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Common fixed point theorems for mappings in fuzzy cone metric spaces

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Abstract

In this paper, an idea of quasicontraction type mapping in fuzzy cone metric space is introduced and some common fixed point theorems of such type of mapping are established.

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1. Introduction

The concept of fuzzy set is introduced by L.A.Zadeh [14] in 1965. After that, to use this concept in topology and analysis different authors have expansively developed the theory of fuzzy sets and its application in different direction.

On the other hand, there have been a number of generalizations of metric spaces (for reference please see [2,4,11,12]) and one such generalization is cone metric space which is introduced by H.Long-Guang et al. [8] which is a generalization of general metric space. In cone metric space, authors replaced the real numbers by ordering real Banach space.

By using their concept, different authors [6,10] established many results of cone metric spaces and fixed point theorems in such spaces.

In an earlier paper [3], the present author introduced an idea of fuzzy cone metric space and established some basic results and fixed point theorems in such spaces.

In this paper, fuzzy real number is considered in the sense of Xiao & Zhu and fuzzy norm in the sense of Felbin [5]. The idea of quasicontraction mapping is introduced in fuzzy cone metric space and established a common fixed point theorem in such space.

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The organization of the paper is as follows: Section 1, comprises some preliminary results which are used in this paper. A common fixed point theorem is established in Section 2.

2. Some preliminary results

Definition 2.1. [13] A mapping $\eta : R \to [0, 1]$ is called a fuzzy real number whose α-level set is denoted by

$$[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}, \ \alpha \in (0, 1],$$

if it satisfies two axioms:

(N1) There exists $t_0 \in R$ such that $\eta(t_0) = 1$.

(N2) For each $\alpha \in (0, 1]$; $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha]$,

where $-\infty < \eta^1_\alpha \leq \eta^2_\alpha < +\infty$. The set of all fuzzy real numbers is denoted by $\mathcal{F}$. Since to each $r \in R$, one can consider $\bar{r} \in \mathcal{F}$ defined by $\bar{r}(t) = 1$ if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$, R can be embedded in $\mathcal{F}$.

Definition 2.2. [13] Let $\eta \in \mathcal{F}$. Then η is called a positive fuzzy real number if $\eta(t) = 0 \ \forall t < 0$. The set of all positive fuzzy real numbers is denoted by $\mathcal{F}^+$.

Note 2.3. Mizumoto and Tanaka [9], Kaleva & Seikkala [7] denote the set of all fuzzy real numbers by E. Kaleva & Seikkala [7] and Felbin [5] denote the set of all non-negative fuzzy real numbers by G and $R^*(I)$ respectively.

A partial ordering "$\preceq$" in E is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0, 1]$ where $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha, b^2_\alpha]$. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0, 1]$.

Propositon 2.4. [9] Let $\eta, \delta \in E(R(I))$ and $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$,

$$[\delta]_\alpha = [a^2_\alpha, b^2_\alpha], \ \alpha \in (0, 1].$$

Then

$$[\eta \bigoplus \delta]_\alpha = [a^1_\alpha + a^2_\alpha, b^1_\alpha + b^2_\alpha]$$

where

$$(x \oplus y)(t) = Sup_{s \in R} min\{x(s), y(t-s)\}, \ t \in R.$$

Definition 2.5. [7] A sequence $\{\eta_n\}$ in E is said to be convergent and converges to η denoted by $\lim_{n\to\infty} \eta_n = \eta$ if $\lim_{n\to\infty} a^n_\alpha = a_\alpha$ and $\lim_{n\to\infty} b^n_\alpha = b_\alpha$ where $[\eta_n]_\alpha = [a^n_\alpha, b^n_\alpha]$ and $[\eta]_\alpha = [a_\alpha, b_\alpha] \ \forall \alpha \in (0, 1]$.

Note 2.6. [7] If $\eta, \delta \in G(R^*(I))$ then $\eta \oplus \delta \in G(R^*(I))$.

Note 2.7. [7] For any scalar t, the fuzzy real number $t\eta$ is defined as $t\eta(s) = 0$ if t=0 otherwise $t\eta(s) = \eta\left(\frac{s}{t}\right)$.

