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Family of Biparameterized Non Logarithmic Fuzzy Uncertainty Measure under Limiting Condition

[1]Rohit Kumar Verma* and [2]Som Kumari

[1]Associate Professor, Department of Mathematics, Bharti Vishwavidyalaya, Durg, C.G., India
[2]Research Scholar Ph.D. Department of Mathematics, Bharti Vishwavidyalaya, Durg, C.G., India

Abstract

The current fuzzy entropy measures are studied in this communication. In order to check their validity, we presented several new generalised exponential fuzzy entropies including one or more real parameters. Our analysis of the proposed measure's four key characteristics as well as a few other ones shows its validity as an entropy.

Keywords: Information Theory.

Introduction

In order to have a global measure of indefiniteness related to the scenario that fuzzy sets show, one can measure the fuzziness of a fuzzy set using its entropy. The sharpness of membership functions is characterised by this kind of measure, which is independent of random experiments. Since it quantifies the uncertainty over the existence or lack of a property over the studied set, it can also be thought of as entropy.

In 1965, Zadeh [13] introduced the concept of fuzzy sets in an effort to address situations where indefiniteness resulting from a type of fundamental uncertainty is a key factor. A characteristic of uncertainty called fuzzyness arises when a set's boundary without a clear cut definition; in other words, a person is neither certainly inside the set nor certainly outside of it. In 1968, Zadeh [14] made the initial attempt to quantify fuzziness. He introduced the weighted Shannon entropy [11], which is an entropy that combines the probability and membership function of a fuzzy event, based on a probabilistic framework. De Luca and Termini [2] created a type of fuzzy set entropy based on Shannon's function and developed axioms that the fuzzy entropy measure must conform to. According to Yager [12], a fuzzy set's entropy measure is the absence of a difference between the fuzzy set and its norm-based negation. Exponential fuzzy entropy is a type of entropy that was developed by Pal and Pal [8, 9] as a way to quantify fuzziness. Fuzzy entropy was described by Hwang and Yang [4] by integrating the ideas of Yager [12] and Pal and Pal [8, 9]. The entropy of De

Luca and Termini [2] has been examined in some parametric generalisations by Kapur [7], Hooda [5], Bhandari and Pal [1], Fan and Ma [3].These parameters provide for some application flexibility, but in the end, the data itself must be used to establish their values. This work proposes a new parametric generalised exponential entropy. The structure of this document is as follows: The axiomatic conditions are verified by a novel fuzzy entropy measure called exponential fuzzy entropy, which is proposed in Section 1. A few characteristics of the suggested measure are examined in Section 2, while section 3 also discusses limiting situations.

Preliminaries

We introduce several fundamental ideas in fuzzy sets and probability theory in this work, which are necessary for the analysis that follows. Let's discuss the probabilistic portion of the prelims.

Let $P = \{p_1, p_2, \ldots\ldots, p_n : p_1 \geq 0, \sum_{i=1}^{n} p_i = 1\}, n \geq 2$ be a collection of probability distributions that are n-complete.Regarding any probability distribution Shannon's entropy[11], or $P = \{p_1, p_2, \ldots\ldots, p_n\}$ is given by

$$H(P) = -\sum_{i=1}^{n} p(x_i) log p(x_i) \quad \textbf{(1)}$$

The Shannon entropy has been the foundation for several generalised entropies that have been published in the literature and found applications in a variety of fields, including economics, statistics, information processing, and computers.

More general (α, β)-entropy was provided by Rathie and Taneja [10], Similarly, Fan and Ma defined (α, β)-fuzzy entropy for a fuzzy set as

$$H_{\alpha\beta}(P) = \frac{1}{(1-\alpha)\beta} \sum_{i=1}^{n} [(P(x_i)^{\alpha} - 1]^{\beta}, \beta \neq 0, \alpha \neq 1, \alpha > 0 \quad \textbf{(2)}$$

After analysing the traditional Shannon information entropy, Pal and Pal [8, 9] introduced an information entropy known as the exponential entropy, which is given by

$$E(P) = \sum_{i=1}^{n} p(x_i)(e^{(1-p(x_i))} - 1) \quad \textbf{(3)}$$

The advantage of exponential entropy over Shannon's entropy is noted by these authors. The exponential entropy has a defined upper bound

$$\lim_{n\to\infty} E(\frac{1}{n}, \frac{1}{n}, \ldots\ldots, \frac{1}{n}) = (e - 1) \quad \textbf{(4)}$$

for the uniform probability distribution $P = (\frac{1}{n}, \frac{1}{n}, \ldots\ldots, \frac{1}{n})$.

which is not the case for Shannon's entropy.

