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STABILITY ANALYSIS OF A DISCRETE-TIME MATHEMATICAL MODEL FOR DEPRESSION IN COLLEGE STUDENTS

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ABSTRACT

A mathematical model of continuous nonlinear system of four differential equations, developed on the basis of Meehl's (1995) taxometric analyses suggest that the youth depression is a dimensional construct and not categorical, is considered [2]. The dimensional construct consists of two levels of vulnerability - high and low, due to 'Biological', 'Social' and 'Psychological' factors. It is assumed that these three factors are the major causes of depression in the college students. In this paper, we apply Micken's discretization method to obtain a discrete-time model described by the difference equations and we observe the positivity of the model together with the existence and stability of the equilibrium points [1].

Key words: Nonlinear differential equations, Discretization, Difference equations, Stability and Equilibrium.

AMS Subject classification: 34A34, 39A12, 39A10 and 39A30.

1 Introduction

We discretize a mathematical model based on the hypothesis that the significant causes of depression in college students between the age group 16 to 24. “Biological factors (i)”, such as genetics, hormones, and changing brain chemistry, “Psychological factors (j)”, such as unique life story and its effect on personal attitudes and thinking habits, and “Social factors (k)”, such as financial position, social status and isolation from support networks; combine to cause different levels of vulnerability to depression for each person [4]. ‘ε’ and ‘μ’ is the per capita rate at which students enter and leave the system, respectively. ‘α’ is the per capita peer pressure rate exerted by the high vulnerability(H) and depressed(D) classes upon the low vulnerability class(L). ‘β’ is the per capita peer pressure rate exerted by L upon H. ‘δ’ is the per capita rate at which students become clinically depressed. ’φ’ is the per capita rate at which depressed students seek treatment. ‘1−φ’ is the per capita rate at which depressed students do not seek treatment. ‘s’ is the probability that treatment is successful. ‘e’ is the probability that the medication has side effects. ‘ψ’ is the per capita rate at which students leave treatment(T) [5,6]. We interpret R0 in terms of the model.

2 Model Equations

Consider the continuous nonlinear system of four differential equations given by

$$\frac{dH}{dt} = i\varepsilon N + j\varepsilon N + k\varepsilon N + \alpha\left(\frac{H + D}{N}\right)L - \beta\left(\frac{L}{N}\right)H - \delta H - \mu H$$

$$\frac{dL}{dt} = (1-i)\varepsilon N + (1-j)\varepsilon N + (1-k)\varepsilon N - \alpha\left(\frac{H + D}{N}\right)L + \beta\left(\frac{L}{N}\right)H - \psi esT - \mu L$$

$$\frac{dD}{dt} = \delta H + \psi e(1-s)T + (1-\varphi)D - \varphi D - \mu D$$

$$\frac{dT}{dt} = \varphi D - (1-\varphi)D - \psi e(1-s)T - \psi esT - \mu T$$

The state vector is defined as $z^T(t)$ = [H L D T]. N(t) denotes the total population of the college students between the age 16-24, at time t. All the above parameters are assumed to be positive. By summing up all the equations in (1), we obtain the total population dynamics,

$$\frac{dN}{dt} = (3\varepsilon - \mu)N$$

It can be seen from (2) that the total population is constant when 3 ε = μ, increases when 3 ε > μ and decreases when 3 ε < μ. This model is discretized by using Micken’s NSFD method, to get

$$\frac{H_{k+1} - H_k}{h} = i\varepsilon N_{k+1} + j\varepsilon N_{k+1} + k\varepsilon N_{k+1} + \alpha\left(\frac{H_{k+1} + D_{k+1}}{N_k}\right)L_k - \beta\left(\frac{L_k}{N_k}\right)H_{k+1} - \delta H_{k+1} - \mu H_{k+1}$$

$$\frac{L_{k+1}-L_k}{h} = (1-i)\varepsilon N_{k+1} + (1-j)\varepsilon N_{k+1} + (1-k)\varepsilon N_{k+1} - \alpha\left(\frac{H_{k+1}+D_{k+1}}{N_k}\right)L_k + \beta\left(\frac{L_k}{N_k}\right)H_{k+1} - \psi esT_{k+1} - \mu L_{k+1}$$

$$\frac{D_{k+1}-D_k}{h} = \delta H_{k+1} + \psi e(1-s)T_{k+1} + (1-\varphi)D_{k+1} - \varphi D_{k+1} - \mu D_{k+1}$$

$$\frac{T_{k+1}-T_k}{h} = \varphi D_{k+1} - (1-\varphi)D_{k+1} - \psi e(1-s)T_{k+1} - \psi esT_{k+1} - \mu T_{k+1}$$

where $h > 0$ denotes the integration step while H_k, L_k, D_k, and T_k denote the value of the highly vulnerable, less vulnerable, depressed and treatment at sampling time $t_k = kh$ which define the discrete-time state vector $z^T{}_k = [\, H_k\ L_k\ D_k\ T_k \,]$. By summing up all the equations in (3), we obtain the dynamics of the total population in discrete-time.