Definition 2.8. [5] (Felbin). Let X be a vector space over R. Let $|| \ || \ : X \to R^*(I)$ and let the mappings

$$L, U \ : [0\,,\,1] \times [0\,,\,1] \to [0\,,\,1]$$

be symmetric, nondecreasing in both arguments and satisfy $L(0, 0) = 0$ and $U(1\,,\,1) = 1$. Write

$$[||x||]_\alpha = [||x||^1_\alpha\,,\, ||x||^2_\alpha]$$

for

$$x \in X,\ 0 < \alpha \leq 1$$

and suppose for all $x \in X,\ x \neq \underline{0}$, there exists $\alpha_0 \in (0\,,\,1]$ independent of x such that for all $\alpha \leq \alpha_0$,

(A) $||x||^2_\alpha < \infty$

(B) $\inf||x||^1_\alpha > 0$.

The quadruple $(X\,,\, ||\ ||,\ L\,,\ U)$ is called a fuzzy normed linear space and $||\ ||$ is a fuzzy norm if

(i) $||x|| = \bar{0}$ if and only if $x = \underline{0}$;

(ii) $||rx|| = |r|||x||,\ x \in X,\ r \in R$;

(iii) for all $x, y \in X$,

 (a) whenever $s \leq ||x||^1_1,\ t \leq ||y||^1_1$ and $s + t \leq ||x + y||^1_1$, $||x + y||(s + t) \geq L(||x||(s)\,,\, ||y||(t))$,

 (b) whenever $s \geq ||x||^1_1,\ t \geq ||y||^1_1$ and $s + t \geq ||x + y||^1_1$, $||x + y||(s + t) \leq U(||x||(s)\,,\, ||y||(t))$

Remark 2.9. [5] Felbin proved that, if $L = \bigwedge$(Min) and $U = \bigvee$(Max) then the triangle inequality (iii) in the Definition 1.1 is equivalent to $||x + y|| \preceq ||x|| \bigoplus ||y||$.

Further $||\ ||^i_\alpha;\ i = 1, 2$ are crisp norms on X for each $\alpha \in (0\,,\,1]$.

Definition 2.10. [3] Let $(E, ||\ ||)$ be a fuzzy real Banach space where $||\ || : E \to R^*(I)$. Denote the range of $||\ ||$ by $E^*(I)$. Thus $E^*(I) \subset R^*(I)$.

Definition 2.11. [3] A member $\eta \in R^*(I)$ is said to be an interior point if $\exists r > 0$ such that

$$S(\eta, r) = \{\delta \in R^*(I)\ : \eta \ominus \delta \prec \bar{r}\} \subset E^*(I).$$

Set of all interior points of $R^*(I)$ is called interior of $R^*(I)$.

Definition 2.12. [3] A subset of F of $E^*(I)$ is said to be fuzzy closed if for any sequence $\{\eta_n\}$ such that $\lim_{n\to\infty} \eta_n = \eta$ implies $\eta \in F$.

Definition 2.13. [3] A subset P of $E^*(I)$ is called a fuzzy cone if

(i) P is fuzzy closed, nonempty and $P \neq \{\bar{0}\}$;

(ii) $a, b \in R,\ a, b \geq 0,\ \eta, \delta \in P \Rightarrow a\eta \oplus b\delta \in P$;

Note 2.14. If $\eta \in P$ then $\ominus\eta \in P \ \Rightarrow \eta = \bar{0}$. For, suppose $[\eta]_\alpha = [\eta^1_\alpha,\ \eta^2_\alpha]$, $\alpha \in (0, 1]$. Since $\eta \in P \subset E^*(I)$, we have $\eta^1_\alpha,\ \eta^2_\alpha \geq 0\,\forall\alpha \in (0, 1]$. Now $[\ominus\eta]_\alpha = [-\eta^2_\alpha,\ -\eta^1_\alpha]$, $\alpha \in (0, 1]$. If $\eta \neq \bar{0}$, then $\eta^1_\alpha,\ \eta^2_\alpha > 0\ \forall\alpha \in (0, 1]$. i.e. $-\eta^2_\alpha \ \leq\ -\eta^1_\alpha < 0\ \forall\alpha \in (0, 1]$. This implies that $\ominus\eta$ does not belong to P. Hence $\eta = \bar{0}$.

