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Probability on Fuzzy Sets

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Abstract

After introducing and developing fuzzy set theory, a lot of studies have been done to combine probability theory and fuzzy set theory. This works, called fuzzy probability, have been developed in some branches. In this article I review basic and fundamental work on fuzzy probability.

Introduction

The notion of fuzzy set has been an attempt to develop a mathematical framework in which to treat systems or phenomena which due to intrinsic indefiniteness, cannot themselves be characterized precisely. In the present paper, an overview of the combination of probability and fuzzy set theory has been discussed. It should be mentioned that, here the focus is only on the fuzzy probability. So, fuzzy statistics, fuzzy random variables, fuzzy stochastic processes, random sets approach to fuzzy sets, etc are not considered.

Preliminaries Probability of Fuzzy Sets

We adopt the conventional fuzzy set representation by its characteristic function. Let I denote the closed unit interval and Ω a background space, then the fuzzy set A defined by its fuzzy characteristic function

$$\mu_A : \Omega \to 1$$

The compliment of A, A^c is defined by $\mu_{A^c}(w) = 1 - \mu_A(w)$.

We define fuzzy set intersection, so that when De Morgan's law is applied the usual relationship on characteristic function holds. Specifically, for fuzzy sets A and B

$$\mu_{A \cup B}(w) = 1 - \mu_{A^c} \cap \mu_{B^c}(w)$$

$$= \mu_A(w) + \mu_B(w) - \mu_{A\cap B}(w) \quad (2.1)$$

If $\mu_{A\cap B}(w)$= min { $\mu_A(w)$, $\mu_B(w)$ }= $\mu_A(w)\,\Lambda\,\mu_B(w)$, then equation (2.1) implies that,

$$\mu_{A\cup B}(w) = \text{max } \{ \mu_A(w)\ \mu_B(w) \} = \mu_A(w)\,\text{V}\,\mu_B(w),$$

whereas if

$$\mu_{A\cap B}(w) = \{ \mu_A(w)\ \mu_B(w) \}$$

then

$$\mu_{A\cup B}(w) = \mu_A(w) + \mu_B(w) - \mu_A(w)\mu_B(w).$$

To define the probability of fuzzy set let Ω be a background space, A, a family of fuzzy set of Ω and Q, a measure on Ω which satisfies

$$\int_\Omega dQ = 1$$

For a given A∈A$_i$ probability of A, P(A) is defined by

$$P(A) = \int_\Omega \mu_A(w)dQ \quad (2.2)$$

since,

$$\mu_{A^c}(w) = 1 - \mu_A(w)$$

$$P(A^c) = 1 - P(A) \quad (2.3)$$

One of the objectives of this work is to study the properties of the probability measure on fuzzy sets, P(A), the left hand side of (2.2). It is obvious that P(A)≥0 for all A∈A$_1$ and if we define

$\mu_\Omega(w) = 1$ then P(Ω) = 1.

A fuzzy set probability calculus with the M&M algebra (minimum of fuzzy set membership functions for fuzzy set intersection and maxima for unions). The probability of the intersection of two fuzzy sets using the M&M algebra is given by

$$P(A\cap B) = \int_\Omega \mu_A(w)\Lambda\mu_B(w)dQ \quad (2.4)$$

and so

$$P(A\cup B) = \int_\Omega \mu_A(w)\vee\mu_B(w)dQ \quad (2.5)$$

Some consequences are $P(A\cup A) = P(A\cap A) = P(A\cap\Omega) = P(A)$ and $P(A\cap\Omega) = 1$. If A and B are fuzzy sets then $P(A\cup B) \geq P(A)\vee P(B)$ and $P(A\cap B) \leq P(A)\wedge P(B)$. It can be shown that traditional probability addition (STP) rule holds. That is, for fuzzy sets, A and B:

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

This rule may be generalized as in standard probability theory to n fuzzy sets. A consequence of the probability addition rule is a fundamental identity for fuzzy set

$$P(A \cap A^c) + P(A \cup A^c) = 1 \tag{2.6a}$$

$$P(A \cap A^c) = P((A \cup A^c)^c) \tag{2.6b}$$

In SPT $P(A \cap A^c)$ is always zero and (2.6a) is usually written in the form $P(A \cup A^c) = P(A) + P(A^c) = 1$, since a standard set and its compliment are disjoint and the probability of the union of disjoint sets is the sum of probabilities of the sets. With fuzzy sets, on the other hand, $P(A \cap A^c)$ need not be zero. Any time $\mu_A(w)$ is strictly between 0 and 1 for some w with nonzero probability $P(A \cap A^c)$ >0.

A fuzzy set probability calculus with PR algebra (probability like addition rule). In this case

$$P(A \cap B) = \int_{\Omega} \mu_A(w)\mu_B(w)dQ$$

and consequently

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

However,

$$P(A \cap A) = P(A^2) \text{ where}$$

$$P(A^2) = \int_{\Omega} \mu_A{}^2(w)dQ$$

and

$$P(A \cap A^c) = P(A) - P(A^2).$$

With the PR algebra it is true that $P(A \cap B^c) = P(A) - P(A \cap B)$ and so it will be seen that SPT independence properties carry over to fuzzy set probability theory (FSPT) if the PR algebra is used.