$$\frac{N_{k+1}-N_k}{h} = (3\varepsilon-\mu)N_{k+1}$$

$$N_{k+1} = \frac{1}{1+(\mu-3\varepsilon)h}N_k$$

which is consistent to the NSFD discretization of equation (2) of the continuous time as in[1]. Rearranging (3) in the following implicit matrix form:

$$Q(z_k)z_{k+1} = z_k + \Gamma_1(i+j+k)\varepsilon hN_{k+1} + \Gamma_2(3-(i+j+k))\varepsilon hN_{k+1} = Rz_k,$$
$$= z_k + \Gamma_1(i+j+k)\varepsilon hN_{k+1} + (\mu-3\varepsilon)h + \Gamma_2(3-(i+j+k))\varepsilon hN_{k+1} + (\mu-3\varepsilon)h$$
$$= Rz_k$$

where

$$\Gamma_1^T = [1 \quad 0 \quad 0 \quad 0],\ \Gamma_2^T = [0 \quad 1 \quad 0 \quad 0],$$

$$R = I_{4\times4} + \frac{(i+j+k)\varepsilon h}{1+(\mu-3\varepsilon)h}\begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \frac{(3-(i+j+k))\varepsilon h}{1+(\mu-3\varepsilon)h}\begin{bmatrix}0&0&0&0\\1&1&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$Q(z_k) = \begin{bmatrix}A&0&-B&0\\C&E&B&-F\\-G&0&I&-J\\0&0&K&P\end{bmatrix} \text{ with}$$

$A = 1+(\mu+\delta)h+(\beta-\alpha)h\frac{L_k}{N_k}$, $B = \alpha h\frac{L_k}{N_k}$, $C = (\alpha-\beta)h\frac{L_k}{N_k}$, $E = 1+\mu h$, $F = \psi esh$, $G = \delta h$,

$I = 1+2\varphi h+ \mu h-h$, $J = \psi e(1-s)h$, $K = (1-2\varphi)h$, $P = 1+(\psi e+\mu)h$.

Note: The matrix $Q(z_k)$ depends explicitly on z_k through entries A, B and C, since they depend on N_k and L_k as (8) shows. Equation (6) gives the value of the state vector z_{k+1}, as

$z_{k+1} = Q(z_k)^{-1}Rz_k$. (8)

3 Existence and stability of equilibrium points

The discrete - time equation system(3) and continuous - time equation system(1) posses the same equilibrium points, given by (1) The only equilibrium point is given by the trivial one $H^* = L^* = D^* = T^* = 0$ when $3\varepsilon \neq \mu$

(2) The total population is constant, i.e. $N(t) = N = N(0) = H(0) + L(0) + D(0) + T(0)$ when $3\varepsilon = \mu$ and there are three potential equilibrium points given by [1]: (a) The trivial one $H^* = L^* = D^* = T^* = 0$, (b) $L^* = N$, $H^* = L^* = D^* = T^* = 0$, (c) The professed endemic equilibrium point :

$$H^* = \frac{\alpha D^* L^*}{(\mu+\delta)N+(\beta-\alpha)L^*},\ L^* = \frac{\alpha H^*+\alpha D^*-\psi esNT^*}{\beta H^*-\mu N},\ D^* = \frac{\delta H^*+\psi e(1-s)NT^*}{\mu+2\varphi-1},\ T^* = \frac{(2\varphi-1)D^*}{\psi e+\mu}$$

