International Journal of Difference Equations
ISSN 0973-6069 Volume 12, Number 1 (2017), pp. 55-70

http://www.ripublication.com

INTUITIONISTIC RANDOM STABILITY OF A QUADRATIC FUNCTIONAL EQUATION ORIGINATING FROM THE SUM OF THE MEDIANS OF A TRIANGLE

M. Arunkumar [1] **and S. Karthikeyan** [2*]

[1] *Department of Mathematics, Government Arts College,*
Tiruvannamalai - 606 603, TamilNadu, India.
E-mail: annarun2002@yahoo.co.in

[2*] *Department of Mathematics, R.M.K. Engineering College,*
Kavaraipettai - 601 206, Tamil Nadu, India.

ABSTRACT

In this paper, the authors proved the intuitionistic random stability of quadratic functional equation

$$f\left(\frac{x+y}{2}-z\right)+f\left(\frac{y+z}{2}-x\right)+f\left(\frac{z+x}{2}-y\right)=\frac{3}{4}\big(f(x-y)+f(y-z)+f(z-x)\big)$$

originating from the sum of the medians of a triangle using direct and fixed point methods.

2010 Mathematics Subject Classification. : 39B52, 32B72, 32B82 .

Key words and phrases: Quadratic functional equation, Generalized Ulam-Hyers stability, intuitionistic random normed space, fixed point method.

1. Introduction

The stability of functional equations had been first raised by S.M. Ulam [46]. In 1941, D. H. Hyers [24] remarked a positive answer to the question of Ulam regards to Banach spaces. In 1950, T. Aoki [3] was considered as the second author to handle this problem for additive mappings.

Eventually Th.M. Rassias [39] succeeded in extending the result of Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $\| x\|^p + \| y\|^p$; $p \in [0,1)$ to be unbounded. While considering the influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Th.M. Rassias is called Hyers-Ulam-Rassias stability one can refer [2, 19, 25].

In 1982, J.M. Rassias [35] followed the innovative approach of the Th.M. Rassias theorem [39] in which he replaced the factor $\| x\|^p + \| y\|^p$ by $\| x\|^p \| y\|^q$ for $p, q \in \mathrm{P}$ with $p+q \neq 1$. The comprehensive state of above results were obtained by P. Gavruta [21] in 1994 by replacing the unbounded Cauchy difference by a general control function $\phi(x, y)$ in the spirit of Rassias approach.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was procured by Ravi et al., [40] in view of the summation of both the sum and the product of two $p-$ norms in the sprit of Rassias approach.

In 2003, V. Radu [11] proposed a new method, successively developed in [12, 13, 14], to obtaining the existence of the exact solutions and the error estimations, based on the fixed point alternative.

The theory of random normed spaces (RN-spaces) is important as a generalization of the deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us with the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. Recently, J.M. Rassias et al. [36] investigated the intuitionistic random stability of the quartic functional equation and C. Park et al. [33] presented the Hyers-Ulam stability of the additive-quadratic functional equation in intuitionistic random normed space.

Very recently, John M. Rassias, M. Arunkumar and S. Karthikeyan [38] investigated the intuitionistic random stability of a quadratic reciprocal functional equation

$$f(x+2y)+f(2x+y)=\frac{f(x)f(y)\left[5f(x)+5f(y)+8\sqrt{f(x)f(y)}\right]}{\left[2f(x)+2f(y)+5\sqrt{f(x)f(y)}\right]^2} \tag{1.1}$$

using direct and fixed point methods.

In 2015, John M. Rassias, M. Arunkumar and S. Karthikeyan [37] proved the solution in vector space and the generalized Ulam-Hyers stability of the ternary quadratic homomorphisms and ternary quadratic derivations between fuzzy ternary Banach algebras associated to the quadratic functional equation

$$f\left(\frac{x+y}{2}-z\right)+f\left(\frac{y+z}{2}-x\right)+f\left(\frac{z+x}{2}-y\right)=\frac{3}{4}\big(f(x-y)+f(y-z)+f(z-x)\big) \tag{1.2}$$

originating from the sum of the medians of a triangle by using direct and fixed point methods.

An application of this functional equation is also studied.

In this paper, we prove the intuitionistic random stability of a quadratic functional equation (1.2) originating from the sum of the medians of a triangle using direct and fixed point methods.

2 Preliminaries of Intuitionistic Random Normed Spaces

In this section, using the idea of intuitionistic random normed spaces introduced by Chang et al. [16], we define the notion of intuitionistic random normed spaces as in [15, 22, 29, 31, 42, 43, 44].

Definition 2.1

A measure distribution function is a function $\mu:\mathbb{R}\to[0,1]$ which is left continuous, non-decreasing on $\mathbb{R}$, $\inf_{t\in\mathbb{R}}\mu(t)=1$ and $\text{suf}_{t\in\mathbb{R}}\mu(t)=1$.

We will denote by D the family of all measure distribution functions and by H a special element of D defined by

$$H(t)=\begin{cases}0, & if\ t\le 0,\\ 1, & if\ t>0.\end{cases} \tag{2.1}$$

If X is a nonempty set, then $\mu:X\to D$ is called a probabilistic measure on X and $\mu(x)$ is denoted by μ_x.

Definition 2.2.