Given a fuzzy cone $P \subset E^*(I)$, define a partial ordering $\leq$ with respect to P by $\eta \leq \delta$ iff $\delta \ominus \eta \in P$ and $\eta < \delta$ indicates that $\eta \leq \delta$ but $\eta \neq \delta$ while $\eta << \delta$ will stand for $\delta \ominus \eta$ $\in$IntP where IntP denotes the interior of P.

The fuzzy cone P is called normal if there is a number $K > 0$ such that for all $\eta, \delta \in E^*(I)$, with $\bar{0} \leq \eta \leq \delta$ implies $\eta \preceq K\delta$. The least positive number satisfying above is called the normal constant of P. The fuzzy cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{\eta_n\}$ is a sequence such that $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_n \leq \cdots \leq \eta$ for some $\eta \in E^*(I)$, then there is $\delta \in E^*(I)$ such that $\eta_n \to \delta$ as $n \to \infty$.

Equivalently, the fuzzy cone P is regular if every decreasing sequence which is bounded below is convergent. It is clear that a regular fuzzy cone is a normal fuzzy cone.

In the following we always assume that E is a fuzzy real Banach space, P is a fuzzy cone in E with IntP $\neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.15. [3] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E^*(I)$ satisfies

(Fd1) $\bar{0} \leq d(x, y)\,\forall x, y \in X$ and $d(x, y) = \bar{0}$ iff $x = y$;

(Fd2) $d(x, y) = d(y, x)\ \ \forall x, y \in X$;

(Fd3) $d(x, y) \leq d(x, z) \oplus d(z, y)\ \ \forall x, y, z \in X$.

Then d is called a fuzzy cone metric and (X, d) is called a fuzzy cone metric space.

Definition 2.16. [3] Let (X, d) be a fuzzy cone metric space. Let$\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\bar{0} << ||c||$ there is a positive integer N such that for all $n > N$, $d(x_n, x) << ||c||$, then $\{x_n\}$ is said to be convergent and converges to x and x is called the limit of $\{x_n\}$. We denote it by $\lim_{n\to\infty} x_n = x$.

Proposition 2.17. [13] Let $\{[a_\alpha\ ,\ b_\alpha],\ \alpha \in (0,1]\}$ be a family of nested bounded closed intervals. Let $\eta : R \to [0,1]$ be a function defined by

$$\eta(t) = \bigvee \{\alpha \in (0,1] :\ t \in [a_\alpha\ ,\ b_\alpha]\}.$$

Then η is a fuzzy real number.

3. Main Results

Definition 3.1. For $F \subset E$ we define fuzzy diameter of F is the fuzzy real number $\delta(F)$ as

$$\delta(F)(t) = \bigvee \{\alpha \in (0,1) :\ t \in [\bigvee_{x \in F} ||x||_\alpha^1\ ,\ \bigvee_{x \in F} ||x||_\alpha^2]\}.$$

Definition 3.2. Let $(X\ ,\ d)$ be a fuzzy cone metric space and let $g, f\ :\ X \to X$. Then g is called fuzzy f-quasicontraction, if for some constant $\lambda \in (0,1)$ and for every $x, y \in X$ there exists

$$u \in C(f;\ x, y) \equiv \{d(fx, fy),\ d(fx, gx),\ d(fx, gy),\ d(fy, gy),\ d(fy, gx)\}$$

such that $d(gx, gy) \leq \lambda u$ (2.2.1).

Lemma 3.3. Let $(X\ ,\ d)$ be a fuzzy cone metric space and P be a normal fuzzy cone with normal constant K. Let $g, f\ :\ X \to X$, g commutes with f and $g(X) \subset f(X)$ and suppose g is fuzzy f-quasicontraction.

Let $x_0 \in X$ and $x_1 \in X$ be such that $g(x_0) = f(x_1)$. Having defined $x_1 \in X$, let $x_{n+1} \in X$ be such that $g(x_n) = f(x_{n+1}) = y_n$.