Definition 1: Assume that $X = (x_1, \dots\dots, x_n)$ is a discrete discourse world. A membership function $\mu_A(x): X \to [0,1]$ characterises a fuzzy set A on X.The degree of x's membership in A is indicated by the value $\mu_A(x)$ of A at $x \in X$.

Definition 2: A fuzzy set A^* is called a sharpened version of fuzzy set A if the following conditions are satisfied:

$$\mu_{A^*}(x) \le \mu_A(x). \quad if\ \mu_A(x) \le 0.5;\ \forall i$$

and

$$\mu_{A^*}(x) \ge \mu_A(x). \quad if\ \mu_A(x) \ge 0.5;\ \forall i$$

Definition 3*:* Assume , $B \in FS(X)$, where $FS(X)$ is the family of all $FS's$ in universe X.

$$A = \{(x, \mu_A(x)) : x \in X\};$$
$$B = \{(x, \mu_B(x)) : x \in X\};$$

then some set operations can be defined as follows:

- $A^C = \{(x, 1 - \mu_A(x), \mu_A(x)) : x \in X\},$
- $A \cup B = \{(x, max(\mu_A(x), \mu_B(x))) : x \in X\},$
- $A \cap B = \{(x, min(\mu_A(x), \mu_B(x))) : x \in X\},$

It appears that Zadeh [14] made the first attempt to quantify the uncertainty related to a fuzzy event in the context of a discrete probability framework by defining the fuzzy set A (weighted) entropy with regard to (X, P) as

$$H(A, P) = -\sum_{i=1}^{n} \mu_A(x_i) p(x_i) \log p(x_i) \quad \textbf{(5)}$$

Four axioms were proposed by De Luca and Termini [2], and it is generally acknowledged that these axioms serve as a standard for characterising any fuzzy entropy. The fuzzy entropy, which indicates the average level of ambiguity or difficulty in determining whether an element belongs to a set or not, is a measure of fuzziness in fuzzy set theory. At minimum, the following axioms should be present in a measure of fuzziness in a fuzzy set:

P1 (Sharpness): $H(A)$ is minimum if and only if A is a crisp set, i.e. $\mu_A(x_i) = 0\ or\ 1\ \forall i$.

P2 (Maximality): $H(A)$ is maximum if and only if A is a most fuzzy set, i.e. $\mu_A(x_i) = .5\ \forall i$.

P3 (Resolution): Fuzzy entropy decreases if set is sharpened. $H(A^*) \le H(A)$, where A^* is a sharpened version of A.

P4 (Symmetry): Fuzzy entropy of a set is same as its complement i.e, $H(A) = H(A^C)$, where A^C is the complement set of A.

Given that the degrees of fuzzyness obtained from $\mu_A(x_i)$ and $(1 - \mu_A(x_i))$ are equal, De Luca and Termini [2] defined fuzzy entropy for a fuzzy set A that corresponds to (1) as

$$H(A) = -\frac{1}{n}\sum_{i=1}^{n}[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))] \quad \textbf{(6)}$$

subsequently, Bhandari and Pal [1] produced a survey on fuzzy set information measures and provided several additional fuzzy entropy measures. In accordance with (2), they have recommended the following action:

$$H_{\alpha\beta}(A) = \frac{1}{(1-\alpha)\beta}\sum_{i=1}^{n}[(P(x_i))^{\alpha} - 1]^{\beta}\ , \beta \neq 0, \alpha \neq 1\ and\ \alpha > 0. \quad \textbf{(7)}$$

Exponential fuzzy entropy for a fuzzy set corresponding to (3) was defined by Pal and Pal [8, 9] as

$$E(A) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}\left[\mu_A(x_i)e^{(1-\mu_A(x_i))} + \left(1-\mu_A(x_i)\right)e^{(1-\left(1-\mu_A(x_i)\right)} - 1\right] \quad \textbf{(8)}$$

In this work, *FS(X)* stands for the set of all fuzzy sets on *X*.
The axiomatic conditions are verified and a generalised fuzzy entropy measure, known as exponential fuzzy entropy, corresponding to equation (4) is proposed in the section 1.