A non-measure distribution function is a function $\nu:\mathbb{R}\to[0,1]$ which is right continuous, non-decreasing on $\mathbb{R}$, $\inf_{t\in\mathbb{R}}\nu(t)=1$ and $\text{suf}_{t\in\mathbb{R}}\nu(t)=1$.

We will denote by B the family of all non-measure distribution functions and by G a special element of B defined by

$$G(t)=\begin{cases}1, & if\ t\le 0,\\ 0, & if\ t>0.\end{cases} \tag{2.2}$$

If X is a nonempty set, then $\nu : X \to D$ is called a probabilistic non-measure on X and $\nu(x)$ is denoted by ν_x.

Lemma 2.3. [8, 20]

Consider the set L^* and the order relation $\leq_{L^*}$ defined by:

$$L^* \left\{(x_1, x_2) \ : \ (x_1, x_2) \ \in \ [0,1]^2 \ and \ x_1 + \ x_2 \leq \ 1\right\},$$
$$(x_1, x_2) \leq_{L^*} \ (y_1, y_2) \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2, \forall (x_1, x_2), \ (y_1, y_2) \in L^*.$$

Then $\left(L^*, \leq_{L^*}\right)$ is a complete lattice.

Definition 2.4. [8]

An intuitionistic fuzzy set $A_{\xi,n}$ in a universal set U is an object $A_{\xi,n} = \left\{(\xi_A(u), \eta_A(u)) \mid u \in U\right\}$ for all $u \in U, \xi_A(u) \in \left[0,1\right]$ and $\eta_A(u) \in \left[0,1\right]$ are called the *membership degree* and the *non-membership degree*, respectively, of u in $A_{\xi,n}$ and, furthermore, they satisfy $\xi_A(u) + \eta_A(u) \leq 1$.

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (0,1)$. Classically, a *triangular norm* $* = T$ on [0,1] is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1, x) = 1 * x = x$ for all $x \in [0,1]$. A *triangular co-norm* $S = \Diamond$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \to [0,1]$ satisfying $S(0, x) = 0 \Diamond x = x$ for all $x \in [0,1]$.

Using the lattice $\left(L^*, \leq_{L^*}\right)$, these definitions can be straightforwardly extended.

Definition 2.5. [8]

A triangular norm ($t-$norm) on L^* is a mapping $\mathrm{T} : (\mathrm{L}^*)^2 \to \mathrm{L}^*$ satisfying the following conditions:

(i) $\left(\forall \in L^*\right)\left(T(x, 1_{L^*}) = x\right)$ (boundary conditions);

(ii) $\left(\forall (x, y) \in \left(L^*\right)^2\right)\left(T(x, y) = T(y, x)\right)$ (commutativity);

(iii) $\left(\forall (x, y, z) \in \left(L^*\right)^3\right)\left(T\left(x, T(y, z)\right) = T\left(T(x, y), z\right)\right)$ (associativity);

(iv) $\left(\forall (x, x', y, y') \in \left(L^*\right)^4\right)\left(x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \Rightarrow T(x, y) \leq_{L^*} T(x', y')\right)$ (monotonically).

If $\left(L^*, \leq_{L^*}, T\right)$ is an Abelian topological monoid with unit 1_{L^*}, then L^* is said to be a *continuous* $t-$*norm*.

Definition 2.6. [8]

A continuous $t-$norms T on L^* is said to be *continuous* $t-$*representable* if there exist a continuous $t-$norm $*$ and a continuous $t-$conorm $\Diamond$ on [0,1] such that, for all

$$x=(x_1,x_2), y=(y_1,y_2)\in L^*, \quad T(x,y)=(x_1*y_1, x_2\Diamond y_2).$$

For example,

$$T(a,b)=\left(a_1,b_1,\ \min\{a_2+b_2,1\}\right)$$
$$\text{and}\quad M(a,b)=\left(\min\{a_1+b_1\},\max\{a_2+b_2\}\right)$$

for all $a=(a_1,a_2), b=(b_1,b_2)\in L^*$ are continuous $t-$representable.

Now, we define a sequence T^n recursively by $T^1=T$ and

$$T^n\left(x^{(1)},...,x^{(n+1)}\right)=T\left(T^{(n-1)}\left(x^{(1)},...,x^{(n)}\right),x^{(n+1)}\right), \forall n\geq 2, x^{(i)}\in L^*.$$

Definition 2.7. [45]

A *negator* on L^* is any decreasing mapping $N:L^*\to L^*$ satisfying $N:(0_{L^*})$ and $N(1_{L^*})=0_{L^*}$. If $N(N(x))=x$ for all $x\in L^*$, then N is called an *involutive negator*. A negator on [0,1] is a decreasing mapping $N:[0,1]\to[0,1]$ satisfying $P_{\mu,\nu}(0) = 1$ and $P_{\mu,\nu}(1) = 0$. N_s denotes the standard negator on [0,1] defined by $N_s(x)=1-x, \forall \in[0,1]$.