Stability of the equilibrium points: When $3\varepsilon = \mu$, we get two nontrivial equilibrium points: the disease-free and the endemic equilibrium points. Now, the nonlinear system (6) is linearized around these equilibrium points and the first Lyapunov theorem (linearization theorem) is used. Thus, the linearized equation reads the inverse of the matrix exists and the constraint $1 + (\mu - 3\varepsilon)h > 0$ [1], $\left(\frac{\partial R^{-1}Q(z_k)z_{k+1}}{\partial z_{k+1}}\right)_{z^*} u_{k+1} = u_k$ where z^* denotes the corresponding equilibrium point and $u_k = z_k - z^*$ is the incremental variable. The results are obtained from the implicit equation (6) because of the advantage that the calculus of the Jacobian matrix is easy[1]. First, we write $Q(z_k)\cdot z_{k+1}$ in terms of z_{k+1} only.

$$Q(z_k)\cdot z_{k+1} = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ -G & 0 & I & -J \\ 0 & 0 & K & P \end{bmatrix} \text{ with}$$

$X_{11} = 1 + (\mu + \delta)h + \frac{(\beta-\alpha)h\ (CH_{k+1}+EL_{k+1}+BD_{k+1}-FT_{k+1})}{N}$, $X_{12} = 0$, $X_{13} = \frac{-\alpha h\ (CH_{k+1}+EL_{k+1}+BD_{k+1}-FT_{k+1})}{N}$, $X_{14} = 0$, $X_{21} = \frac{(\alpha-\beta)h\ (CH_{k+1}+EL_{k+1}+BD_{k+1}-FT_{k+1})}{N}$, $X_{22} = E$, $X_{23} = \frac{\alpha h\ (CH_{k+1}+EL_{k+1}+BD_{k+1}-FT_{k+1})}{N}$, $X_{24} = -F$.

The Jacobian is given by,

$$\left(\frac{\partial R^{-1}Q(z_k)z_{k+1}}{\partial z_{k+1}}\right) = R^{-1}\cdot\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ -G & 0 & I & -J \\ 0 & 0 & K & P \end{bmatrix}$$

with $Y_{11} = 1 + (\mu + \delta)h + \frac{(\beta-\alpha)h\ (2CH^*+EL^*+BD^*-FT^*)}{N} - \frac{\alpha h\ (CD^*)}{N}$, $Y_{12} = \frac{(\beta-\alpha)h\ (EH^*)}{N} - \frac{\alpha h\ (ED^*)}{N}$, $Y_{13} = \frac{(\beta-\alpha)h\ (BH^*)}{N} - \frac{\alpha h\ (CH^*+EL^*+2BD^*-FT^*)}{N}$, $Y_{14} = 0$, $Y_{21} = \frac{(\alpha-\beta)h\ (2CH^*+EL^*+BD^*-FT^*)}{N} + \frac{\alpha h\ (CD^*)}{N}$, $Y_{22} = E + \frac{(\beta-\alpha)h\ (EH^*)}{N} - \frac{\alpha h\ (ED^*)}{N}$, $Y_{23} = \frac{(\beta-\alpha)h\ (BH^*)}{N} - \frac{\alpha h\ (CH^*+EL^*+2BD^*-FT^*)}{N}$, $Y_{24} = -F$.

Around the disease free equilibrium point ($L^* = N$, $H^* = L^* = D^* = T^* = 0$),

$$R^{-1}\cdot\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ -G & 0 & I & -J \\ 0 & 0 & K & P \end{bmatrix}$$

with $Z_{11} = 1 + (\mu + \delta)h + (\beta - \alpha)hE$, $Z_{12} = 0$, $Z_{13} = -\alpha hE$, $Z_{14} = 0$, $Z_{21} = (\alpha - \beta)hE$, $Z_{22} = E$, $Z_{23} = -\alpha hE$, $Z_{24} = 0$.

The reduced system is given by,

$$\begin{bmatrix} E & -\alpha hE & 0 \\ 0 & I & -J \\ 0 & K & P \end{bmatrix}\begin{bmatrix} u_{2,k+1} \\ u_{3,k+1} \\ u_{4,k+1} \end{bmatrix} = \begin{bmatrix} u_{2,k} \\ u_{3,k} \\ u_{4,k} \end{bmatrix}$$

The system mentioned above will be asymptotically stable provided that all the eigenvalues of the left-hand side matrix are strictly greater than 1 in absolute value[1]. The eigenvalues of are given by, $\lambda_1 = E = 1+\mu h$, $\lambda_2 = \frac{(P+I)+\sqrt{(P+I)^2-4(IP+JK)}}{2}$, $\lambda_3 = \frac{(P+I)-\sqrt{(P+I)^2-4(IP+JK)}}{2}$.