Now for $n \in N$ we set,

$$O(x_0, n) = \{y_0,\ y_1,\ y_2, \ldots, y_n\}$$

and

$$O(x_0\ ,\ \infty) = \{y_0,\ y_1,\ y_2, \ldots\}.$$

Then there exists $n_0 \in N$ such that the following hold:

(i) If $i, j, n \in N,\ n > n_0$ and $n_0 < i,\ j \geq n$ then $d(y_i\ ,\ y_j) \prec \delta(O(x_0; n))$.

(ii) If $n \in N$ and $n > n_0$ then

$$\delta_\alpha^r(O(x_0; n)) = max\{d_\alpha^r(y_0, y_k),\ d_\alpha^r(y_i, y_j) :\ 1 \leq k \leq n,\ 1 \leq i,\ j \leq r_0\},\ \alpha \in (0,1)$$

and $r = 1, 2$.

(iii) If $n \in N$ and $n > n_0$ then

$$\delta_\alpha^r(O(x_0; n)) \leq \left\{ \frac{k}{1 - k^2\lambda^{n_0}} d_\alpha^r(y_0, y_{n_0+1}),\ \lambda k \delta_\alpha^r(O(x_0; n_0)), d_\alpha^r(y_0, y_l) :\ 1 \leq l \leq n_0 \right\},$$

$\alpha \in (0,1)$ and $r = 1, 2$.

(iv) If $n \in N$ and $n > n_0$ then

$$\delta_\alpha^r(O(x_0;\infty)) \leq \left\{ \frac{k}{1-k^2\lambda^{n_0}} d_\alpha^r(y_0, y_{n_0+1}),\ \lambda k \delta_\alpha^r(O(x_0;n_0)),\ d_\alpha^r(y_0, y_l) :\ 1 \leq l \leq n_0 \right\},$$

$\alpha \in (0,1)$ and $r = 1,2$.

(v) For $n \in N$ and $n \geq n_0 + 1$ we have

$$d(y_n\ ,\ y_{n-1}) \preceq K\lambda^{n_0}\delta(O(x_0;\infty)).$$

(vi) Sequence $\{y_n\}$ is Cauchy and for $m > n > n_0 + 1$ we have

$$d(y_n\ ,\ y_m) \preceq K\frac{\lambda^n}{1-\lambda}\delta(O(x_0;\infty)).$$

Proof. Suppose that $n_0 \in N$ is such that $max\{\lambda^{n_0}K,\ \lambda^{n_0}K^2\}\ < 1$.

(i) Suppose that $i, j, n \in N,\ n > n_0$ and $n_0 < i,\ j \leq n$. There exists

$$\eta_1 \in \{d(gx_{i-1}, gx_{j-1}),\ d(gx_{i-1}, gx_i),\ d(gx_{i-1}, gx_j),\ d(gx_{j-1}, gx_j),\\ d(gx_{j-1}, gx_i)\} \subset O(x_0;n)$$

such that $d(y_i, y_j) = d(gx_i, gx_j) \leq \lambda\eta_1$. Now, there exists $\eta_2 \in O(x_0;n)$ such that $\eta_1 \leq \lambda\eta_2$. Hence $d(y_i, y_j) \leq \lambda^2\eta_2$. Thus after n_0 steps we get,

$$d(y_i, y_j) \leq \lambda^{n_0}\eta_{n_0}$$

for some $\eta_{n_0} \in O(x_0;n)$. Since P is normal, thus

$$d(y_i, y_j) \preceq K\lambda^{n_0}\eta_{n_0} \prec \delta((x_0;n)) \tag{2.3.1}$$

(ii) From (i) we get,

$$d_\alpha^r(y_i, y_j) <\ \delta_\alpha^r(O(x_0;n))$$

for $r = 1,2$ and $\alpha \in (0,1)$. Thus it follows that, for $n \in N$ and $n > n_0$;

$\delta_\alpha^r(O(x_0;n)) = max\{d_\alpha^r(y_0, y_k), d_\alpha^r(y_i, y_j) :\ 1 \leq k \leq n, 1 \leq i,\ j \leq n_0\}, \alpha \in (0,1)$

and $r = 1,2$.