Definition 2.8. [45]

Let μ and ν be measure and non- measure distributions functions from $X\times(0,+\infty)$ to [0,1] such that $\mu_x(t)+\nu_x(t)\leq 1$ for all $x\in X$ and all $t>0$. The triple $\left(X,P_{\mu,\nu},T\right)$ is said to be an *intuitionistic* random *normed space* (briefly IRN-space) if X is a vector space, T is a continuous $t-$representable and $P_{\mu,\nu}$ is a mapping $X\times(0,+\infty)\to L^*$ satisfying the following conditions: for all $x,y\in X$ and $t,s>0$,

$(IRN1)$ $P_{\mu,\nu}(x,0) = 0_{L^*}$;

$(IRN2)$ $P_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x=0$;

$(IRN3)$ $P_{\mu,\nu}(\alpha x,t) = P_{\mu,\nu}\left(x,\dfrac{t}{|\alpha|}\right)$ for all $\alpha\neq 0$;

$(IRN4)$ $P_{\mu,\nu}(x+y,t+s) \geq_{L^*} \mathrm{T}\left(P_{\mu,\nu}(x,t),P_{\mu,\nu}(y,s)\right)$.

In this case, $P_{\mu,\nu}$ is called an *intuitionistic random norm*. Here $P_{\mu.\nu}(x,t)=\left(\mu_x(t),\nu_x(t)\right).$,

Example 2.9. [45]

Let $\left(x,\|.\|\right)$ be a normed space. Let $T(a,b)=(a_1,b_1, \min\,(a_2+b_2,1))$ for all $a=(a_1,a_2),\ b=(b_1,b_2)\in L^*$ and μ,ν be measure and non-measure distributions functions defined by

$$P_{\mu.\nu}(x,t)=\left(\mu_x(t),\nu_x(t)\right)=\left(\frac{t}{t+\|t\|},\frac{\|x\|}{t+\|x\|}\right),\forall t\in R^+.$$

Then $(X,P_{\mu,\nu},T)$ is an IRN-sapce.

Definition 2.10. [45]

A sequence $\{x_n\}$ in an IRN-space $(X,P_{\mu,\nu},T)$ is called a *Cauchy sequence* if, for any $\varepsilon>0$ and $t>0$, there exists $n_0\in N$ such that $P_{\mu.\nu}(x_n-x_m,t)>L^*\left(N_s(\varepsilon),\varepsilon\right),\forall n,m\geq n_0,$ where N_s is the standard negator.

Definition 2.11. [45]

The sequence $\{x_n\}$ is said to be *convergent* to a point $x\in X$

(denoted by $x_n\xrightarrow{P_{\mu.\nu}}x$) if $P_{\mu.\nu}(x_n-x,t)\to 1_{L^*}$ as $n\to\infty$ for every $t>0$.

Definition 2.12. [45]

An IFN-space $(X,P_{\mu,\nu},T)$ is said to be *complete* if every Cauchy sequence in X is convergent to a point $x\in X$.

Now, we use the following notation for a given mapping $f:X\to Y$

$$\Delta(x,y,z)=f\left(\frac{x+y}{2}-z\right)+f\left(\frac{y+z}{2}-x\right)+f\left(\frac{z+x}{2}-y\right)-\frac{3}{4}\left(f\left(x-y\right)+f\left(y-z\right)+f\left(z-x\right)\right)$$

for all $x,y,z\in X$.

3. STABILITY RESULTS: DIRECT METHOD

In this section, the authors present the generalized Ulam-Hyers stability of the functional equation (1.2) in IRN- space using direct method.

Hereafter throughout this paper, let us consider X be a linear space and $(Y, P_{\mu,\nu}, M)$ be a complete intuitionistic random normed space.

Theorem 3.1. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there are $\xi, \zeta : X^3 \to D^+$, $\xi(x,y,z)$ is denoted by $\xi_{x,y,z}$ and $\zeta(x,y,z)$ is denoted by $\zeta_{x,y,z}$, furthur, $\left(\xi_{x,y,z}(t), \zeta_{x,y,z}(t)\right)$ is denoted by $P'_{\xi,\zeta}(x,y,z,t)$ with the property

$$P_{\mu,\nu}\left(\Delta(x,y,z),t\right) \geq_{L^*} P'_{\xi,\zeta}\left(x,y,z,t\right) \tag{3.1}$$

for all $x, y, z \in X$ and all $t > 0$. If

$$T_{i=1}^{\infty} P'_{\xi,\zeta}\left(2^{i+n}x, 2^{i+n}x, -2^{i+n}x, 2^{2(i+n)}t\right) = 1_{L^*} \tag{3.2}$$

and $$\lim_{n\to\infty} P'_{\xi,\zeta}\left(2^n x, 2^n x, -2^n x, 2^{2n}t\right) = 1_{L^*} \tag{3.3}$$

for all $x \in X$ and all $t > 0$ then there exists a unique quadratic mapping $Q : X \to Y$ satisfies the inequality

$$P_{\mu,\nu}\left(f(x) - Q(x), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(2^i x, 2^i x, -2^i x, 2^{i+1}t\right) \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Replacing (x,y,z) by $(x,x,-x)$ in (3.1), we get

$$P_{\mu,\nu}\left(\frac{f(2x)}{2^2} - f(x), \frac{t}{2^2}\right) \geq_{L^*} P'_{\xi,\zeta}\left(x,x,-x,t\right) \tag{3.5}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^n x$ and using $(IRN3)$ in the above equation, we have