2. Our Work

Section 1 Exponential Fuzzy Entropy

The formal definition that follows is what we do next:

Definition 4: Let the fuzzy set *A* be the collection a fuzzy set with membership values $\mu_A(x_i)$, where *i = 1, 2,..., n*, is defined on the discrete universe of discourse $X = (x_1, \ldots, x_n)$. The exponential fuzzy entropy that corresponds to (7) is defined as

$$E_{\alpha\beta}(A) = \frac{1}{(1-\alpha)\beta}\sum_{i=1}^{n}\left[exp\{\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}\}^{\beta} - 1\right] - 1\ , \alpha > 0, \beta \neq 0. \quad \textbf{(9)}$$

We prove the measure (9) satisfies measure of fuzzy entropy.
From the definition, symmetry is evident.We demonstrate that (9) satisfies the properties (1) through (4).
P1(Sharpness): let $E_{\alpha\beta}(A) = 0$, then

$$\sum_{i=1}^{n}\left[exp\{\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}\}^{\beta} - 1\right] - 1 = 0 \quad \textbf{(10)}$$

Now $\alpha > 0, for\ i = 1,2, \ldots, n$. (11) will now hold whether $\mu_A(x_i) = 0\ or\ \mu_A(x_i) = 1$.
Next conversely, in the event where A is a crisp set, either $\mu_A(x_i) = 0\ or\ \mu_A(x_i) = 1\ for\ all\ i = 1,2, \ldots, n$.It provides

$$\sum_{i=1}^{n}\left[exp\{\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}\}^{\beta} - 1\right] - 1 = 0 \quad \textbf{(11)}$$

i.e,

$$E_{\alpha\beta}(A) = 0$$

Hence $E_{\alpha\beta}(A)$ if and only if A a crisp set.

P2 (Maximality): Now we show that fuzzy entropy is maximum at $\mu_A(x_i) = 0.5$

Let $E_{\alpha\beta}(A) = \sum_{i=1}^{n} f(\mu_A(x_i))$ **(12)**

Where

$$f\big(\mu_A(x_i)\big) = \left[exp\{\mu_A^{\alpha}(x_i) + \big(1-\mu_A(x_i)\big)^{\alpha}\}^{\beta} - 1\right] - 1$$

assume that $\mu_A(x_i) = x$ then above equation is

$f(x) = \sum_{i=1}^{n} exp\big\{[x^{\alpha} + (1-x)^{\alpha}]^{\beta} - 1\big\} - 1$ **(13)**

Now differentiating (13) with respect to *x*, we get

$$f'(x) = \frac{\alpha}{1-\alpha}(x^{\alpha-1} - (1-x)^{\alpha-1})\{x^{\alpha} + (1-x)^{\alpha}\}^{\beta-1} exp[\{x^{\alpha} + (1-x)^{\alpha}\}^{\beta} - 1]$$ **(14)**

Let $0 \le x < .5$ then

$f'(x) > 0, \quad 0 < \alpha < 1 \; as\ also\ for\ \alpha > 1$ **(15)**

Similarly, for $0.5 < x \le 1$, we have

$f'(x) < 0, \quad 0 < \alpha < 1 \; as\ also\ for\ \alpha > 1$ **(16)**

and for $x = 0.5$, then

.0Thus, the function $f(\mu_A(x_i))$ is concave and has a global maximum at $\mu_A(x_i) = 0.5$. Therefore, A must be the most fuzzy set for $E_{\alpha\beta}(A)$ to be maximum, meaning that $\mu_A(x_i) = 0.5. for\ all\ i = 1,2, \ldots, n$.

P3 (Resolution): Given that $H_{\alpha\beta}(A)$ is a decreasing function of $\mu_A(x_i)$ in the range (0.5,1) and an increasing function of $\mu_A(x_i)$ in the range [0,0.5],

Therefore

$$\mu_{A^*}(x_i) \le \mu_A(x_i) \Rightarrow E_{\alpha\beta}(A^*) \le E_{\alpha\beta}(A) \quad in\ [0,0.5),$$

and

$$\mu_{A^*}(x_i) \ge \mu_A(x_i) \Rightarrow E_{\alpha\beta}(A^*) \ge E_{\alpha\beta}(A) \quad in\ (0.5,1],$$

Hence

$$E_{\alpha\beta}(A^*) \le E_{\alpha\beta}(A).$$

P4 (Symmetry): It is obvious from the definition,

$$E_{\alpha\beta}(A) = E_{\alpha\beta}(A^C).$$

This prove the measure (9) satisfies measure of fuzzy entropy.
In the next section,we examine a few characteristics of $E_{\alpha\beta}(A)$, the exponential fuzzy entropy.