(iii) There are three cases may arise.

Case I. If

$$\delta_\alpha^1(O(x_0;n)) = d_\alpha^1(y_0, y_{k(\alpha)})$$

for some $k(\alpha) \in N$ such that $1 \leq k(\alpha) \leq n_0$ then

$$\delta^1_\alpha(O(x_0; n)) \leq \ max\{d^1_\alpha(y_0, y_l) : \ 1 \leq l \leq n_0\}.$$

Similarly $\delta^2_\alpha(O(x_0; n)) \leq \ max\{d^2_\alpha(y_0, y_l) : \ 1 \leq l \leq n_0\}$.

Case II. If

$$\delta^1_\alpha(O(x_0; n)) = d^1_\alpha(y_0, y_{k(\alpha)})$$

for some $k(\alpha) \in N$ such that $n_0 < \ k(\alpha) \leq n$ then

$$d(y_0, y_k) \leq d(y_0, y_{n_0+1}) \oplus d(y_{n_0+1}, y_k).$$

Since P is normal we have,

$$d(y_0, y_k) \preceq K\{d(y_0, y_{n_0+1}) \oplus d(y_{n_0+1}, y_k)\}.$$

i.e.

$$\delta^r_\alpha(y_0, y_k) \leq K\{d^r_\alpha(y_0, y_{n_0+1}) + d^r_\alpha(y_{n_0+1}, y_k)\} \ \ \alpha \in (0, 1), \ r = 1, 2.$$

By using (ii), it follows that,

$$\delta^r_\alpha(O(x_0; n)) \leq K d^r_\alpha(y_0, y_{n_0+1}) + \lambda^{n_0} K^2 \delta^r_\alpha(O(x_0; n))$$

i.e.

$$\delta^r_\alpha(O(x_0; n)) =\leq \frac{K}{1 - \lambda^{n_0} K^2} d^r_\alpha(y_0, y_{n_0+1})$$

for $r = 1, 2$.

Case III. If

$$\delta^r_\alpha(O(x_0; n)) = d^r_\alpha(y_i, y_j)$$

for some $i, j \in N$ with $1 \leq i, \ j \leq n_0$ then

$$d^r_\alpha(y_i, y_j) \leq \lambda \eta^1_\alpha$$

for some $\eta \in \{d(a, b) : \ a, b \in O(x_0, n_0)\}$. Hence in this case

$$\delta^r_\alpha(O(x_0; n)) \leq \lambda K \delta^r_\alpha(O(x_0; n_0)), \ r = 1, 2, \ \alpha \in (0, 1).$$

Thus (iii) holds.

(iv) From (iii), (iv) follows.

(v) For $n \geq n_0 + 1$ we get,

$$d(y_n, y_{n-1}) \leq \lambda \eta_{n,n-1}$$

for some

$$\eta_{n,n-1} \in \{d(y_{n+1}, y_n),\ d(y_{n+1}, y_{n-1}),\ d(y_n, y_{n-1}),\ \bar{0}\}.$$

Again

$$d(y_{n+1}, y_n) \leq \lambda \eta_{n,n+1}$$

for some

$$\eta_{n,n+1} \in \{d(y_{n+2}, y_{n+1}),\ d(y_{n+2}, y_n),\ d(y_{n+1}, y_n),\ \bar{0}\}.$$

Further, $d(y_{n+1}, y_{n-1}) \leq \lambda \eta_{n-1,n+1}$ where

$\eta_{n-1,n+1} \in \{d(y_{n+2}, y_n), d(y_{n+2}, y_{n+1}), d(y_{n+2}, y_{n-1}), d(y_n, y_{n-1}), d(y_n, y_{n-1})\}$.