$$P_{\mu,\nu}\left(\frac{f(2^{n+1}x)}{2^{2(n+1)}} - \frac{f(2^n x)}{2^{2n}}, \frac{t}{2^{2(n+1)}}\right) \geq_{L^*} P'_{\xi,\zeta}\left(2^n x, 2^n x, -2^n x, t\right) \tag{3.6}$$

for all $x \in X$ and all $r > 0$ which implies that

$$P_{\mu,\nu}\left(\frac{f(2^{n+1}x)}{2^{2(n+1)}} - \frac{f(2^n x)}{2^{2n}}, \frac{t}{2^{n+1}}\right) \geq_{L^*} P'_{\xi,\zeta}\left(2^n x, 2^n x, -2^n x, 2^{n+1}t\right) \tag{3.7}$$

holds for all $x \in X$ and all $t > 0$. As $1 > \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$, by the triangular inequality it follows

$$P_{\mu,\nu}\left(\frac{f(2^n x)}{2^{2n}} - f(x), t\right) = T_{i=0}^{n-1}\left\{P_{\mu,\nu}\left(\frac{f(2^{i+1}x)}{2^{2(i+1)}} - \frac{f(2^i x)}{2^{2i}}, \sum_{i=0}^{n-1}\frac{1}{2^{i+1}}t\right)\right\}$$
$$\geq_{L^*} T_{i=0}^{n}\left\{P'_{\xi,\zeta}\left(2^i x, 2^i x, -2^i x, 2^{i+1}t\right)\right\} \tag{3.8}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\left\{\frac{f(2^n x)}{2^{2n}}\right\}$, replacing x by $2^m x$ in (3.8) , we obtain

$$P_{\mu,\nu}\left(\frac{f(2^{m+n}x)}{2^{2(m+n)}} - \frac{f(2^m x)}{2^{2m}}, t\right) \geq_{L^*} T_{i=1}^{n}\left\{P'_{\xi,\zeta}\left(2^{i+m}x, 2^{i+m}x, -2^{i+m}x, 2^{i+2m+1}t\right)\right\} \tag{3.9}$$

for all $x \in X$ and all $t > 0$ and all $m, n \geq 0$. Since the right hand-side of the inequality tends

to 1_{L^*} as m tends to infinity, the sequence $\left\{\frac{f(2^n x)}{2^{2n}}\right\}$ is a Cauchy sequence. Therefore, we may define the mapping $Q: X \to Y$ by

$$P_{\mu,\nu}\left(Q(x) - \frac{f(2^n x)}{2^{2n}}, t\right) \to 1_{L^*} \quad as \quad n \to \infty, t > 0$$

for all $x \in X$. Now, we prove that Q satisfies (1.2). Replacing (x, y, z) by $\left(2^n x, 2^n y, 2^n z\right)$ in (3.1), we get

$$P_{\mu,\nu}\left(\frac{1}{2^{2n}}\Delta\left(2^n x, 2^n y, 2^n z\right), t\right) \geq_{L^*} P'_{\xi,\zeta}\left(2^n x, 2^n y, 2^n z, 2^{2n}t\right) \tag{3.10}$$

for all $x, y, z \in X$ and all $t > 0$. Letting $n \to \infty$ in the above inequality and using the definition of $Q(x)$,

we see that Q satisfies (1.2) for all $x, y, z \in X$. Finally, to prove the uniqueness of the quadratic function $Q(x)$ subject to (3.4), let us assume that there exists another quadratic function $R(x)$ which satisfies (3.4). Obviously, we have $Q(2^n x) = 2^{2n}Q(x)$ and $R(2^n x) = 2^{2n}R(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence, it follows from (3.4) that

$$P_{\mu,\nu}\left(Q(x) - R(x), t\right) \geq_{L^*} P_{\mu,\nu}\left(Q(2^n x) - R(2^n x), 2^{2n}t\right)$$
$$\geq_{L^*} T\left\{P_{\mu,\nu}\left(Q(2^n x) - f(2^n x), \frac{2^{2n}}{2}t\right), P_{\mu,\nu}\left(f(2^n x) - R(2^n x), \frac{2^{2n}}{2}t\right)\right\}$$
$$\geq_{L^*} T\left(T_{i=1}^{\infty}\left(P'_{\xi,\zeta}\left(2^{i+n}x, 2^{i+n}x, -2^{i+n}x, 2^{i+2n+1}t\right)\right), T_{i=1}^{\infty}\left(P'_{\xi,\zeta}\left(2^{i+n}x, 2^{i+n}x, -2^{i+n}x, 2^{i+2n+1}t\right)\right)\right) \tag{3.11}$$

for all $x \in X$ and all $t > 0$. By letting $n \to \infty$ in (3.11), we find that $Q = R.$ This completes the proof.

From Theorem 3.1, we obtain the following corollary concerning the Hyers-Ulam-Rassias and JMRassias stabilities for the functional equation (1.2).

