - \sigma) \leq 0$$

and

$$-y(n_k^*) + \frac{k_0 c}{2a_0} y(n_k) \leq 0.$$

This implies eventually

$$\frac{y(n_k^* - \sigma)}{y(n_k^*)} \leq \left(\frac{2a_0}{k_0 c}\right)^2,$$

which is a contradiction with (34). The proof is complete.

4. SOME EXAMPLES

Example 4.1. Consider the following first order nonlinear neutral delay difference equation

$$\Delta \left[\left(\frac{n+1}{n} \right) x(n) - \frac{1}{n} x(n-1) \right] + \left(1 + \frac{1}{n} + \frac{1}{n+1} \right) x(n-2)(1+x^2(n-2)) = 0; n \geq 2. \quad (38)$$

where $\tau = 1$, $\sigma = 2$, $r(n) \equiv 1$, $a(n) = \frac{n+1}{n}$, $p(n) = \frac{1}{n}$, $q(n) = 1 + \frac{1}{n} + \frac{1}{n+1}$ and

$$f(x(n-2)) = x(n-2)(1+x^2(n-2))$$

We see that $a_0 = \frac{3}{2}$, $\frac{p(n)}{a(n)} \leq \frac{1}{3} < 1$, and $k_0 = 1$. Also,

$$\begin{aligned} & \sum_{s=2}^{\infty} q(n) \ln \left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} q(s) \right) \\ &= \sum_{s=2}^{\infty} q(n) \ln \left(\frac{2e}{3} \sum_{s=n+1}^{n+2} q(s) \right) \\ &= \sum_{s=2}^{\infty} q(n) \left[1 + \ln \left(2 + \frac{1}{n+1} + \frac{1}{n+3} \right) \right] \\ &> \sum_{n=2}^{\infty} q(n) \\ &= \sum_{n=2}^{\infty} \left(1 + \frac{1}{n} + \frac{1}{n+1} \right) \\ &= \infty. \end{aligned}$$

Hence all the conditions of Theorem 3.1 are satisfying and so every solution of (38) is oscillatory. One of such solution of (38) is $x(n) = (-1)^n$.

Example 4.2. Consider the following first order neutral delay difference equation

$$\Delta \left[\frac{n}{n+1} \left(\frac{n+1}{n} x(n) - \frac{1}{n} x(n-1) \right) \right] + \frac{1}{3} \left[\frac{(n+3)}{(n+2)} + \frac{(n+2)}{(n+1)} \right] \times \\ x(n-2)(2+x^2(n-2)) = 0; \quad n \geq 3, 4, 5, \dots \quad (39)$$

Here

$$r(n) = \frac{n}{n+1}, \quad q(n) = \frac{1}{3} \left[\frac{n+3}{n+2} + \frac{n+2}{n+1} \right], \tau = 1, \sigma = 2 \text{ and} \\ f(x(n-2)) = x(n-2)(2+x^2(n-2)).$$

clearly $k_0 = 2$, $a(n) \leq \frac{4}{3}$ and $\frac{p(n)}{a(n)} = \frac{1}{n+1} \leq \frac{1}{4} < 1$. Also

$$\liminf_{n \rightarrow \infty} \sum_{s=n}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} = \liminf_{n \rightarrow \infty} \sum_{s=n}^{n+2} \frac{1}{3} \left(\frac{s^2+2s-3}{s^2-4} + \frac{s^2+s-2}{s^2-s-2} \right) \\ = 2 > 0.$$

Again

$$\sum_{n=3}^{\infty} \frac{q(n)}{r(n-\sigma)} \ln \left(\frac{ek_0}{a_0} \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} \right) \\ = \sum_{n=3}^{\infty} \frac{1}{3} \left(\frac{n^2+2n-3}{n^2-4} + \frac{n^2+n-2}{n^2-n-2} \right) \\ \ln \left(\frac{2e}{\frac{4}{3}} \sum_{s=n+1}^{n+2} \left(\frac{s^2+2s-3}{s^2-4} + \frac{s^2+s-2}{s^2-s-2} \right) \right) \\ > \frac{1}{3} \sum_{n=3}^{\infty} \left(\frac{n^2+2n-3}{n^2-4} + \frac{n^2+n-2}{n^2-n-2} \right) = \infty.$$

Then by Theorem 4.2, every solution of (39) is oscillatory. One of such solution of (39) is $x(n) = (-1)^n$.

REFERENCES

- [1] R.P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, Marcel Dekker, New York, 1999.
- [2] M.P. Chen, B.S. Lalli and J.S. Yu, Oscillation in neutral delay difference equations with variable coefficients, *Comput. Math. Appl.*, 29(1995), 5-11.
- [3] S. Elayadi, *An Introduction to Difference Equations*, Third Edition, Springer, New York, 2005.
- [4] D.A. Georgiou, E.A. Grove and G. Ladas, Oscillations of neutral difference equations, *Appl. Anal.*, 33(1989), 243-253.
- [5] I.Gyori and G.Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.
- [6] W.G. Kelly and A.C. Peterson, *Difference Equations; An Introduction with Application*, Academic Press, New York, 1991.
- [7] I.G.E. Kordonis and CH. G. Philos, Oscillation of neutral difference equations with periodic coefficients, *Comput. Math. Appl.*, 33(7) (1997), 11-27.
- [8] W.T. Li, Oscillation criteria for first-order neutral nonlinear difference equations with variable coefficients, *Appl. Math. Lett.*, 10(1997), 101-106.
- [9] W. T. Li and J. R. Yan, Oscillation of first order difference equations with nonlinear neutral terms, *Dynam. Cont. Disc. Impul. Sys.*, 6(1999), 337-344.
- [10] Q. Li, C. Wang, F. Li, H. Liang and Z. Zhang, Oscillation of sublinear difference equations with positive neutral term, *J. Appl. Math. and Comput.*, 20(2006), 305-314.
- [11] X. H. Tang, J. S. Yu and D. H. Peng, Oscillation and nonoscillation of neutral difference equations with positive and negative coefficients, *Comput. Math. Appl.*, 39(2000), 169-181.
- [12] G. Zhang, Oscillation for nonlinear neutral difference equations, *Appl. Math. E-Notes*, 2(200), 22-24.