So,

$$d(y_n, y_{n-1}) \leq \lambda^2 \eta^{(2)}_{n,n-1}$$

where

$$\eta^{(2)}_{n,n-1} \in \{d(y_n, y_{n-1}),\ d(y_{n+1}, y_n),\ d(y_{n+1}, y_{n-1}),\ d(y_{n+2}, y_{n+1}),\ d(y_{n+2}, y_n),\ d(y_{n+2}, y_{n-1}),\ \bar{0}\}.$$

We continue in similar way and after n_0 steps we have,

$$d(y_n, y_{n-1}) \leq \lambda^{n_0} \eta^{n_0}_{n,n-1} \tag{2.3.2}$$

where

$$\eta^{(n_0)}_{n,n-1} \in \bigcup_{j=0}^{n_0} \bigcup_{i=0}^{j} \{d(y_{n+j}, d_{n-1+i})\}.$$

Hence

$$d(y_n, y_{n-1}) \preceq K \lambda^{n_0} \delta(O(x_0; \infty)).$$

(v) By (2.3.2), for $m > n > n_{0+1}$ we have,

$$d(y_n, y_m) \leq d(y_n, y_{n+1}) \oplus d(y_{n+1}, y_{n+2}) \oplus \cdots \oplus d(y_{m-1}, y_m) \leq \sum_{k=0}^{m-n-1} \lambda^{n+k} \eta^{(n+k)}_{n+k+1,n+k}.$$

Since P is normal, we have,

$$d(y_n, y_m) \preceq K \lambda^n \sum_{k=0}^{m-n-1} \lambda^k \eta^{(n+k)}_{n+k+1,n+k} \preceq K \frac{\lambda^n}{1-\lambda} \delta(O(x_0; \infty)).$$

This implies that $d(y_n, y_m) \to \bar{0}$ as $m, n \to \infty$. So $\{y_n\}$ is a Cauchy sequence.

Theorem 3.4. Let $(X\ ,\ d)$ be a complete fuzzy cone metric space and P be a normal fuzzy cone. Let $f\ :\ X \to X$, f^2 be continuous, $g\ :\ f(X) \to X$ be such that $g(f(X)) \subset f^2(X)$ and $f(g(x)) = g(f(x))$ whenever both sides are defined. Further suppose that there exists $\lambda \in (0,1)$ such that (2.2.1) holds for every $x, y \in f(X)$. Then f and g have a common fixed point.

Proof. Suppose $x_0 \in f(X)$. We define a sequence $\{x_n\}$ in $f(X)$ such that

$$f(x_{n+1}) = g(x_n) = y_n.$$

Let

$$fy_n = fgx_n = gfx_n = gy_{n-1} = z_n.$$

By Lemma 2.3, for $n < m$ we have

$$d(z_n\ ,\ z_m) \preceq K\frac{\lambda^n}{1-\lambda}\delta(O(x_0;\infty)).$$

This implies that $\{z_n\}$ is a Cauchy sequence in X and since X is complete $\exists z \in X$ such that $z_n \to z$ as $n \to \infty$. By using (2.2.1), it follows that

$$d(f^2z_n, gfz_n) = d(gfz_{n-1}\ ,\ gfz_n) \preceq \lambda\eta_1$$

for some

$$\eta_1 \in \{d(gfz_{n-2}\ ,\ gfz_{n-1}),\ d(gfz_{n-1}\ ,\ gfz_n),\ d(gfz_{n-2}\ ,\ gfz_n),\ \bar{0}\}.$$

As in the proof of Lemma 2.3, for each $n \in N$,

$$d(f^2z_n\ ,\ gfz_n) = d(gfz_{n-1}\ ,\ gfz_n) \preceq \lambda^{n-2}\eta_{n-2}$$

for some

$$\eta_{n-2} \in \{d(gfz_i\ ,\ gfz_j):\ 0 \le i \le n-1,\ 1 \le j \le n\}.$$

Hence $d(f^2z_n\ ,\ gfz_n) \to \bar{0}$ as $n \to \infty$ and thus $f^2z\ =\ gfz$. Now from (2.2.1) we get,

$$d(g(gfz)\ ,\ gfz) \preceq \lambda u$$

where $x = gfz$ and $y = fz$. If $u = d(fx\ ,\ fy)$ then we have,

$$\begin{aligned}
d(g(gfz)\ ,\ gfz) &\preceq \lambda d(f(gfz)\ ,\ f(fz)) \\
&= \lambda d(g(f^2z)\ ,\ f^2z) \text{ (since } g(f(x)) = f(g(x))) \\
&= \lambda d(g(gfz)\ ,\ gfz) \\
&\Rightarrow (\lambda-1)d(g(gfz)\ ,\ gfz) \in P \\
&\Rightarrow d(g(gfz)\ ,\ gfz) = \bar{0} \text{ (since } \lambda - 1 < 0) \\
&\Rightarrow g(gfz)\ =\ gfz.
\end{aligned}$$