Corollary 3.2 Suppose that a function $f: X \to Y$ satisfies the inequality

$$P_{\mu,\nu}\left(\Delta(x,y,z),t\right) \geq_{L^*} \begin{cases} P'_{\xi,\zeta}\left(\varepsilon,t\right), & \\ P'_{\xi,\zeta}\left(\varepsilon\left\{\|x\|^s+\|y\|^s+\|z\|^s\right\},t\right), & s \neq 2; \\ P'_{\xi,\zeta}\left(\varepsilon\|x\|^s\|y\|^s\|z\|^s,t\right), & s \neq \frac{2}{3}; \\ P'_{\xi,\zeta}\left(\varepsilon\left\{\|x\|^s\|y\|^s\|z\|^s+\left\{\|x\|^{3s}+\|y\|^{3s}+\|z\|^{3s}\right\}\right\},t\right), & s \neq \frac{2}{3} \end{cases} \tag{3.12}$$

for all all $x, y, z \in X$ and all $t > 0$, where ε, s are constants with $\varepsilon > 0$. Then there exists a unique quadratic mapping $Q: X \to Y$ such that

$$P_{\mu,\nu}\left(f(x)-Q(x),t\right) \geq_{L^*} \begin{cases} P'_{\xi,\zeta}\left(\varepsilon,|3|t\right), \\ P'_{\xi,\zeta}\left(\dfrac{3\varepsilon}{\left|2^2-2^s\right|}\|x\|^s,t\right), \\ P'_{\xi,\zeta}\left(\dfrac{\varepsilon}{\left|2^2-2^{3s}\right|}\|x\|^{3s},t\right), \\ P'_{\xi,\zeta}\left(\dfrac{4\varepsilon}{\left|2^2-2^{3s}\right|}\|x\|^{3s},t\right) \end{cases} \tag{3.13}$$

for all $x \in X$ and all $t > 0$.

4. STABILITY RESULTS: FIXED POINT METHOD

In this section, using the fixed point alternative approach, we prove the generalized Ulam - Hyers stability of the functional equation (1.2) in intuitionistic random normed spaces.

Now, we will recall the fundamental results in fixed point theory.

Theorem 4.1 *(Banach Contraction Principle)*

Let (Ω, d) *is a non-Archimedean generalized complete metric space and consider a mapping* $T: \Omega \to \Omega$ *which is strictly contractive mapping, that is*

(A1). $d(Tx, Ty) \leq L d(x, y)$, for all $x, y \in T$ for some (Lipschitz constant) $L < 1$. Then,

(i) The mapping T has one and only fixed point $x^* = T(x^*)$;

(ii)The fixed point for each given element x^* is globally attractive, that is

(A2). $lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in \Omega$;

(iii) One has the following estimation inequalities:

(A3). $d(T^n x, x^*) \le \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \ge 0, \forall\ x \in \Omega;$

(A4). $d(x, x^*) \le \frac{1}{1-L}\ d(x, x^*), \forall\ x \in \Omega.$.

Theorem 4.2 *[13](The alternative of fixed point)*

Suppose that for a complete generalized metric space (A, d) *and a strictly contractive mapping* $T : \mathrm{A} \to \mathrm{A}$ *with Lipschitz constant* L. *Then, for each given element* $x \in \mathrm{A}$, *either*

$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \ge 0,$$

or

$(B2)$ there exists a natural number n_0 such that:

(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \ge n_0$;

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in \mathrm{A} : d(T^{n_0} x, y) < \infty\}$;

(iv) $d(y^*, y) \le \frac{1}{1-L}\ d(y, Ty)$ for all $y \in Y$.

Using above fixed point theorems to prove the stability results, we define the following constant $$\delta_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1 \end{cases}$$

and Ω is the set such that

$$\Omega = \{g \mid g : X \to Y, g(0) = 0\}.$$

Theorem 4.3. Let $f : X \to Y$ be a mapping for which there exist a function $\xi, \zeta : X^3 \to D^+$ with the condition

$$T_{i=1}^{\infty} P'_{\xi,\zeta}\left(2^{i+n}x, 2^{i+n}x, -2^{i+n}x, 2^{i-1+2n}t\right) = 1_{L^*} \tag{4.1}$$

and $\lim_{n\to\infty} P'_{\xi,\zeta}\left(2^n x, 2^n x, -2^n x, 2^{2n} t\right) = 1_{L^*}$ (4.2)

and satisfying the functional inequality

$$P_{\mu,\nu}\left(\Delta(x,y,z),t\right) \geq_{L^*} P'_{\xi,\zeta}\left(x,y,z,t\right),\ \forall\ x,y,z \in X,\ t>0. \tag{4.3}$$

If there exists L such that the function

$$x \to \beta(x) = \frac{x}{2}, \frac{x}{2}, -\frac{x}{2} \tag{4.4}$$

has the property

$$P'_{\xi,\zeta}\left(\frac{L}{\delta_i^2}\beta(\delta_i x), r\right) = P'_{\xi,\zeta}\left(\beta(x),t\right), \quad \forall\ x \in X,\ t>0, \tag{4.5}$$

then there exists unique quadratic function $Q: X \to Y$ satisfying the functional equation (1.2) and

$$P_{\mu,\nu}\left(f(x)-Q(x),t\right) \geq_{L^*} P'_{\xi,\zeta}\left(\left(\frac{L^{1-i}}{1-L}\right)\beta(x),t\right),\ \forall\ x \in X, t>0. \tag{4.6}$$