Section 2: Properties of Exponential Fuzzy Entropy
The following properties of the exponential fuzzy entropy measure are:
Theorem 1: $E_{\alpha\beta}(A \cup B) + E_{\alpha\beta}(A \cap B) = E_{\alpha\beta}(A) + E_{\alpha\beta}(B) \quad For\ A, B \in FS(X)$
Proof: Let

$$X_+ = \{x: x \in X, \mu_A(x_i) \geq \mu_B(x_i)\} \quad \textbf{(18)}$$
$$X_- = \{x: x \in X, \mu_A(x_i) < \mu_B(x_i)\} \quad \textbf{(19)}$$

where the fuzzy membership functions of A and B are denoted by $\mu_A(x)$ and $\mu_B(x)$, respectively.

$$E_{\alpha\beta}(A \cup B) = \frac{1}{(1-\alpha)\beta}\sum_{i=1}^{n}\left[exp\{\mu_{A\cup B}^{\alpha}(x_i) + (1-\mu_{A\cup B}(x_i))^{\alpha}\}^{\beta} - 1\right] - 1 \quad \textbf{(20)}$$

$$E_{\alpha\beta}(A \cup B) = \frac{1}{(1-\alpha)\beta}\sum_{i=1}^{n}(\left[exp\left\{\mu_{A}^{\alpha}(x_i) + \left(1-\mu_{A}(x_i)\right)^{\alpha}\right\}^{\beta} - 1\right] + \left[exp\left\{\mu_{B}^{\alpha}(x_i) + \left(1-\mu_{B}(x_i)\right)^{\alpha}\right\}^{\beta} - 1\right]) - 2 \quad \textbf{(21)}$$

$$E_{\alpha\beta}(A \cap B) = \frac{1}{(1-\alpha)\beta}\sum_{i=1}^{n}\left[exp\{\mu_{A\cap B}^{\alpha}(x_i) + (1-\mu_{A\cap B}(x_i))^{\alpha}\}^{\beta} - 1\right] - 1 \quad \textbf{(22)}$$

$$E_{\alpha\beta}(A \cap B) = \frac{1}{(1-\alpha)\beta}\sum_{i=1}^{n}\left[exp\{\mu_{B}^{\alpha}(x_i) + (1-\mu_{B}(x_i))^{\alpha}\}^{\beta} - 1\right] + [exp\{\mu_{A}^{\alpha}(x_i) + (1-\mu_{A}(x_i))^{\alpha}\}^{\beta} - 1] - 2 \quad \textbf{(23)}$$

Adding (21) and (23) we obtain,

$$E_{\alpha\beta}(A \cup B) + E_{\alpha\beta}(A \cap B) = E_{\alpha\beta}(A) + E_{\alpha\beta}(B)$$

This proves the theorem.

Section 3: Limiting and Particular cases

a) When $\alpha \to 1$ *and* $\beta \to 0$ in equation (9), we obtain

$$E(A) = \frac{-1}{n}\sum_{i=1}^{n}[\mu_A(x_i)\log(\mu_A(x_i)) + (1-\mu_A(x_i))\log(1-\mu_A(x_i)]$$

Which is De-Luca and Termini logarithmic entropy.

b) When $\beta \to 0$ in equation (9), we obtain

$$E_{\alpha\beta}(A) = \frac{1}{(1-\alpha)}\sum_{i=1}^{n}[ln\{\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}\} - 1]\,, \alpha > 0.$$

Which is Bhandari and Pal exponential entropy.

Conclusion

In this study exponential fuzzy entropy is a new entropy measure introduced in the context of fuzzy set theory .Also investigated are a few of this measure's characteristics.The logarithmic entropy of De-Luca and Termini [2] and the exponential entropy of Pal and Pal [9] are generalised by this metric.We note that because real parameters α and β are present, newly developed measures offer more application flexibility.

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