Thus gfz is a fixed point of g. Also, $f(gfz) = g(f^2z) = g(gfz) = gfz$. Hence gfz is a common fixed point of f and g. ■

The following example justifies the Theorem 2.4.

Example 3.5. Let us consider the Banach space $(E\ ,\ ||\ ||')$ where $E = R$ and $||x||' = |x|\ \forall x \in E$. Define $||\ ||:\ E \to E^*(I)$ by

$$||x||(t) = \begin{cases} 1 & \text{if } t > ||x||' \\ 0 & \text{if } t \leq ||x||' \end{cases}$$

Then $[||x||]_\alpha = [||x||'\ ,\ ||x||']\ \forall \alpha \in (0, 1]$. It can be verified that, (i) $||x|| = \bar{0}$ iff $x = \underline{0}$ (ii) $||rx|| = |r|||x||$ (iii) $||x + y|| \preceq ||x|| \oplus ||y||$. Thus $(E, ||\ ||)$ is a complete fuzzy normed linear space. Define $P = \{\eta \in E^*(I)\ :\ \eta \succeq \bar{0}\}$. Then P is a fuzzy cone as well as normal fuzzy cone with normal constant 1. Now choose the ordering $\leq$ as $\preceq$ and define $d : E \times E \to E^*(I)$ by $d(x, y) = ||x - y||\ \forall x, y \in E$. Then it is easy to verify that d satisfies the conditions (Fd1) to (Fd3). Hence $(E\ ,\ d)$ is a fuzzy cone metric space. Define $f, g :\ E \to E$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in (-\infty, -1] \\ x & \text{if } x \in (-1, 1) \\ -1 & \text{if } x \in [1, \infty) \end{cases}$$

and

$$g(x) = \begin{cases} \dfrac{1+x}{2} & \text{if } x \in (-1,\ 0] \\ \dfrac{1-x}{2} & \text{if } x \in (0, 1) \\ 0 & \text{if } x \in (-\infty, -1] \cup [1, \infty) \end{cases}$$

Then

$$f^2(x) = \begin{cases} 1 & \text{if } x \in [1, \infty) \\ x & \text{if } x \in (-1, 1) \\ -1 & \text{if } x \in (-\infty, -1] \end{cases}$$

Thus f^2 is continuous. Also $f(g(x)) = g(f(x))\ \forall x \in E$ and $gf(X) \subset f^2(X)$. Now we show that g is f-quasicontraction. For, take $x, y \in (0, 1)$ then

$$d(fx, fy) = d(x, y) = |x - y|,\ d(fx, gx) = d\left(x, \frac{1-x}{2}\right) = \left|\frac{3x-1}{2}\right|,$$

$$d(fx, gy) = d\left(x, \frac{1-y}{2}\right) = \left|\frac{2x+y-1}{2}\right|,\ d(fy, gy) = d\left(y, \frac{1-y}{2}\right) = \left|\frac{3y-1}{2}\right|,$$

$$d(fy, gx) = d\left(y, \frac{1-x}{2}\right) = \left|\frac{2y+x-1}{2}\right|,\ d(gx, gy) = d\left(\frac{1-x}{2}, \frac{1-y}{2}\right) = \left|\frac{x-y}{2}\right|.$$

So

$$d(gx, gy) = \left|\frac{x-y}{2}\right| \leq 1.|x-y| = 1.d(fx, fy).$$

Thus all the hypothesis of Theorem 2.4 hold. Now

$$\begin{aligned} f(x) = g(x) &= x \\ \Rightarrow \frac{1-x}{2} &= x \\ \Rightarrow x &= \frac{1}{3}. \end{aligned}$$

Thus $\frac{1}{3}$ is the common fixed point of f and g.

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