Proof. Let d be a general metric on Ω, such that

$$d(g,h) = inf\left\{K \in (0,\infty) \mid P_{\mu,\nu}\left(g(x)-h(x),t\right) \geq_{L^*} P'_{\xi,\zeta}\left(K\beta(x),t\right), x \in X,\ t>0\right\}.$$

It is easy to see that (Ω, d) is complete. Define $\Gamma: \Omega \to \Omega$ by $\Gamma g(x) = \frac{1}{\delta_i^2} g(\delta_i x)$, for all $x \in X$. Now for all $g,h \in \Omega$, we have

$$\begin{aligned}
d(g,h) \leq K \Rightarrow \quad & P_{\mu,\nu}\left(g(x)-h(x),t\right) \geq_{L^*} P'_{\xi,\zeta}\left(K\beta(x),t\right),\ x \in X,\\
\Rightarrow\ & P_{\mu,\nu}\left(\frac{1}{\delta_i^2} g(\delta_i x) - \frac{1}{\delta_i^2} h(\delta_i x), t\right) \geq_{L^*} P'_{\mu,\nu}\left(K\beta(\delta_i x), \delta_i^2 t\right),\ x \in X,\\
\Rightarrow\ & P_{\mu,\nu}\left(\frac{1}{\delta_i^2} g(\delta_i x) - \frac{1}{\delta_i^2} h(\delta_i x), t\right) \geq_{L^*} P'_{\mu,\nu}\left(\frac{K}{\delta_i^2}\gamma(\delta_i x), t\right),\ x \in X, \qquad (4.7)\\
\Rightarrow\ & P_{\mu,\nu}\left(Tg(x)-Th(x),t\right) \geq_{L^*} P'_{\mu,\nu}\left(KL\beta(x),t\right),\ x \in X,\\
\Rightarrow\ & d(Tg,Th) \leq LK.
\end{aligned}$$

This gives $d(Tg,Th) \leq Ld(g,h)$, for all $g,h \in \Omega$, i.e., T is a strictly contractive mapping of Ω with Lipschitz constant L. Replacing (x,y,z) by $(x,x,-x)$ in (4.3), we get

$$P_{\mu,\nu}\left(f(2x)-2^2 f(x),t\right) \geq_{L^*} P'_{\xi,\zeta}\left(x,x,-x,t\right),\ \forall x \in X, t>0. \tag{4.8}$$

Using $(IFN3)$ in (4.5), we arrive

$$P_{\mu,\nu}\left(\frac{f(2x)}{2^2}-f(x),t\right)\geq_{L^*} P'_{\xi,\zeta}\left(x,x,-x,2^2t\right),\quad \forall \quad x\in X, t>0. \tag{4.9}$$

With the help of (4.5), when $i=0$, it follows from (4.8), that

$$P_{\mu,\nu}\left(\frac{f(2x)}{2^2}-f(x),t\right)\geq_{L^*} P'_{\xi,\zeta}\left(L\beta(x),t\right),\quad \forall\, x\in X, t>0.$$

$$\Rightarrow d(f,\Gamma f)\leq L=L^1=L^{1-i}. \tag{4.10}$$

Replacing x by $\frac{x}{2}$ in (4.8) and using (IRN3), we obtain

$$P_{\mu,\nu}\left(f(x)-2^2 f\left(\frac{x}{2}\right),t\right)\geq_{L^*} P'_{\xi,\zeta}\left(\frac{x}{2},\frac{x}{2},-\frac{x}{2},t\right),\quad \forall\, x\in X, t>0. \tag{4.11}$$

With the help of (4.5), when $i=1$, it follows from (4.11), that

$$P_{\mu,\nu}\left(f(x)-2^2 f\left(\frac{x}{2}\right),t\right)\geq_{L^*} P'_{\xi,\zeta}\left(\beta(x),t\right),\quad \forall\, x\in X, t>0.$$

$$\Rightarrow d(\Gamma f,f)\leq 1=L^0=L^{1-i}. \tag{4.12}$$

Then from (4.10) and (4.12), we can conclude

$$d(f,\Gamma f)\leq L^{1-i}<\infty.$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point Q of Γ in Ω such that

$$\lim_{n\to\infty} P_{\mu,\nu}\left(\frac{1}{\delta_i^{2n}} f\left(\delta_i^n x\right)-Q(x),t\right)\to 1_{L^*,}\qquad \forall x\in X, t>0. \tag{4.13}$$

Replacing (x,y,z) by $\left(\delta_i^n x,\delta_i^n y,\delta_i^n z\right)$ in (4.3), we arrive

$$P_{\mu,\nu}\left(\delta_i^{2n}\Delta\left(\delta_i^n x,\delta_i^n y,\delta_i^n z\right),r\right)\geq_{L^*} P'_{\xi,\zeta}\left(\delta_i^n x,\delta_i^n y,\delta_i^n z,\delta_i^{2n}t\right),\quad \forall\, x,y,z\in X, t>0. \tag{4.14}$$

By proceeding the same procedure as in the Theorem 3.1, we can prove the function, $Q:X\to Y$ satisfies the functional equation (1.2). By fixed point alternative, since Q is unique fixed point of Γ in the set

$$\Delta=\left\{f\in\Omega \mid d(f,Q)<\infty\right\},$$

such that

$$P_{\mu,\nu}\left(f(x)-Q(x),t\right)\geq_{L^*} P'_{\xi,\zeta}\left(K\beta(x),t\right),\quad \forall\, x\in X, t>0. \tag{4.15}$$