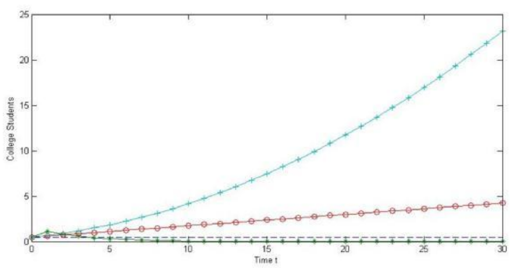
$R_0 = \frac{(P+I)+\sqrt{(P+I)^2-4(IP+JK)}}{2}$

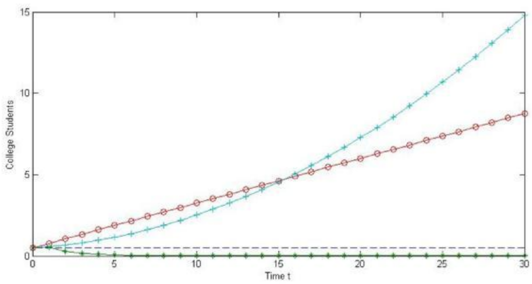
The linearized system is globally stable around the disease-free equilibrium point, $u_{1,k}, u_{2,k}, u_{3,k}, u_{4,k}$, $\rightarrow 0$ and $L_k \rightarrow N$ as $k\rightarrow\infty$,while the non linear system is locally stable around the equilibrium point [1]. The discrete - time model (3) is locally asymptotically stable around the disease - free equilibrium point when $3\varepsilon = \mu$ if $R_0 < 1$ [1]. If $3\varepsilon = \mu$ and $R_0 > 1$ then the solution of the discrete - time model (3) tends to the endemic equilibrium point (or) some variables of the discrete - time model (3) oscillate[1]. The disease-free equilibrium point is unstable when $R_0 > 1$ by which the equilibrium point must be an endemic equilibrium point. In other case, if any of the variables does not converge to a finite limit it must oscillate [1].

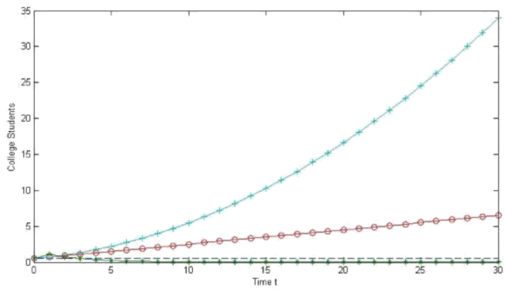
4 Numerical simulations

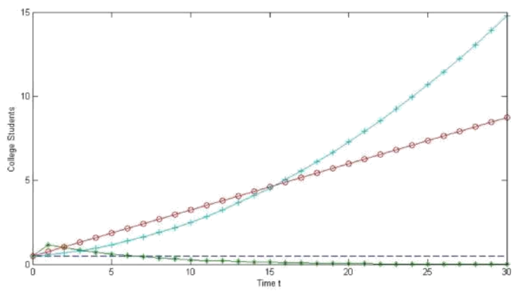
We have provided the numerical simulation using MATLAB. The results are obtained for the hypothetical values of the parameters.

We have provided the numerical simulation using MATLAB. The results are obtained for the hypothetical values of the parameters. In figure 1, i,j,k = 0.33, $\varepsilon,\mu = 0.5$, $\alpha = 0.65$,$\beta = 0.90$,$\delta = 0.40$,$\varphi = 0.655$,$\psi = 0.554$,s = 0.80,e = 0.30; in figure 2, i,j,k = 0.33, $\varepsilon,\mu = 0.5$,$\alpha = 0.30$, $\beta = 0.90$, $\delta,\varphi,\psi = 0.55$,s = 0.60,e = 0.35; in figure 3, i,j,k = 0.33, $\varepsilon,\mu = 0.5$,$\alpha = 0.55$,$\beta = 0.90$,$\delta = 0.25$,$\varphi = 0.655$,$\psi = 0.554$,s = 0.80,e = 0.30; in figure 4, i,j,k = 0.33, $\varepsilon,\mu = 0.5$,$\alpha = 0.90$, $\beta = 0.30$, $\delta,\varphi,\psi = 0.55$,s = 0.60,e = 0.35.









5 Conclusion

By increasing the value of α, we see that the rate of depression is increased. Also, by increasing the value of β, we see that the rate of less vulnerable students is increased. Therefore, by managing the parameters of the peer pressure, i.e. α and β, we can control depression to much extent. Another important parameter which is vital to manage is the treatment parameter φ.

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