Independence and Conditional Probability

M&M algebra: A notion of fuzzy set independence was suggested by Zadeh, where he defined two fuzzy sets A and B to be independent if

$$P(A \cap B) = P(A).P(B) \tag{3.1}$$

We will now investigate the consequences of this definition and point out some of its peculiarities. Then an alternative definition of independence will be suggested that is formulated in terms of conditional probability.

With the independence defined by (3.1), it is possible to show that a fuzzy set A and its complement may be independent. If A and B are independent, it may be shown that A and B^c need not be also. The reason is that $P(A \cap B^c) \neq P(A) - P(A \cap B)$, in general with the M&M algebra. Thus for fuzzy sets independence need not imply that of their complements.

In SPT, the conditional probability of a set A given B, denoted by P(A/B), is defined by

$$P(A/B)=\frac{P(A\cap B)}{P(B)} \tag{3.2}$$

for P(B)>0. In SPT this expression is equivalent to

$$P(A/B)=\frac{P(A\cap B)}{P(A\cap B)+P(A^c\cap B)} \tag{3.3}$$

since for standard sets $B=(A\cap B)\cup(A^c\cap B)$. In fact, a SPT sample space partition $\{A\}_{i=1}$ may be used in the denominator of (3.2) to obtain a third equivalent form:

$$P(A/B)=\frac{P(A\cap B)}{\sum_{i=1}^{\infty}P(A_i\cap B)} \tag{3.4}$$

With fuzzy sets and the M&M algebra, it may be shown that these three alternatives, (3.2), (3.3) and (3.4) are not equivalent.

We adopt (3.2) as the definition of fuzzy set conditional probability, also proposed by Zadeh, and derived some consequences.

Since we are using the M&M algebra it is easy to see that P(A/A) = 1 and P(A/Ω) = P(A). If A and B are independent then P(A/B) = P(A). This motivated our choice in the definition of conditional probability. Since the independence of A and B need not imply that of A^c and B, then it does not follow that $P(A^c/B) = P(A^c)$ when A and B are independent even though P(A/B) = P(A). In addition, with the M&M algebra conditional probability equality hold:

$$P(A\cup B/C)=P(A/C)+P(B/C)-P(A\cap B/C) \text{ for P(C) > 0.}$$

It is clear that there are undesirable side effects with the assumed definitions of independence and conditional probability in the M&M algebra case. An alternative that avoids some of these (but presents others) would be to first define conditional probability and then use this definition to define independence, so that if A and B are independent, so are A^c and B. To do this, we define for fuzzy sets A and B, P(A) > 0, the conditional probability of A given B to be that equation (3) namely,

$$P(A/B)=\frac{P(A\cap B)}{P(A\cap B)+P(A^c\cap B)}$$

It is now evident that $P(A/B) + P(A^c/B) = 1$, which was not true with definition (2).

Next, we define the fuzzy set A to be independent of the fuzzy set B if P(A/B)=P(A), P(B)>0. In other words, P(A) must satisfy,

$$P(A) = \frac{P(A \cap B)}{P(A \cap B) + P(A^c \cap B)}$$

for P(B) > 0.

With this definition of independence it can be shown that if A is independent of B then A^c is also independent of B. However, it is also true that if A is independent of B then B need not be independent of A. These alternative definitions also allow the possibility to have a fuzzy set and its complement to be nontrivially independent.

PR algebra: The consequences of the independence (1) in the PR case are more inline with SPT. It may be shown that if the fuzzy sets A and B are independent, so are A, B^c and A^c, B^c. It is also possible in this case to have a fuzzy set A and its complement to be independent and such that $P(A \cap A^c) > 0$.

For the PR algebra $P(A \cap A) = P(A^2)$, so the definition of conditional probability (2) implies $P(A/A) = P(A^2)/P(A) \leq 1$, where as P(A/A) = 1 in the M&M case. In contrast with the M&M case since now the independence of two fuzzy sets A and B implies that of their complements, it is true that P(A/B) + P(A) implies $P(A^c/B) = P(A^c)$. In addition, we say that $P(A/B) + P(A^c/B) = 1$, which was observed not to hold in general in the M&M case. In fact the alternative approach to conditional probability and independence is equivalent to the present one in the PR case.

Conclusion

Fuzzy systems, including fuzzy logic and fuzzy set theory, provide a rich and meaningful addition to standard logic. The mathematics generated by these theories is consistent, and fuzzy logic may be a generalization of classic logic. The applications which may be generated from or adapted to fuzzy logic are wide-ranging, and provide the opportunity for modeling of conditions which are inherently imprecisely defined, despite the concerns of classical logicians. Many systems may be modeled, simulated, and even replicated with the help of fuzzy systems, not the least of which is human reasoning itself.

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