Again using the fixed point alternative, we obtain

$$d\left(f,Q\right)\le \frac{1}{1-L}d\left(f,\Gamma f\right) \qquad \Rightarrow d\left(f,Q\right)\le \frac{L^{1-i}}{1-L}.$$

Hence, we have

$$P_{\mu,\nu}\left(f(x)-Q(x),t\right)\ge_{L^*} P'_{\xi,\zeta}\left(\left(\frac{L^{1-i}}{1-L}\right)\beta(x),t\right),\ \forall\, x\in X, t>0. \tag{4.16}$$

This completes the proof of the theorem.

From Theorem 4.3, we obtain the following corollary concerning the stability for the functional equation (1.2).

Corollary 4.4 Suppose that a function $f:X\to Y$ satisfies the inequality

$$P_{\mu,\nu}\left(\Delta(x,y,z),t\right)\ge_{L^*}\begin{cases} P'_{\xi,\zeta}\left(\varepsilon,t\right), & \\ P'_{\xi,\zeta}\left(\varepsilon\left\{\|x\|^s+\|y\|^s+\|z\|^s\right\},t\right), & s\ne 2;\\ P'_{\xi,\zeta}\left(\varepsilon\|x\|^s\|y\|^s\|z\|^s,t\right), & s\ne \frac{2}{3};\\ P'_{\xi,\zeta}\left(\varepsilon\left\{\|x\|^s\|y\|^s\|z\|^s+\left\{\|x\|^{3s}+\|y\|^{3s}+\|z\|^{3s}\right\}\right\},t\right), & s\ne\frac{2}{3}\end{cases} \tag{4.17}$$

for all all $x,y,z\in X$ and all $t>0$, where ε,s are constants with $\varepsilon>0$. Then there exists a unique quadratic mapping $Q:X\to Y$ such that

$$P_{\mu,\nu}\left(f(x)-Q(x),t\right)\ge_{L^*}\begin{cases} P'_{\xi,\zeta}\left(\varepsilon,|3|t\right), \\ P'_{\xi,\zeta}\left(\frac{3\varepsilon}{\left|2^2-2^s\right|}\|x\|^s,t\right), \\ P'_{\xi,\zeta}\left(\frac{\varepsilon}{\left|2^2-2^{3s}\right|}\|x\|^{3s},t\right), \\ P'_{\xi,\zeta}\left(\frac{4\varepsilon}{\left|2^2-2^{3s}\right|}\|x\|^{3s},t\right)\end{cases} \tag{4.18}$$

for all $x\in X$ and all $t>0$.

REFERENCES

[1] Abasalt Bodaghi and Sang Og Kim, Approximation on the quadratic reciprocal functional equation, Journal of Function Spaces, http://dx.doi.org/10.1155/2014/532463.

[2] J. Aczel and J. Dhombres, Functional Equations in Several Variables, Cambridge Univ, Press, 1989.

[3] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2 (1950), 64-66.

[4] M. Arunkumar and S. Karthikeyan, Solution and Intuitionistic Fuzzy Stability of n-Dimensional Quadratic Functional Equation: Direct and Fixed Point Methods, Int. J. of Advanced Mathematical Sciences, 2 (1) (2014), 21-33.

[5] M. Arunkumar and S. Karthikeyan, Solution and stability of a reciprocal functional equaton originating from n-consecutive terms of a harmonic progression: Direct and _xed Point methods, Int. J. of Information Science and Intelligent System, 3(1) (2014), 151-168.

[6] M. Arunkumar, S. Karthikeyan and S. Ramamoorthi, Generalized Ulam-Hyers stability of n-dimensional cubic functional equation in FNS and RNS: various methods, Middle-East J of Scienti_c Research, 24 (S2) (2016), 386-404.

[7] M. Arunkumar, John M. Rassias and S. Karthikeyan, Stability of a Leibniz type additive and quadratic functional equation in intuitionistic fuzzy normed spaces, Advances in theoretical and applied mathematics, 11 (2016), No.2, 145-169.

[8] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1986), 87-96.

[9] T. Bag and S.K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11 (3) (2003), 687-705.

[10] T. Bag and S.K. Samanta, Fuzzy bounded linear operators, Fuzzy Sets and Systems, 151 (2005) 513-547.

[11] L. C_adariu and V. Radu, Fixed points and the stability of Jensen's functional equation, J. Inequal. Pure Appl. Math., 4 (2003), no. 1, Article 4, 7 pp. (electronic).

[12] L. C_adariu and V. Radu, Fixed point methods for the generalized stability of functional equations in a single variable, Fixed Point Theory and Applications, 2008, Article ID 749392, (2008), 15 pages.

[13] L.C_adariu V. Radu, A generalized point method for the stability of Cauchy functional equation, In Functional Equations in Mathematical Analysis, Th. M. Rassias, J. Brzdek (Eds.), Series Springer Optimization and Its Applications, 52, 2011.

[14] L. C_adariu and V. Radu, On the stability of the Cauchy functional equation: a _xed point approach, in Iteration theory (ECIT '02), 43{52, Grazer Math. Ber., 346, Karl-Franzens-Univ. Graz, Graz.

[15] S.S. Chang, Y. Cho, Y. Kang, Nonlinear Operator Theory in Probabilistic Metric Spaces, Nova Science Publishers Inc., New York (2001).

[16] S.S. Chang, J.M. Rassias, R. Saadati, The stability of the cubic functional equation in intuitionistic random normed spaces, Appl Math Mech. 31, 2126 (2010).

[17] S.C. Cheng and J.N. Mordeson, Fuzzy linear operator and fuzzy normed linear spaces, Bull. Calcutta Math. Soc., 86 (1994), 429-436.

[18] S.Czerwik, On the stability of the quadratic mappings in normed spaces, Abh.Math.Sem.Univ Hamburg., 62 (1992), 59-64.

[19] S. Czerwik, Functional Equations and Inequalities in Several Variables, World Scienti_c, River Edge, NJ, 2002.

[20] G. Deschrijver and E.E. Kerre, On the relationship between some extensions of fuzzy set theory, Fuzzy Sets and Systems, 23 (2003), 227-235.

[21] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., 184 (1994), 431-436.

[22] O. Hadzic, E. Pap, Fixed Point Theory in PM-Spaces, Kluwer Academic, Dordrecht (2001).

[23] K. Hensel, Uber eine neue Begrundung der Theorie der algebraischen Zahlen, Jahres Deutsch Math Verein.,6 (1897), 83-88.

[24] D.H. Hyers, On the stability of the linear functional equation, Proc.Nat. Acad.Sci.,U.S.A., 27 (1941) 222-224.

[25] S.M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.

[26] Pl. Kannappan, Functional Equations and Inequalities with Applications, Springer Monographs in Mathematics, 2009.

[27] A.K. Katsaras, Fuzzy topological vector spaces II, Fuzzy Sets and Systems, 12 (1984), 143-154.

[28] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica, 11 (1975), 326-334.

[29] S. Kutukcu, A. Tuna, A.T. Yakut, Generalized contraction mapping principle in intuitionistic Menger spaces and application to di_erential equations, Appl Math Mech., 28 (2007), 799-809. doi:10.1007/s10483-007-0610-z.

[30] B. Margolis and J. B. Diaz, A _xed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc., Vol.126 (1968), no.74, 305-309.

[31] D. Mihet, V. Radu On the stability of the additive Cauchy functional equation in random normed spaces,J Math Anal Appl., 343 (2008), 567-572.

[32] A.K. Mirmostafaee and M.S. Moslehian, Fuzzy versions of Hyers-Ulam-Rassias theorem, Fuzzy Sets and Systems, Vol. 159 (2008), no.6, 720-729.

[33] C. Park, Madjid Eshaghi Gordji, M. Ghanifard and H. Khodaei, Intuitionistic random almost additive-quadratic mappings, Advances in Difference Equations, 2012, 2012:152.

[34] J.H. Park, Intuitionistic fuzzy metric spaces, Chaos, Solitons and Fractals, 22 (2004),1039-1046.

[35] J.M. Rassias, On approximately of approximately linear mappings by linear mappings, J. Funct. Anal. 46, (1982) 126-130.

[36] J.M. Rassias, R. Saadati, G. Sadeghi and J. Vahidi, On nonlinear stability in various random normed spaces, J. of inequalities and applications, 2011, 2011:62.

[37] J. M. Rassias, M. Arunkumar and S. Karthikeyan, Stability of a quadratic functional equation originating from sum of the medians of a triangle in fuzzy ternary Banach algebras: Direct and fixed point methods, Int. J. of Math. and its Appl. 3 (2015), no. 4A, 45-64.

[38] J. M. Rassias, M. Arunkumar and S. Karthikeyan, Ulam-Hyers Stability of Quadratic Reciprocal Functional Equation in Intuitionistic Random Normed spaces: Various Methods, Malaya J. Mat. 5 (2) (2017), 293-304.

[39] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc.Amer.Math. Soc., 72 (1978), 297-300.

[40] K. Ravi, M. Arunkumar and J.M. Rassias, On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, Int. J. of Mathematical Sciences, Autumn 2008 Vol.3, No. 08, 36-47.

[41] K. Ravi, J.M. Rassias, Abasalt Bodaghi and B.V. Senthil Kumar, Intuitionistic Fuzzy Stability of a Reciprocal Quadratic Functional Equation, Int. J. of Applied Science and Mathematics, Volume 1, No. 1, 2014, 9-14.

[42] R. Saadati and J.H. Park, On the intuitionistic fuzzy topological spaces, Chaos, Solitons and Fractals, 27 (2006), 331-344.

[43] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, Elsevier, North Holand, New York, 1983.

[44] A.N. Serstnev, On the notion of a random normed space, Dokl Akad Nauk SSSR, 149 (1963), 280-283.

[45] S.Shakeri, Intuitionstic fuzzy stability of Jensen type mapping, J. Nonlinear Sci. Appli. Vol.2 No. 2 (2009), 105-112.

[46] S.M. Ulam, Problems in Modern Mathematics, Science Editions,Wiley, NewYork, 1964 (Chapter VI, Some Questions in Analysis: 1, Stability).